DOCUMENT RESUME

ED 389 534 SE 057 177

AUTHOR	Owens, Douglas T., Ed.; And Others
TITLE	Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (17th, Columbus, Ohio, October 21-24, 1995). Volumes 1 and 2: Plenary Lectures, Discussion Groups, Research Papers, Oral Reports, and Poster Presentations.
INSTITUTION	ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Columbus, Ohio.; International Group for the Psychology of Mathematics Education. North American Chapter.
SPONS AGENCY	Office of Educational Research and Improvement (ED), Washington, DC.
PUB DATE	Oct 95
CONTRACT	RR93002013
NOTE	821p.; For individual research papers, see SE 057 178-270. For the 1994 proceedings, see ED 383 533-534.
AVAILABLE FROM	ERIC/CSMEE, 1929 Kenny Road, Columbus, OH 43210-1080.
PUB TYPE	Collected Works - Conference Proceedings (021) -- Reports - Research/Technical (143) -- Information Analyses - ERIC Clearinghouse Products (071)
EDRS PRICE	MF05/PC33 Plus Postage.
DESCRIPTORS	Algebra; Constructivism (Learning); Cultural Differences; *Educational Change; Elementary Secondary Education; Epistemology; Functions (Mathematics); Geometry; Higher Education; Language Role; Learning Modalities; *Mathematics Education; Problem Solving; Rational Numbers; Research Methodology; Student Evaluation; *Teacher Education; *Technology; Visualization; Whole Numbers

ABSTRACT

In this conference proceedings the overarching theme of research on teaching and learning mathematics in diverse settings and the subthemes of diversity, constructivism and algebra are achieved in the plenary papers. The plenary papers and authors include "Constructivist, Emergent, and Sociocultural Perspectives in the Context of Developmental Research" (Paul Cobb & Erna Yackel); "Fairness in Dealing: Diversity, Psychology, and Mathematics Education" (Suzanne K. Damarin); and "A Research Base Supporting Long Term Algebra Reform?" (James J. Kaput). Included in these Proceedings are 84 research reports, two discussion groups, 40 oral reports and 43 poster presentation entries. The one-page synopses of discussion groups, oral reports and poster presentations are organized by topic along with the research reports. Papers are grouped under the following subject headings: advanced mathematical thinking, algebraic thinking, assessment, cognitive modalities, curriculum reform, epistemology, functions and graphs, geometric thinking, language and mathematics, probability and statistics, problem solving, rational number concepts, research methods, social and cultural factors, student beliefs and attitudes, teacher beliefs and attitudes, teacher change, teacher conceptions of mathematics, teacher education, teacher understanding of student understanding, technology, visualization, and whole numbers. An alphabetical list of addresses of authors is included in the appendix in Volume 2 with page numbers of their report or synopsis. For the first time the electronic mail address is included in this address list. (MKR)

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Proceedings of the
Seventeenth Annual Meeting

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Psychology of Mathematics Education

Volume 1: Plenary Lectures, Discussion Groups, Research Papers, Oral Reports, and Poster Presentations

October 21-24, 1995
The Ohio State University
Columbus, Ohio, U.S.A.

ERIC Clearinghouse for Science, Mathematics,
and Environmental Education

SE 057 177

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Proceedings of the
Seventeenth Annual Meeting

North American Chapter
of the International Group
for the

Psychology of Mathematics Education

Volume 1: Plenary Papers, Discussion Groups, Research Papers, Oral Reports, and Poster Presentations.

PME-NA XVII

October 21-24, 1995
The Ohio State University
Columbus, Ohio, U.S.A.

Editors:
Douglas T. Owens
Michelle K. Reed
Gayle M. Millsaps

Published by:
The ERIC Clearinghouse for Science,
Mathematics, and Environmental Education
Columbus, OH

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Cite as:

Owens, D. T., Reed, M. K., and Millsaps, G. M. (Eds.). (1995). *Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Document development:

David L. Haury, *Editor in Chief*
Linda A. Milbourne, *Copyeditor*
J. Eric Bush, *Cover Design and Page Layout*

Accession number: SE 057 177

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- A rationale for development of the document, including identification of target audience and the needs served.
- A vita and a writing sample.

This publication was funded in part by the Office of Educational Research and Improvement, U. S. Department of Education under contract no. RR93002013. Opinions expressed in this publication do not necessarily reflect the positions or policies of OERI or the Department of Education.

History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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1995	PME 19	Brazil	SE 057 135

Preface

This program began with a meeting of interested volunteers in November 1994 at Baton Rouge during the PME-NA meeting. The results of the suggestions made were taken to a meeting of the local program committee at The Ohio State University where the overarching theme of research on teaching and learning mathematics in diverse settings and the subthemes of diversity, constructivism and algebra were selected. These emphases are achieved in the plenary papers. Constructivism from a social perspective in the paper by Paul Cobb and Erna Yackel takes account of diverse learning idiosyncrasies. Fairness in dealing with diversity in characteristics and background of the learners was addressed in the paper by Suzanne Damarin. Reform in algebra toward making algebra more accessible to all students is addressed in the paper by James Kaput. No reaction paper to the Cobb and Yackel paper was requested because the opening plenary session ended with questions posed by the speaker, Paul Cobb, for roundtable discussions. Reactions to the paper by Suzanne Damarin were prepared by Ruth Cossey and Edward Silver from their perspectives of the work in which they are involved. Reactions to Jim Kaput's paper were written by Gail Burrill and Elizabeth Phillips who are involved in aspects of change in algebra curriculum.

Included in these *Proceedings* are 84 research reports, two discussion groups, 40 oral reports and 43 poster presentation entries. The one-page synopses of discussion groups, oral reports and poster presentations are organized by topic along with the research reports following the pattern begun with the *Proceedings* of the 1994 PME-NA meeting. Proposers expressed first choice: 124 research reports (2 withdrawals), 12 oral presentations, 35 poster presentations, and two discussion groups. Proposals for all categories were blind reviewed by three reviewers with expertise in the topic of the submission. Cases of disagreement among reviewers were refereed by subcommittees of the Program Committee at The Ohio State University. This process resulted in rejection or reassignment of about 31% of the research report proposals and about 25% overall.

For the first time, the submissions for the Proceedings were made on disk. These *Proceedings* were produced by the ERIC/CSMEE staff. The format of the papers were adjusted to make them uniform. As papers were assigned to topic areas for the table of contents, possible secondary or tertiary topic areas were noted. Thus, most papers are included with more than one descriptor in the index in the appendix in Volume 2. An alphabetical list of addresses of authors is included in the appendix in Volume 2 with page numbers of their report or synopsis. For the first time the electronic mail address is included in this address list. In the case of multiple authors, submissions were made with presenting author(s) name(s) underlined.

The editors wish to express thanks to all those who submitted proposals, the reviewers, the 1995 Program Committee, the PME-NA Steering Committee for making the program an excellent contribution to ongoing research and discussions of psychology and mathematics education; Dean Nancy Zimpher, College of Education, and the administration of the Department of Educational Theory and Practice at The Ohio State University, for their support; The Mathematics Education faculty and graduate students for their endless committee work; and the ERIC/CSMEE staff, especially Director David Haury, Linda Milbourne, and J. Eric Bush for the production of these *Proceedings*.

Douglas T. Owens
Michelle K. Reed
Gayle M. Millsaps
October, 1995

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Plenary Papers

CONSTRUCTIVIST, EMERGENT, AND SOCIOCULTURAL PERSPECTIVES IN THE CONTEXT OF DEVELOPMENTAL RESEARCH[1]

Paul Cobb, Vanderbilt University
Erna Yackel, Purdue University Calumet

Our overall intent is to clarify relationships between the psychological constructivist, sociocultural, and emergent perspectives by grounding them in our attempts to understand what might be happening in a variety of teaching and learning situations. In the first part of the paper, we therefore outline an interpretive framework that we have developed in the course of a classroom-based research project. At the level of classroom processes, the framework involves an emergent approach in which psychological constructivist analyses of individual activity are coordinated with interactionist analyses of classroom interactions and discourse. At the level of school and societal processes, the perspective taken is broadly sociocultural and focuses on the influence of individuals' participation in culturally-organized practices. In the second part of the paper, the framework is taken as background against which to compare and contrast the three theoretical perspectives. We discuss how the emergent approach augments the psychological constructivist perspective by making it possible to locate analyses of individual students' constructive activities in social context. In addition, we consider the purposes for which the emergent and sociocultural perspectives might be appropriate and observe that they together span the individual students' activities, the classroom community, and broader communities of practice.

It can be argued that one of the most significant developments in educational research during the past decade has been the increasingly prominent role played by constructivist and sociocultural approaches. In contrast to the initial claims made by adherents to each perspective for the hegemony of their own views, there appears to be a growing consensus that the perspectives are at least partially complementary (Cobb, 1994; Confrey, 1995; Hatano, 1993; Smith, in press; Steffe, 1995). Our interest in the relationship between sociocultural theory and various forms of constructivism is pragmatically based and stems from our involvement in a classroom-based research and development project. In the course of our work with teachers and their students, we addressed a variety of issues by drawing on several different interpretive perspectives. The views we will advance in this paper about the relationships between interpretive perspectives are therefore rooted in our activity of attempting to understand what might be going on in a range of specific teaching and learning situations.

In the first part of this paper, we describe the interpretive framework that we currently use when analyzing teachers' and students' activity. Our intention in doing so is to ground the proposed relationships between perspectives in the settings from which they first emerged. At the level of classroom processes, this framework represents an emergent or social constructivist approach that evolved from an initial psychological constructivist position. The framework was subsequently extended beyond the classroom level to the school and societal levels by drawing on sociocultural theory. In the course of the discussion, we justify the framework by indicating the unanticipated problems that we found ourselves addressing and the interpretive stances that we eventually took. In the second part of the paper, this presentation of the framework is then used as a backdrop against which to compare and contrast psychological constructivist, emergent, and sociocultural perspectives.

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[1] The analysis reported in this paper was supported by the National Science Foundation under grant No. RED-9353587. The opinions expressed do not necessarily reflect the views of the Foundation. The authors are grateful to Janet Bowers for numerous helpful comments on an earlier draft of this paper.

The approach we will take of attempting to ground theory in practice reflects the view that the relationship between theory and practice is reflexive (Cobb, in press; Simon, 1995). Theory is seen to grow out of practice and to feed back to inform and guide practice. This approach can be contrasted with more traditional styles of presentation in which the basic principles or tenets of theoretical positions are stated, and then implications are deduced for practice. As Schön (1983) observes, this rhetorical style elevates theory over practice and enacts a positivist epistemology of practice, thereby devaluing the relation between theory and practice as it is lived by reflective practitioners (Ball, 1993; Lampert, 1990; Simon & Blume, 1994). Further, it positions researchers and practitioners in superior and subordinate roles as producers and consumers of theory. In contrast, alternative approaches that attempt to ground theory in practice tend to position researchers and practitioners in more collaborative roles and to treat their areas of expertise as complementary (Nicholls & Hazzard, 1993). Approaches of this type also acknowledge the importance of developing a basis for communication between researchers and practitioners. As a consequence, they seem to have greater potential to contribute to current reform efforts.

The Interpretive Framework

The interpretive framework we will outline was developed in the course of an ongoing program of developmental research (Gravemeijer, 1994). The basic developmental research cycle consists of two closely related phases (see Figure 1). At the most global level, our goal has been to investigate ways of supporting elementary school students' conceptual development in mathematics. As part of this process, we and our colleagues have developed both sequences of instructional activities for students and an approach to professional development for teachers. The general methodology employed is that of the classroom teaching experiment conducted in collaboration with a practicing teacher who is a member of the research and development team. In the past nine years,we have completed a series of these experiments at the first-, second-, and third-grade levels. It became apparent in the first of these experiments that the individualistic psychological constructs that we had intended to use to account for mathematical learning were inadequate for our purposes. As a consequence, one of our primary theoretical objectives became that of exploring ways to account for students' mathematical development *as it occurs in the social context of the classroom.* Analyses of this type are central to the second of the two phases of the development research cycle shown in Figure

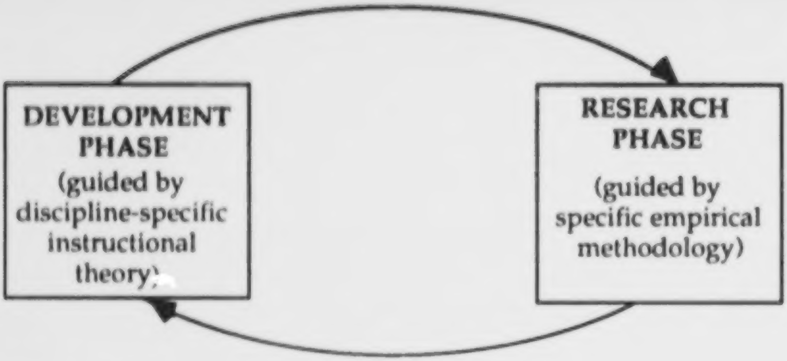
DEVELOPMENT PHASE
(guided by discipline-specific instructional theory)
RESEARCH PHASE
(guided by specific empirical methodology)

Figure 1. Phases of the Developmental Research Cycle[2]

1 and feed back to inform ongoing development efforts. The interpretive framework we will outline should be viewed as a response to this issue of accounting for learning in social context. Although our focus has been on students' development, there is some indication that the framework might be appropriate for analyses of teachers' socially-situated activity (Simon, 1995). However, we want to avoid the essentialist implication that the framework might somehow capture the structure of individual and collective activity independent of history, situation, and purpose. Our strongest claim is that we have found the framework useful when attempting to support change at the classroom and school levels.

The Classroom Level

It is important to clarify that when we speak of the classroom level, we do not mean this as a physical location. Instead, our intent is to indicate that explanations are formulated in terms of processes that occur in the classrooms—individual interpretations and actions, and face-to-face interactions and discourse. Thus, these explanations of students' activity in the classroom do not make reference to their participation in practices outside of the classroom. The interpretive framework at this level is shown in Figure 2.

The column heading "Social Perspective" refers to an interactionist perspective on communal or collective classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988). The column heading "Psychological Perspective" signifies a psychological constructivist perspective on individual students' (or the teacher's) activity as they participate in and contribute to the development of these communal processes (von Glasersfeld, 1984, 1992). The version of social constructivism to which we sub-

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[2] The domain-specific instructional theory referred to is that of Realistic Mathematics Education developed at the Freudenthal Institute (Streefland, 1991; Treffers, 1987). This developmental research cycle is, in many ways, analogous to the mathematical teaching cycle described by Simon (1995).

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

Figure 2. An Interpretive Framework for Analyzing Individual and Collective Activity at the Classroom Level.

scribe is called the *emergent approach* or the *emergent perspective* and involves the explicit coordination of interactionism and psychological constructivism (Cobb & Bauersfeld, 1995). In the following paragraphs, we outline the framework by discussing social norms, then sociomathematical norms, and finally classroom mathematical practices.

Social norms. When we conducted our first classroom teaching experiment during the 1986-87 school year, we initially viewed learning in almost exclusively psychological constructivist terms. This methodology was in fact devised as an extension of the constructivist teaching experiment in which the researcher interacts one-on-one with a single child and attempts to influence the child's constructive activities (Cobb & Steffe, 1983; Steffe, 1983). In the case of the constructivist teaching experiment, the goal is to account for the child's development of increasingly powerful mathematical ways of knowing by analyzing the cognitive restructurings he or she makes while interacting with the researcher. In a similar manner, we intended to account for individual children's learning in the classroom by analyzing the conceptual reorganizations they made while interacting with the teacher and their peers. With hindsight, it is apparent that the relation between social interaction and children's mathematical development implicit in this approach was neo-Piagetian. We assumed that conflicts in individual students' mathematical interpretations might give rise to internal cognitive conflicts, and that these would precipitate mathematical learning (cf. Doise & Mugny, 1984; Perret-Clermont, 1980). In this account, social interaction was viewed as a catalyst for otherwise autonomous cognitive development in that it influenced the process of mathematical development but not its products, increasingly sophisticated mathematical ways of knowing.

The first unanticipated issue that we addressed in the classroom teaching experiment arose within the first few days of the school year. The second-grade teacher with whom we collaborated engaged her students in both collaborative small group work and whole class discussions of their mathematical interpretations and solutions. However, it soon became apparent that the teacher's expectation that the children would publicly explain how they had actually interpreted and solved tasks ran counter to their prior experiences of class discussions in school. The students had been in traditional classrooms during their first-grade year and seemed to take it for granted that they were to infer the response the teacher had in mind rather than to articulate their own understandings. The teacher coped with this conflict between her own and the students' expectations by initiating a process that we subsequently came to term the renegotiation of classroom social norms. Examples of social norms for whole class discussions that became explicit topics of conversation included explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives in situations where a conflict in interpretations or solutions has become apparent. In general, social norms can be seen to delineate the classroom participation structure (Erickson, 1986; Lampert, 1990).

A detailed account of the renegotiation process has been given elsewhere (Cobb, Yackel, & Wood, 1989). For our purposes, it suffices to note that a social norm is not a psychological construct that can be attributed to any particular individual, but is instead a joint social construction. As a consequence, we would object to accounts framed in individualistic terms in which the teacher is said to establish or specify social norms for students. To be sure, the teacher is necessarily an institutionalized authority in the classroom (Bishop, 1985). However, in our view, the most the teacher can do is express that authority in action by initiating and guiding the renegotiation process. The students also have to play their part in contributing to the establishment of social norms.[3] One of our primary contentions is in fact that in making these contributions, students reorganize their individual beliefs about their own role, others' roles, and the general nature of mathematical activity (Cobb et al., 1989). As a consequence, we take these beliefs to be the psychological correlates of the classroom social norms.

It is important to clarify that, in the view we are advancing, neither the social norms nor individual students' beliefs are given primacy over the other. Thus, it is neither a case of a change in social norms causing a change in students' beliefs, nor a case of students first reorganizing their beliefs and then contributing to the evolution of social norms. Instead, social norms and beliefs are seen to be reflexively related such that neither exists independently of the other. We can further clarify our position by building on Whitson's (in press) observation that what are seen are human processes that can be analyzed in either psychological or social terms depending on the issues at hand. A social analysis conducted from the interactionist perspective documents the evolution of social norms, and an analysis conducted from the psychological constructivist perspective documents students' reorganization of their beliefs. The social constructivist or emergent approach to which we subscribe draws on both these analyses and treats them as complementary. In this joint perspective, the social norms are seen to evolve as students reorganize their beliefs and, conversely, the reorganization of these beliefs is seen to be enabled and constrained by the evolving social norms.

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[3] Cooney's (1985) analysis of Fred, a beginning mathematics teacher, provides an excellent illustration of this point. In our terms, Cooney documents the difficulties that Fred encountered when he attempted to initiate the renegotiation of social norms and institute a problem solving approach.

Sociomathematical norms. Thus far, in describing our initial interest in classroom social norms, we have explained why we found it necessary to go beyond an exclusively individualistic psychological perspective. We should again stress that we did not analyze social norms as an end in itself. Instead, our overriding motivation was to account for students' mathematical development as it occurred in the social context of the classroom. In this regard, one aspect of our analysis of social norms that proved disquieting was that it was not specific to mathematics, but applied to almost any subject matter area. For example, one would hope that students would challenge each other's thinking and justify their own interpretations in science and literature lessons as well as in mathematics lessons. As a consequence, our focus shifted in subsequent analyses to the normative aspects of whole class discussions that are specific to students' mathematical activity (Lampert, 1990; Voigt, 1995; Yackel & Cobb, in press). Examples of such *sociomathematical norms* include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation.

As part of the process of guiding the development of an inquiry approach to mathematics in their classrooms, the teachers with whom we have worked regularly asked the students if anyone had solved a task a different way and then questioned contributions that they did not consider were mathematically different. It was while analyzing classroom interactions in these situations that sociomathematical norms first emerged as an explicit focus of interest. The analysis indicated that, on the one hand, the students did not know what would constitute a mathematical difference until the teacher and other students accepted some of their contributions but not others. Consequently, in responding to the teacher's requests for a different solution, the students were both learning what counts as a mathematical difference and helping to interactively constitute what counts as a mathematical difference in their classroom. On the other hand, the teachers in these classrooms were themselves attempting to develop an inquiry form of practice and had not, in their prior years of teaching, asked students to explain their thinking. Consequently, the experiential basis from which they attempted to anticipate students' contributions was extremely limited. Further, they had not necessarily decided in advance what would constitute a mathematical difference. Instead, the teachers seemed to clarify their own understanding of mathematical difference as they interacted with their students. Viewed in this way, the sociomathematical norm of mathematical difference appeared to emerge in the course of joint activity via a process of often implicit negotiation. A similar conclusion also holds for the other sociomathematical norms we have analyzed (Yackel & Cobb, in press).

The analysis of sociomathematical norms has proved to be pragmatically significant in that it has helped us understand the process by which the teachers with whom we have collaborated fostered the development of intellectual autonomy in their classrooms. This issue is particularly significant to us in that the development of student autonomy was a goal of our work in classrooms that was explicitly stated from the outset. We originally characterized intellectual autonomy in terms of students' awareness of and willingness to draw on their own intellectual capabilities when making mathematical decisions and judgments. This view of intellectual autonomy was contrasted with intellectual heteronomy wherein students rely on the pronouncements of an authority to know how to act appropriately (Kamii, 1985; Piaget, 1973). As part of the process of supporting the growth of autonomy, the teachers with whom we have worked guided the development of a community of validators in their classrooms. In doing so, they necessarily had to encourage the devolution of responsibility (cf. Brousseau, 1984). However, their students could assume these responsibilities only to the extent that they had developed personal ways of judging that enabled them to know-in-action both when it was appropriate to make a mathematical contribution and what constituted an acceptable contribution. This required, among other things, that the students could judge what counted as a different mathematical solution, an insightful mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation. However, these were precisely the types of judgments that they and the teacher negotiated when establishing sociomathematical norms. We therefore inferred that students constructed specifically-mathematical beliefs and values that enabled them to act as increasingly autonomous members of classroom mathematical communities as they participated in the renegotiation of sociomathematical norms (Yackel & Cobb, 1993). These beliefs and values, it should be noted, are psychological constructs and constitute what the National Council of Teachers of Mathematics (1991) calls a mathematical disposition. We view them as the psychological correlates of the sociomathematical norms and consider the two to be reflexively related (see Figure 2).

It is apparent from the account we have given that we revised our conception of intellectual autonomy in the course of the analysis. At the outset, we defined autonomy in psychological terms as a characteristic of individual activity. However, by the time we had completed the analysis, we came to view autonomy as a characteristic of an individual's participation in a community. Thus, although the development of autonomy continues to be a central pragmatic goal for us, we have redefined our view of what it means to be autonomous by going beyond our original psychological constructivist position. This shift in perspective has enabled us to be more effective in helping teachers support the development of autonomy in their classrooms (McClain, 1995).

Classroom mathematical practices. The third aspect of the interpretive framework, that concerning classroom mathematical practices, was motivated by the realization that one can talk of the mathematical development of a classroom community as well as of individual children. For example, in the second-grade classrooms in which we have worked, various solution methods that involve counting by ones are established mathematical practices at the beginning of the school year. Some of the students are also able to develop solutions that involve the conceptual creation of units of ten and one. However, when they do so, they are obliged to explain and justify their interpretations of number words and numerals. Later in the school year, solutions based on such interpretations are taken as self-evident by the classroom community. The activity of interpreting number words and numerals in this way has become an established mathematical practice that no longer stands in need of justification. From the students' point of view, numbers simply are composed of tens and ones—it is a mathematical truth.

This illustration from the second-grade classrooms describes a global shift in classroom mathematical practices that occurred over a period of several weeks. An example of a detailed analysis of evolving classroom practices can be found in Cobb et al. (in press). We contend that analyses of this type are appropriate for the purposes and interests of developmental research in that they document instructional sequences as they are realized in interaction in the classroom. They therefore draw together the two general phases of developmental research, instructional development and classroom-based research, and feed back to inform ongoing development efforts (see Figure 1).

Analyses of this type are also of theoretical significance in that they bear directly on the issue of accounting for mathematical learning as it occurs in the social context of the classroom. Viewed against the background of classroom social and sociomathematical norms, the mathematical practices established by the classroom community can be seen to constitute the immediate, local situations of the students' development. Consequently, in identifying sequences of such practices, the analysis documents the evolving social situations in which students participate and learn. Individual students' mathematical conceptions and activities are taken as the psychological correlates of these practices, and the relation between them is considered to be reflexive. In particular, students actively contribute to the evolution of classroom mathematical practices as they reorganize their individual mathematical activities and, conversely, these reorganizations are enabled and constrained by the students' participation in the mathematical practices.

As a point of clarification, we should stress that psychological analyses typically reveal qualitative differences in individual children's mathematical interpretations even as they participate in the same mathematical practices. In general, analyses conducted from the psychological constructivist perspective bring out the heterogeneity in the activities of members of a classroom community. In contrast, social analyses of classroom mathematical practices conducted from the interactionist perspective bring out what is jointly established as the teacher and students coordinate their individual activities. In drawing on these two analytic perspectives, the emergent approach focuses on both the individual and the community. This approach seeks to analyze both the development of individual minds and the evolution of the local social worlds in which those minds participate (Balacheff, 1990).

Summary. We pause to make two points about the interpretive framework as we have outlined it thus far. The first point concerns a possible misinterpretation. In the past, we have sometimes been interpreted as saying that students' mathematical activity is essentially psychological and individualistic, but is constrained by social and cultural processes such as social norms. We therefore emphasize that we consider students' mathematical activity to be social through and through in that it develops as they participate in classroom mathematical practices. More generally, our intent is not to classify the teacher's and students' activities into those that are intrinsically individual and those that are intrinsically communal. Instead, our proposal is to *coordinate* analyses of classroom processes that are conducted in psychological and in social terms.

The second point we want to make is methodological and concerns the notion of replicability in the context of developmental research. The results of developmental research consist of a variety of products and analyses. These include sequences of instructional activities and analyses of students' learning in social context as the sequences are realized in interaction in classrooms. On the one hand, the assumption that productive patterns of learning can occur when an instructional sequence is enacted in other classrooms is central to the developmental research enterprise. On the other hand, the conception of the teacher as one who continually adjusts his or her plans on the basis of ongoing assessments of students' understandings implies that complete replicability is neither desirable nor, perhaps, possible (cf. Ball, 1993; Simon, 1995). The enactment of an instructional sequence is therefore assumed to involve experimentation on the part of the teacher in the course of which the sequence as intended by its developers is deliberately revised and modified.

Taking account of this formulation of the issue, we observe that educational research is replete with more than its share of wildly disparate and irreconcilable findings. The primary source of difficulty in our view is that the independent variables of traditional experimental research are relatively superficial and have little to do with either context or meaning. Such approaches are difficult to justify if one follows Lemke (in press) and considers that the ecology of the classroom is semiotic and involves meaning-making in which one thing is taken as a sign for another. Lemke calls systems with semiotic ecologies ecosocial systems. From this point of view, students are seen to always perceive, act, and learn by participating in the self organization of a system which is larger than themselves—the community of practice established in the classroom. Learning can therefore be characterized as "an aspect of self organization, not just of the human organism as a biological system, but of ecosocial systems in which the organism functions as a human being" (Lemke, in press). It is precisely this sense of participation in an evolving community of practice that is typically ignored in traditional educational research.

These considerations lead us to suggest that the relevant concept is commensurability rather than replicability. The difficulty is not so much that past findings have been disparate, but that they have been irreconcilable—it has not been possible to account for differences in findings when different groups of students received supposedly the same instructional treatment. The challenge as we see it is not that of replicating instructional sequences by ensuring that they are enacted in the same way in different classrooms. Instead, it is to develop ways of analyzing both instructional sequences and students' participation in them as they are realized in interaction in different classrooms. In this regard, we note that the framework as we have outlined it thus far illustrates one possible way to organize analyses of both the classroom ecosocial system and the activity of the students (and teacher) who contribute to its development. For example, we have suggested that the constructs of social norms, sociomathematical norms, and classroom mathematical practices address aspects of the classroom microculture that are relevant to the purposes of developmental research. An analysis of classroom events organized in this way therefore might relate the emerging patterns of students' learning to their participation in sequences of instructional activities as they are realized in interaction. In addition, the teacher's role in guiding the development of both the classroom ecosocial system and the activity of the children who participate in it could become an explicit object of analysis, as could the broader institutional contexts in which such systems are embedded.

We should clarify that the intent of these comments is neither to recommend that others should necessarily use the specific framework we have outlined nor to claim that this framework resolves the commensurability issue once and for all. Instead, it is to illustrate the potential contribution of a framework of this type that is concerned with context and meaning. In particular, such a framework might support greater precision in developmental research by making it possible to compare, contrast, and relate different enactments of instructional sequences. This in turn would facilitate disciplined, systematic inquiry that embraces the messiness and complexity of the classroom.

The School and Societal Levels

In the course of our ongoing research and development activities, we have often been able to develop explanations that proved adequate for our purposes by referring solely to classroom processes. These analyses focus on the classroom ecosocial system as it is portrayed in the framework shown in Figure 2. There have, however, been occasions when we have found it essential to take account of the broader institutional contexts in which such systems are embedded. The elaborated version of the interpretive framework shown in Figure 3 synthesizes our reflections on these experiences. The central box replicates Figure 2 and corresponds to practices at the classroom level. The next box corresponds to practices at the school level, and the outermost box to practices at the societal level. In the following paragraphs, we provide a grounding for the elaborated framework. Later in the paper, we will reflect back on the discussion and use it as a setting in which to clarify distinctions between the emergent and sociocultural perspectives.

School level. The need to take account of broader institutional contexts first became apparent to us when we attempted to account for our experiences of working with approximately 50 first-, second-, and third-grade teachers at two action research sites. One of these sites was rural/suburban whereas the other served an almost exclusively inner-city student population. Our overall goal was to help these teachers revise the way in which they taught mathematics. To this end, we formulated an initial approach to teacher development at the rural/suburban site, where it proved to be reasonably successful. Our first priority when working with the teachers at this site was to help them make aspects of their textbook-based instruction prob-

Social Perspective	Psychological Perspective
General societal norms	Beliefs about what constitutes normal or natural development in mathematics
General school norms	Conception of the child in school—beliefs about own and others' role in school
Classroom social norms	Beliefs about own role, others' roles, and general nature of mathematics in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

Figure 3. An Elaboration of the Interpretive Framework to the School and Societal Levels

lematic. We reasoned that only then would they have reason and motivation to want to reform their instructional practices while working with us. To this end, we used videorecordings of both individual interviews and classroom episodes to explore the consequences of traditional instruction. We have previously documented the success of this approach at the rural/suburban site. We observed, for example, that the teachers:

> began to differentiate between correct adherence to accepted procedures and [children's] mathematical activity that expressed conceptual understanding.
>
> As the teachers began to question the adequacy of textbook instructional activities and their current ways of teaching, they were then willing to consider alternative instructional activities designed to encourage meaningful mathematical activity. In doing so, they demonstrated the value they placed on children's mathematical sense-making. We did not have to convince them that children should learn with understanding. Rather, they had assumed that this kind of learning was occurring in their classrooms. *A shared desire to facilitate meaningful learning and a general concern for children's intellectual and social welfare*

> *constituted the foundation upon which we and the teachers began to mutually construct a consensual domain.* (Cobb, Wood, & Yackel, 1990, p. 140, emphasis added)

With our support during the school year, the 20 teachers referred to in the above passage radically revised the way they taught mathematics.

Shortly after this passage was written, we began working at the inner-city site. It soon became apparent that our initial approach to teacher development was not viable at that site. For the most part, an exploration of the consequences of traditional instruction did not lead these teachers to question their primarily drill-based approaches. It therefore appeared that whereas the teachers at the rural/suburban site assumed without question that students should learn mathematics with understanding, the beliefs and values of the teachers at the inner city site did not appear to be in conflict with traditional instructional practices. Subsequent efforts to support these teachers were more successful than our initial attempts in that several of them did develop forms of practice that were compatible with current reform recommendations in mathematics education. However, as we have documented elsewhere, the process by which these teachers reorganized their practices differed significantly from that of the teachers at the rural/suburban site (Feikes, 1992; Yackel & Cobb, 1993). This again indicates that there were differences in the two groups of teachers' underlying beliefs and values.

In reflecting on these experiences, we have come to realize that assumptions that we initially considered to be self-evident in fact reflect our culturally-specific beliefs and values. After working with teachers at the rural/suburban site, we had written that "a shared desire to facilitate meaningful learning and a general concern for children's intellectual and social welfare" constituted the foundation on which we and the teachers developed a basis for communication. At the time we wrote this statement, we assumed unquestioningly that engaging children in what for us counts as meaningful learning would necessarily be viewed as contributing to their welfare. However, our experiences at the inner-city site have led us to reconsider this assumption.

An analysis of observations made at the inner-city site during both classroom mathematics lessons and teacher induction sessions indicates that these teachers were deeply concerned about their students' intellectual and social welfare. However, there were crucial differences in what counted as intellectual and social welfare at the two sites (Yackel & Cobb, 1993). In particular, strictly enforced discipline seemed to be highly valued by teachers and administrators at the inner-city site. In addition, we did not observe instances where rules were discussed with students. Thus, although there were discussions of whether a rule had been violated, neither the appropriateness of the rules nor reasons for complying with them seemed to be topics of conversation.

In accounting for these differences between the two sites, we have come to the view that *what it means to be a child in school* is constituted by pedagogical communities (Banks, 1995; Walkerdine, 1988). Therefore this notion does not therefore appear to be fixed and universal, but is instead continually regenerated by the members of a pedagogical community as they participate in the practices of schooling. At the inner city site, for example, to be a child in school was to follow specific rules and instructions. Further, to understand was to be able to verbalize relevant rules. Consequently, adults showed their concern for children's welfare by helping them learn to follow and verbalize rules. There was therefore no conflict at this site between the consequences of traditional mathematics instruction and the institutionalized views about what it means to be a child in school. This in turn implies that the teachers had no reason to revise their instructional practices.[4]

It is apparent that in the course of this discussion of our experiences at the two sites, we have viewed the teachers as representatives of particular communities of practice. As we will see, this approach of characterizing individuals in terms of community membership is typical of the sociocultural perspective. With regard to the implications of the analysis, we observe that core beliefs and values implicit in current reform recommendations are compatible with those of the teachers at the rural/suburban site but conflict with those of teachers at the inner-city site. This, for us, raises the very real possibility that reform efforts in which mathematics educators assume that their culturally-situated commitments are universal might well result in even greater disparities in the types of mathematics education that children experience than is currently the case. We therefore join Apple (1992) in calling for mathematics educators to explicate the ideological assumptions underpinning their reform recommendations. Only then might we be able to guard against the possibility that we will unknowingly foster even greater inequities.

Societal level. The grounding for the discussion of practices at this level is provided by an analysis reported by Yang and Cobb (1995). At the outset, our goal was simply to build on previous investigations of the mathematics achievement of Asian and American students by comparing the arithmetical learning of children in Taiwan and the U.S. However, in the course of the analysis, we came to the view that children in the two countries were participating in very different types of learning activities, and that these activities were culturally organized at the societal level.

With regard to the specifics of the investigation, the analysis covered preschool through second grade and dealt with arithmetical developments up to and including the construction of place value conceptions. Consistent with previous investigations, an analysis of videorecorded individual interviews indicated that there were significant differences in the quality of the two groups of children's arithmetical conceptions that favored the Chinese children in Taiwan (cf. Stevenson & Lee, 1990). In addition, an interactional analysis of classroom videorecordings made in the two countries indicated that there were important differences in the obligations that the children had to fulfill to appear competent (cf. Stigler, Fernandez, & Yoshida, 1992). However, the most relevant differences for our current purposes were those between the sequences of learning activities in which the children in the two countries participated. These sequences were identified by analyzing textbooks and by interviewing parents and teachers of the kindergarten, first-, and second-grade students. The issues addressed in these interviews included the types of learning activities that the teachers and parents considered most important for children's arithmetical development, the specific concepts and methods that children were expected to develop, the extent to which children needed either assistance or directed instruction, and the parents' and teachers' expectations for childrens' competencies at various age and grade levels.

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[4] We have been asked on several occasions whether the differences between the school communities reflect differences in the wider communities in which they were embedded. It would be inappropriate for us to address this issue for ethical reasons that pertain to the nature of the relationships we established with teachers and administrators at the two sites. As a consequence, a level corresponding to the wider community beyond the school is not included in Figure 3.

The analysis indicated that there were important differences in the teachers' and parents' expectations for both the learning routes that the children would follow and the competencies they would develop, and in the extent to which the adults believed that it was necessary to provide direct support. In addition, there appeared to be differences in the internal consistency and coherence of the sequences of learning activities in the two countries. The American learning activities appeared to involve a major discontinuity in that the children's initial experiences in situations involving single-digit numbers did not appear to constitute a basis for their subsequent construction of place value conceptions. Significantly, the American teachers and parents considered that place value was a challenging concept and that it should be delayed until the second grade. In addition, the American teachers unequivocally stated that direct instruction was required. By way of contrast, the culturally-organized learning activities in Taiwan did not appear to have such contradictions. Further, the Chinese parents and teachers treated place value conceptions as relatively unproblematic developments that should begin in kindergarten. The tasks they posed and the questions they asked both seemed to reflect the view that it is natural for children to conceptualize numbers as composed of tens and ones at a relatively early point in their arithmetical development. In addition, they did not consider that this relatively easy developmental stage required direct instruction.

It is apparent from the analysis that the culturally-organized learning activities in which the Chinese students participated tended to enable the development of conceptual understanding in arithmetic to a far greater extent than did the learning activities in which the American students participated. Further, these differences in learning activities appeared to both corroborate and be supported by differences in the American and Chinese parents' and teachers' beliefs about what constitutes normal or natural development when children learn arithmetic. For example, the American parents and teachers had good reasons for believing that place value was a relatively late development. This belief in turn sustained pedagogical practices in which place value was experienced as a relatively challenging concept. Similarly, the Chinese teachers' and parents' beliefs were both expressed in and corroborated by the culturally-organized learning activities in which they and the children participated. It therefore seems reasonable to characterize these two contrasting sets of beliefs about normal development as culturally-situated social constructions that are reflexively verified in practice.

The general position that we have arrived at is consistent with sociocultural approaches in that the American and Chinese children's contrasting arithmetical competencies are accounted for in terms of their participation in differing sequences of culturally-organized learning activities (cf. Cole, 1990; Lave & Wenger, 1991; Rogoff, 1994). Explanations of this type can be contrasted with an alternative orientation consistent with mainstream American psychology in which culture is treated as a cluster of variables that influences the course of essentially individualistic psychological processes. It should also be noted that the characterization of beliefs about psychological development as social constructions applies as much to widely accepted academic theories as it does to so-called folk theories (Lave, 1988; Newman, Griffin, & Cole, 1989). This, of course, does not imply that academic theories are mere myths or fictions, or that they are nothing more than arbitrary social conventions. Our point is instead that these theories are culturally-situated and that their development is guided by particular concerns and interests (Barnes, 1977). In our own case, for example, we have come to see the emergent approach we have outlined as grounded locally in the practices of developmental research, and as located more globally in an encompassing activity system that constitutes schooling in the U.S.

Summary. The interpretive framework shown in Figure 3 emerged relative to our purposes and offers a way to organize analyses conducted from different perspectives. It was in fact with these analyses in mind that we have previously discussed possible relationships between theoretical perspectives (e.g., Cobb, 1994). The order in which we have described the various perspectives, starting with a psychological constructivist perspective and working our way out, retraces the developments in our thinking over the last several years. The discussion of the framework at the classroom level focused on the relation between psychological constructivism and the emergent perspective. The sociocultural perspective came to the fore when we considered practices at the school and societal levels. In the remainder of this paper, we step back to compare and contrast these theoretical orientations more directly.

Coordinating Perspectives

Psychological Constructivism and the Emergent Perspective

We have seen that the emergent perspective augments psychological constructivism by coordinating it with interactionism. The relationship between psychological constructivism and interactionism can be clarified by considering, as an illustration, a situation in which a researcher is interacting with one student. To the extent that a psychological constructivist analysis takes account of the interaction, the focus is on the student's interpretations of the teacher's actions. An analysis of this type is made from the perspective of the researcher, who is inside the interaction and is concerned with the ways in which the student modifies his or her activity. In contrast, an interactional analysis is made from the outside, from the point of view of an observer rather than of a participant in the interaction. From this perspective, the focus is on regularities in the student's and researcher's interactions, and on the consensual meanings that emerge between them, rather than on the student's personal interpretations. As Voigt (1994) makes clear, these consensual meanings are not psychological elements that capture the partial match of individual interpretations, but are instead located at the level of interaction. Examples of such constructs illustrated during the discussion of the interpretive framework include social norms, sociomathematical norms, and classroom mathematical practices.

Despite claims made to the contrary, we contend that researchers who typically take an individualistic focus are not conducting an interactional analysis merely because the students whose activity they are analyzing happen to be interacting with others. The researcher is conducting a psychological analysis as long as he or she focuses on the activity of each of the interacting individuals and fails to take their joint activity as an explicit object of analysis (Blumer, 1969). By the same token, it should be clear that the emergent approach does not merely involve bolting a social component onto an otherwise unchanged psychological approach. Instead, the relation between the interactionist and psychological constructivist perspectives is considered to be reflexive. The characterization of learning as an individual constructive activity is therefore relativized in that these constructions are seen to occur as students participate in and contribute to the practices of the local community.

The comments that we have made thus far do not delegitimize psychological analyses of, say, interviews or one-on-one teaching sessions. However, we do question whether such analyses capture individual students' conceptual understandings independently of situation and purpose. From the emergent perspective, interviews are viewed as social events in which the researcher and child negotiate their roles, their interpretations of tasks, and their understanding of what counts as a legitimate solution and an adequate explanation (Mischler, 1986; Schoenfeld, 1987; Voigt, 1995). As a consequence, we believe that it is important to view the students' activity as being socially situated even in settings such as interviews that are typically associated with psychological paradigms. The psychological analysis would then be conducted against the background of a social analysis that documents the interactively constituted situation in which the student is acting.

We have argued that the emergent approach is consistent with the purposes of classroom-based developmental research. We have also clarified that analyses conducted in line with this approach can give greater weight to either the psychological or the interactionist perspective depending on the issues and purposes at hand. In each case, one perspective comes to the fore against the background of the other. This reciprocity between the psychological and the social in turn serves to differentiate the emergent approach from sociocultural approaches.

Emergent and Sociocultural Perspectives

The emergent and sociocultural perspectives have a number of points in common. For example, both reflect the view that learning and understanding are inherently social and cultural activities. The two positions therefore reject the view that social interactions serve as a catalyst for otherwise autonomous intellectual development. Further, both attend to the role of symbols and artifacts in conceptual development. However, whereas the emergent perspective subsumes psychological constructivism, the sociocultural perspective constitutes an alternative to approaches that attribute a primary role to individual students' constructive activities.

We have seen that from an emergent perspective, learning is a constructive process that occurs while participating in and contributing to the practices of the local community. In the case of the interpretive framework, for example, students were seen to actively construct their mathematical ways of knowing as they participated in the mathematical practices of the classroom community. The link between collective and individual processes in this approach is therefore indirect in that participation enables and constrains learning, but does not determine it. Participation is therefore seen to constitute the conditions for the possibility of learning (Krummheuer, 1992). In contrast, a Vygotskian perspective such as that advanced by van Oers (in press) treats the link between collective processes and individual processes as a direct one: The qualities of students' thinking are *generated by* or *derived from* the organizational features of the social activities in which they participate. This conjectured direct linkage allows sociocultural theorists to be more directive when making instructional recommendations. For example, van Oers (in press) suggests that students should imitate culturally-established mathematical practices when they interact with the teacher or more capable peer. He goes on to argue that help should be gradually withdrawn so that students take over functions they could not initially perform alone, thereby internalizing the cultural activity. This recommendation instantiates Vygotsky's frequently cited general genetic law of cultural development.

> Any higher mental function was external and social before it was internal. It was once a social relationship between two people.... We can formulate the general genetic law of cultural development in the following way. Any function appears twice or on two planes....It appears first between people as an intermental category, and then within the child as an intramental category. (1960, pp. 197-8)

The contrasting emphases of the sociocultural and emergent perspectives are reflected in differing characterizations of the teacher's role. In sociocultural accounts, the teacher is typically portrayed as a representative of society who supports students' discursive reconstruction of culturally-approved meanings (cf. Forman, in press). This view leads to a treatment of negotiation that is partially at odds with emergent accounts of communication. From the emergent perspective, negotiation is a process of mutual *adaptation* that gives rise to shifts and slides of meaning as the teacher and students coordinate their individual activities, in the process of constituting the practices of the classroom community. However, from the sociocultural perspective, negotiation is a process of mutual *appropriation* in which the teacher and students continually co-opt or use each others' contributions (Newman, Griffin, & Cole, 1989). The teacher is therefore typically expected to insert culturally-approved insights that students can co-opt, and to appropriate students' actions into the wider system of mathematical practices that they are to reconstruct. In this account, the teacher negotiates with students in order to mediate between their personal meanings and established cultural meanings. However, in the emergent approach, it is the local classroom community rather than the mathematical practices institutionalized by wider society that are taken as the immediate point of reference. From this point of view, the teacher negotiates with students in order to initiate and guide both students' individual constructions and the evolution of consensual mathematical meanings so that they become increasingly compatible with culturally-approved meanings. In general, whereas sociocultural approaches frame instructional issues in terms of the *transmission* of culture from one generation to the next, the emergent perspective is concerned with the *emergence* of individual and collective meanings in the classroom.

A further contrast between the two perspectives concerns the treatment of semiotic mediation. It is important to clarify that the emergent approach fully accepts Vygotsky's (1987) fundamental insight that semiotic mediation is crucially involved in students' conceptual development. The issue under consideration is that of explaining the nature of this involvement. In line with its central tenets, sociocultural accounts of semiotic mediation give precedence to social and cultural processes over individual psychological processes. For example, in one line of explanation most directly associated with Vygotsky, cultural tools such as conventional mathematical symbols are said to be internalized and to become cultural tools for thinking (Davydov & Radzikhovskii, 1985; Rogoff, 1990). In a second line of explanation associated with Leont'ev (1978), individuals are said to appropriate cultural tools to their own activity. Both formulations distinguish between students' personal meanings and sociohistorically developed cultural meanings inherent in the appropriate use of cultural tools. Further, both approaches contend that students will develop particular culturally-approved meanings as they learn to use language and other semiotic means appropriately (cf. Forman, in press). These approaches therefore characterize symbols as primary vehicles of the enculturation process in that they serve as carriers of meaning from one generation to the next when students use them while engaging in culturally-organized activities (van Oers, in press). It was in this sense that Vygotsky referred to symbols as "objective tools" (Bauersfeld, 1995). The underlying metaphor is again that of transfer or transmission in that learning is characterized as a process in which students inherit the cultural meanings that constitute their intellectual bequest from prior generations.

In the alternative emergent perspective, learning is viewed as a process of both active individual construction and enculturation. Further, processes of signification are considered to be integral to both classroom mathematical practices and the activities of students who participate in them. For example, the mathematical practices established by a classroom community might involve reasoning with physical materials, pictures, diagrams, computer graphics, or notations. An analysis of classroom mathematical practices can in fact delineate emerging chains of signification (Walkerdine, 1988) that constitute what Lemke (in press) calls the semiotic ecology of the classroom (Cobb et al., in press). When attention shifts from the interactionist to the psychological constructivist perspective, the physical materials, symbols, and notations that students use are viewed as constituent parts of their individual activities rather than as external tools (Bateson, 1973; Dewey, 1977; Prawat, in press). As a consequence, the use of particular materials and symbols is considered to profoundly influence both the nature of the mathematical capabilities that students develop and the processes by which they develop them.

We contend that the account of signification offered by the emergent approach is better suited to the purposes of developmental research in that it provides greater precision than sociocultural approaches. For example, a sociocultural analysis of a classroom teaching experiment might account for students' learning in terms of their appropriation or internalization of particular semiotic means. The difficulty from our point of view is that such an analysis does not specify in any detail the evolving social situation of the students' development by analyzing instructional sequences as they are realized in interaction in a particular classroom. In addition, this approach has difficulty in accounting for qualitative differences in individual children's mathematical interpretations except to the extent that they can be tied to the students' participation in different out-of-school communities of practice (Confrey, 1995; Hanks, 1991). In contrast, we illustrated when discussing the interpretive framework that an emergent approach addresses both of these issues. Analyses developed from this perspective therefore have implications for both the revision of instructional sequences and the development of follow-up teaching experiments (Cobb et al., in press).

In this discussion, we have questioned the relatively common view that a sociocultural stance must be adopted if the central role of language and other semiotic means are to be addressed. As an alternative, we have suggested that an emergent approach is appropriate for some purposes in that it admits a psychological constructivist view of learning but sees it as inextricably bound up with processes of signification (cf. Confrey, 1995; Kaput, 1991; Pirie & Kieren, 1994; Sfard, 1991; Thompson, 1992). An emergent analysis might in fact be said to "unpack" appropriation processes posited by sociocultural theorists by specifying how they are realized in interaction by members of specific classroom communities. What, at the global level of the reproduction of culture, is viewed as a process of transmission becomes, at the local level of the classroom community, a process of emergence in which students' constructive activities and the practices in which they participate are considered to be reflexively related.

Thus far, we have focused on situations in which an emergent approach might be particularly appropriate. We turn now to consider the sociocultural perspective and do so by first discussing an analysis reported by Crawford (in press). In proposing to view "conscious behavior as a reflection of the socio-cultural environment in which an individual functions," Crawford makes it clear that she is taking a strong sociocultural perspective. One of her primary interests is to understand situations in which "there are conflicts and inconsistencies between the values and priorities of cultural experience at home and at school." As an illustration, she discusses the conflicts that arise when children growing up in traditional Aboriginal communities in Australia participate in school mathematics activities.

> The resistance of many Aboriginal students to learning mathematics in schools has been interpreted as lack of ability by many educators. In fact, for many Aboriginal people, the value conflicts that arise as a result of the world view that is implicit in the elementary mathematics curriculum are substantial barriers to learning....[For example,] the very high priority given in Western culture to quantity and to quantifiable variables was not supported by everyday activities and modes of categorical thinking in traditional Aboriginal communities. (Crawford, in press)

Crawford goes on to observe that

> Aboriginal communities find the educational practice, used frequently by teachers of mathematics, of asking students questions when the answer is already known to the teacher, extremely puzzling and distasteful.

In addition, there are "substantial differences between Aboriginal and non-Aboriginal categorical thinking even about such perceptually grounded concepts as color." As a consequence, for Aboriginal children, "the primary colors were not immediately evident as a means of classification [of counters and other manipulative materials]."

We find Crawford's analysis compelling and suggest that, for her purposes, it would be counter-productive to "unpack" the process by which Aboriginal children appropriate the values and priorities of their communities. In the analysis, these children are portrayed as "carriers" of the culturally-based understandings of their communities. The vantage point that Crawford seems to adopt is therefore that of an observer located outside the cultural group. From this perspective, thought and activity within a cultural group appear to be relatively homogeneous when compared with differences between groups. This, it will be recalled, was also the perspective that we took when conducting the school-level and societal-level analyses. In the case of the teachers at the two action research sites, for example, we viewed them as representatives of different pedagogical communities whose activity reflected the priorities and values of those communities. Similarly, in the comparison of the arithmetical learning in Taiwan and the U.S., the children, teachers, and parents in the two countries were viewed as "carriers" of distinct systems of cultural beliefs and values. In the course of the analysis, we did in fact point out the qualitative differences in the mathematical activity of children within each of the two national groups (Yang & Cobb, 1995). However, these observations were tangential to the major emphasis of the analysis and merely served to illustrate the possibility of unpacking sociocultural processes, thereby focusing on the constructive activities of individual children.

Crawford subsequently clarifies that situations involving tensions in individuals' needs, expectations, and goals are not limited to conflicts between home and school experience, but also include attempts to reform instruction. In such cases, the tension is between the needs, experiences, and goals of the innovators and the teachers, or between those of the teachers and the students. For example, in the school level analysis, our interactions with the teachers at the inner-city site can be characterized in terms of a conflict between our own and the teachers' culturally-situated beliefs about what it means to be a child in school. Further, our experiences of working with the teachers at both action research sites can be seen to involve a tension between our own and the teachers' views about the general nature of mathematical activity in school. In this regard, Crawford observes that teachers tend to teach in the ways in which they were taught. She accounts for this phenomenon in sociocultural terms by contending that future teachers internalize attitudes and beliefs about how mathematics is learned and about the role of the teacher from their own participation as students in the culturally-organized activities of schooling. In conducting an emergent analysis, we, for our part, would attempt to "unpack" this internalization process. It can be noted, for example, that the beliefs and attitudes to which Crawford refers are the psychological correlates of classroom social and sociomathematical norms. Consequently, from an emergent perspective, future teachers are seen to actively construct the beliefs, suppositions, and assumptions that subsequently find expression in their pedagogical activity when, as students, they participate in the negotiation of classroom social and sociomathematical norms. In this account, a global process of internalization from the sociocultural environment is recast as one of negotiation and individual construction at the classroom level. The issue for us is not which of these two accounts gets things right. Instead, it is to consider the situations in which one type of analysis or the other might be more helpful. In our view, the precision of the emergent account is appropriate for certain purposes. However, in other situations, the globalness of sociocultural accounts has it own advantage. In this respect, the two theoretical perspectives can be seen to complement each other. The sociocultural approach that Crawford illustrates focuses on the social and cultural bases of personal experience whereas analyses developed from the emergent perspective account for the constitution of social and cultural processes by actively cognizing individuals.

Conclusion

We have used Crawford's work as a paradigm case to illustrate the relevance of sociocultural approaches to issues of cultural diversity and of reform at a more global level. It should be clear from the discussion that we consider both sociocultural and emergent perspectives to be viable positions. We would also note that a central notion common both to these two perspectives and to psychological constructivism is that of activity. Differences between the perspectives concern the positioning of the researcher and thus the way in which activity is framed.

In psychological constructivist approaches, the analytical position taken by the researcher is inside an ongoing interaction, and the focus is on the ways in which individual students reorganize their activity. The emergent approach coordinates analyses of this type with those conducted from the interactionist perspective. We have suggested that the analytical position taken in this latter perspective is that of an observer of ongoing interactions located outside the local community but inside the broader cultural community. From this vantage point, individual activity is seen to be situated within the practices of a local community such as that constituted by the teacher and students in the classroom. In contrast, the positioning of the sociocultural theorist is outside the cultural group. From this perspective, individual activity is situated in broad sociocultural practices, and learning is characterized as a process of internalization or appropriation while participating in these practices.

In the course of the discussion, we have clarified that the emergent approach coordinates the psychological constructivist and interactionist perspectives. This led us to suggest that analyses whose primary purpose is psychological should be conducted against the background of an interactionist analysis of the social situation in which the student is acting. The contrasts we drew between the emergent and sociocultural perspectives paid particular attention to the kinds of issues that analyses conducted from each perspective might reasonably address. In addition, we considered how the two perspectives might complement each other. These possibilities are worth pursuing in our view given that the perspectives together span the individual student's activity, the classroom community, and broader communities of practice. The interpretive framework we have outlined represents one attempt to achieve such a coordination.

References

Apple, M. W. (1992). Do the standards go far enough? Power, policy, and practice in mathematics education. *Journal for Research in Mathematics Education, 23*, 412-431.

Balacheff, N. (1990). Future perspectives for research in the psychology of mathematics education. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 135-148). Cambridge, MA: Cambridge University Press.

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*, 373-397.

Banks, J. A. (1995). The historical reconstruction of knowledge about race: Implications for transformative teaching. *Educational Researcher, 24*(2), 15-25.

Barnes, B. (1977). *Interests and the growth of knowledge*. London: Routledge and Kegan Paul.

Bateson, G. (1973). *Steps to an ecology of mind*. London: Paladin.

Bauersfeld, H. (1995). The structuring of the structures: Development and function of mathematizing as a social practice. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 137-159). Hillsdale, NJ: Lawrence Erlbaum Associates.

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H-G. Steiner & A. Vermandel (Eds.), *Foundations and methodology of the discipline of mathematics education* (pp. 174-188). Antwerp, Belgium: Proceedings of the TME Conference.

Bishop, A. (1985). The social construction of meaning—A significant development for mathematics education? *For the Learning of Mathematics, 5*(1), 24-28.

Blumer, H. (1969). *Symbolic interactionism: Perspectives and method.* Englewood Cliffs, NJ: Prentice-Hall.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H-G. Steiner (Ed.), *Theory of mathematics education* (pp. 110-119). Occasional paper 54. Bielefeld, Germany: Institut für Didaktik der Mathematik.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13-20.

Cobb, P. (in press). Continuing the conversation: A response to Smith. *Educational Researcher*.

Cobb, P., & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *Emergence of mathematical meaning: Interaction in classroom cultures* (pp. 1-16). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (in press). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition theory: Social, semiotic, and neurological perspectives.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14*, 83-94.

Cobb, P., Wood, T., & Yackel, E. (1990). Classrooms as learning environments for teachers and researchers. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 125-146). Journal for Research in Mathematics Education Monograph No. 4. Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117-148). New York: Springer-Verlag.

Cole, M. (1990). Cognitive development and formal schooling: The evidence from cross-cultural research. In L. C. Moll (Ed.), *Vygotsky and education: Instructional implications and applications of sociohistorical psychology* (pp. 89-110). New York: Cambridge University Press.

Confrey, J. (1995). How compatible are radical constructivism, sociocultural approaches, and social constructivism? In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 185-225). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cooney, T. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16*, 324-336.

Crawford, K. (in press). Cultural processes and learning: Expectations, actions, and outcomes. In P. Nesher, L. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Davydov, V. V., & Radzikhovskii, L. A. (1985). Vygotsky's theory and the activity-oriented approach in psychology. In J. V. Wertsch (Ed.), *Culture, communication, and cognition: Vygotskian perspectives* (pp. 35-65). New York: Cambridge University Press.

Dewey, J. (1977). In J. A. Boydston (Ed.), *John Dewey, The middle works, 1899-1924* (Vol. 2, pp. 362-363). Carbondale, IL: Southern Illinois University Press.

Doise, W., & Mugny, G. (1984). *The social development of the intellect*. Oxford: Pergamon.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *The handbook of research on teaching* (3rd ed., pp. 119-161). New York: Macmillan.

Feikes, D. (1992). *Beliefs and belief changes of two elementary teachers in a problem-centered second grade mathematics projects*. Unpublished doctoral dissertation, Purdue University, West Lafayette, IN.

Forman, E. (in press). Forms of participation in classroom practice: Implications for learning mathematics. In P. Nesher, L. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Gravemeijer, K. (1994). Educational development and developmental research. *Journal for Research in Mathematics Education, 25*, 443-471.

Hanks, W. F. (1991). Foreword. In J. Lave & E. Wenger, *Situated learning: Legitimate peripheral participation* (pp. 13-26). Cambridge, MA: Cambridge University Press.

Hatano, G. (1993). Time to merge Vygotskian and constructivist conceptions of knowledge acquisition. In E. A. Forman, N. Minick, & C. A. Stone (Eds.), *Contexts for learning: Sociocultural dynamics in children's development* (pp. 153-166). New York: Oxford University Press.

Kamii, C. (1985). *Young children reinvent arithmetic: Implications of Piaget's theory*. Columbia, NY: Teacher College Press.

Kaput, J. J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Constructivism in mathematics education* (pp. 53–74). Dordrecht, The Netherlands: Kluwer.

Krummheuer, G. (1992). *Lernen mit "Format": Elemente einer interaktionistischen Lerntheorie*. Weinheim, Germany: Deutscher Studien Verlag.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*, 29–63.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life.* Cambridge, MA: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* Cambridge, MA: Cambridge University Press.

Lemke, J. L. (in press). Cognition, context, and learning: A social semiotic perspective. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition theory: Social, semiotic, and neurological perspectives.* Hillsdale, NJ: Erlbaum.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality.* Englewood Cliffs, NJ: Prentice-Hall.

McClain, K. (1995, April). *An Analysis of the Teacher's Proactive Role in Initiating and Guiding the Development of Productive Mathematical Discourse.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.

Mishler, E. G. (1986). *Research interviewing: Context and narrative.* Cambridge, MA: Harvard University Press.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school.* Cambridge, MA: Cambridge University Press.

Nicholls, J. G., & Hazzard, S. P. (1993). *Education as adventure: Lessons from the second grade.* New York: Teachers College Press.

Perret-Clermont, A. N. (1980). *Social interaction and cognitive development in children.* New York: Academic Press.

Piaget, J. (1973). *To understand is to invent.* New York: Grossman.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics, 26,* 61-86.

Prawat, R. S. (in press). Misreading Dewey: Reform, projects, and the language game. *Educational Researcher.*

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context.* Oxford: Oxford University Press.

Rogoff, B. (1994). Developing understanding of the idea of communities of learners. *Mind, Culture, and Activity, 1,* 209-229.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-216). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schön, D. (1983). *The reflective practitioner.* New York: Basic Books.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1–36.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 114-145.

Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*, 472-494.

Smith, E. (in press). Where is the mind? "Knowing" and "knowledge" in Cobb's constructivist and sociocultural perspectives. *Educational Researcher.*

Steffe, L. P. (1983). The teaching experiment methodology in a constructivist research program. In M. Zweng, T. Green, J. Kilpatrick, H. Pollak, & M. Suydam (Eds.), *Proceedings of the Fourth International Congress on Mathematical Education* (pp. 469-471). Boston: Birkhauser.

Steffe, L. P. (1995). Alternative epistemologies: An educator's perspective. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 489-523). Hillsdale, NJ: Lawrence Erlbaum Associates.

Stevenson, H. W., & Lee, S. Y. (1990). (Eds.), Contexts of achievement: A study of American, Chinese, and Japanese children. *Monographs of the Society for Research in Child Development, 55*, 181.

Stigler, J., Fernandez, C., & Yoshida, M. (1992, August). *Children's thinking during mathematics instruction in Japanese and American elementary school classrooms.* Paper presented at the International Congress on Mathematical Education, Québec, Canada.

Streefland, L. (1991). *Fractions in realistic mathematics education. A paradigm of developmental research.* Dordrecht, The Netherlands: Kluwer.

Thompson, P. W. (1992). Notations, principles, and constraints: Contributions to the effective use of concrete manipulatives in elementary mathematics. *Journal for Research in Mathematics Education, 23*, 123–147.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction—The Wiskobas Project.* Dordrecht, The Netherlands: Reidel.

van Oers, B. (in press). Learning mathematics as meaningful activity. In P. Nesher, L. Steffe, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies in Mathematics, 26*, 273-298.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *Emergence of mathematical meaning: Interaction in classroom cultures* (pp. 163-201). Hillsdale, NJ: Lawrence Erlbaum Associates.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17–40). New York: Norton.

von Glasersfeld, E. (1992). Constructivism reconstructed: A reply to Suchting. *Science and Education, 1*, 379–384.

Vygotsky, L. S. (1960). *Razvitie vysshikh psikhicheskikh funktsii [The development of the higher mental functions].* Moscow: Akad. Ped. Nauk. RSFSR.

Vygotsky, L. S. (1987). Thinking and speech. In R. W. Rieber & A. S. Carton (Eds.), *The collected works of Vygotsky, L. S. (Vol. 1). Problems of general psychology.* New York: Plenum.

Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality.* London: Routledge.

Whitson, J. A. (in press). Cognition as a semiotic process: Grounding, mediation, and critical reflective transcendence. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition theory: Social, semiotic, and neurological perspectives.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Yackel, E., & Cobb, P. (1993, April). *Classroom microcultures reform in mathematics education.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA.

Yackel, E., & Cobb, P. (in press). Sociomath norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education.*

Yang, M. T-L., & Cobb, P. (1995). A cross-cultural investigation into the development of place value concepts in Taiwan and the United States. *Educational Studies in Mathematics, 28,* 1-33.

FAIRNESS IN DEALING: DIVERSITY, PSYCHOLOGY, AND MATHEMATICS EDUCATION

Suzanne K. Damarin, The Ohio State University

> But constructivism as a pedagogical orientation has to be embedded in an ethical or political framework.
>
> *Nel Noddings, 1993, p. 159*

This paper lies at the interpretive intersection of several lines of research, some of them quite familiar to mathematics educators, and some of them probably less so, or at least familiar only to some. Among the familiar discourses are cognitive constructivism, social construction, situated learning, and the psychological study of differences between groups defined by gender, race, ethnicity, and other variables used to categorize persons. The less familiar literatures and discourses include cultural studies, feminist research and theorizing, and postmodern social sciences, among others. Within the space inscribed by these theories, discourses, and research traditions and findings, I excavate issues surrounding the basic questions of what knowledge and approaches can be applied in order to increase the "fairness in dealing" with all students in and through mathematics curriculum, instruction, assessment, and related activities. And, what are the implications for research in mathematics education, particularly research which invokes and pursues knowledge categorized as psychological?

I choose to use the term "fairness in dealing" specifically to displace the more traditional ideas of equity. Educational equity (if it is achieved) connotes various forms of measured equality in the selection, preparation, treatment, achievement, and/or career tracks of groups of students categorized by sex, race, ethnicity, or class (Fennema, 1990). The equity concept is limited in several regards. First, it is measure dependent, and different measures often yield different assessments of the extent to which equity is achieved; thus, while the quantitative nature of equity reports lend them the aura of scientific truth, the construct validity is questionable. Secondly, equity and its measures cannot take into account phenomena such as the accumulating evidence that even many girls and women who have achieved excellence in mathematics often feel that some unfairness was involved. Third, measured equity is a post hoc concept, measured after the fact of preparation, achievement, et cetera, and offers mathematics educators no guidance as to how to work towards its achievement. A review of the literature reveals a fourth concern: "equity" and "excellence" are often regarded as pitted against each other by educational policy makers and philosophers, with the implication that one must be sacrificed for the other. As others have observed, if equity and excellence were consistent in practice, we would see at least a few examples of "excellence" in urban schools. The idea that in a democratic society excellence *must entail* equity is lost in the operational uses of the terms "equity" and "excellence".

Periodically throughout American educational history, arguments for equity in mathematics and/or science education have been confounded with arguments concerning the need of capitalists to increase labor pools (see Cohen, 1982; Damarin, 1993a); thus, the idea of equity in children's access to education is confused with the idea of giving employers access to trained workers. While there are clear relationships between education and employment, this conceptual confusion deflects focus from the current educational needs of learners to future needs of employers. Current reform efforts in education and schooling in general, and in mathematics education as guided by the *Standards* (NCTM, 1989) in particular, focus on the construction of knowledge by the individual student. But, the measures used in gathering data for equity reports usually lead in directions opposite to the "authentic assessment" of the *Standards*. Finally, (and perhaps consequent upon the issues outlined above) educational activities directed toward (and by) the current equity construct have not led to the kinds or magnitude of change intended with the inception of systematic equity work in the 1970s. In a multicultural democratic society such as ours, the goal of universal education requires a rethinking and recommitment to the education, in particular the mathematical education, of all students. Toward this end, and in recognition of the power of language to inhibit or promote change, consider the idea of "fairness in dealing."

Fairness in Dealing

The term "fairness in dealing" is one of the definitions for equity supplied by *Webster's Ninth Collegiate Dictionary*. Although "dealing" and "fairness" are difficult words to define operationally, "dealing" conveys ideas of continuity in action, reciprocity (dealing cannot be accomplished by a single actor), and negotiation to resolution; "fairness" entails openness, honesty, full disclosure, and often the setting aside of knowledge or information that might bias one. When Myra and David Sadker (Sadker and Sadker, 1994) titled their recent book *Failing at Fairness: How Our Schools Cheat Girls*, they used "fairness" in this sense. Theirs is not an equity report *per se*, but a summing up of more than two decades of observational research focused on the schooling of girls; the book documents "a curriculum of sexist school lessons becoming secret mind games played against female children, our daughters, tomorrow's women" (p.1). Since 1973 when Myra Sadker published her first book, *Sexism in School and Society*, (Frazier and Sadker, 1973) and launched the contemporary study of gender in education, a growing cohort of educational and psychological researchers has conducted a great deal of research on gender in general and on mathematics and gender, in particular (see Fennema, 1993; Fennema and Hart, 1994). Despite the accumulated findings of the latter research, and despite earnest efforts by many mathematics teachers, curriculum designers, policy makers, and teacher educators to articulate research findings into classrooms, the recent findings of the Sadkers, the AAUW study of schooling and girls (Wellesley Center, 1992; also, Orenstein, 1994), and other comprehensive studies indicate that little has changed for girls or women in mathematics classes. But, the absence of some changes can motivate others, and in the area of gender and mathematics there is apparent today a worldwide movement (Kaiser and Rogers, 1995) to change the conceptual bases and paradigms of research on gender and mathematics, incorporating feminist philosophy and theories, feminist studies in psychology, and ideas of fairness. This paper reflects, and perhaps contributes to, that change.

Considering Race and Ethnicity I

Although my assignment in this paper is to address diversity in general, the primary discussions and arguments are based in sex/gender/feminism for several reasons. First, much of my own work and, therefore, my greatest knowledge and my habitual focus are on gender. Secondly, the body of research and theory directed specifically toward mathematics and gender is larger, and perhaps more varied in its theoretical bases, than work on mathematics and race[1] (or mathematics and class[2]). Third, I argue below that the sex/gender/ feminisms based discussions presented here have clear analogues in the area of mathematics and race, as well as mathematics and ethnicity.

Before making that argument, it is important to note that white feminists have been rightly criticized for ignoring and/or denying racial differences between and among women in much of their work I do not deny the validity of this claim, nor its importance; I regret any and all participation on my part in this silencing. Certainly sex/gender and race operate both differently and interactively in the larger society. Just as research and writing on women have often focused on white women, research and writings on race and ethnicity have often ignored the multiplicity of races and ethnicities. In this paper, my race-based examples focus on African Americans. This choice reflects my desire to display the depth and richness of findings concerning a particular marginalized group. I have no doubt that a comparable set of examples and arguments could be given with respect to a different group, nor that in its details the discussion would vary with culture.

For both race and sex/gender, however, the domains of "mathematical ability" and "mathematics performance" have functioned as areas in which "demonstration of difference" has been used both as a rationalization for, and a tool in, the continuing suppression/ oppression of individuals based solely upon their race and/or sex. This structural and operational sameness is at the crux of the analogical moves in this paper. Moreover, the "red thread" that runs through the feminist analysis discussed below is the importance of recognizing and valuing lived experiences and epistemological standpoints in the psychological and educational study of cognition and in the teaching of mathematics. That these experiences and standpoints vary with both sex and race in relevant ways is a major point, and the basis of the analogies and comparisons between gender and race.

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[1] In a recent paper, Ladson-Billings and Tate (in press) argue that, in contrast with gender and class for which there is extensive theoretical work, race has not been adequately theorized. These authors propose and argue for a critical race theory grounded in the ownership of property.

[2] This paper will not address issues of mathematics and class in any depth. The interested reader is referred to Mellin-Olsen (1987), and Frankenstein (1987, 1995) for insightful discussions. Elsewhere, (Damarin 1993b, 1994a), I discuss some issues of class in relation to situated cognition; the arguments there are related to those of this paper.

Feminisms, Psychology, Gender, and Mathematics: A (very) Brief History

Although all feminist research and theorizing begins with the goal of improving the lot of women in the world, beyond this common aim feminism is not singular in its underlying assumptions, beliefs, methods, and goals. Instead, diverse feminists work within a range of perspectives and frameworks — liberal feminism, socialist feminisms of several sorts, radical feminisms, black womanist theories, and postmodern feminism among them.[1] Until recently, most research in gender and mathematics was carried out under the assumptions and using the methods associated with liberal feminism which assumes (basically) that the larger structures (e.g., capitalism, the scientific establishment, educational systems) and concepts (e.g., mathematics, science, research, evidence) of current society are stable, essential, and appropriate. Liberal feminists "work within the system", attempting to improve the lot of women within conceptual, experiential, and political systems which are otherwise left unchanged.

For research on gender and mathematics, liberal feminist researchers using current concepts and methods of psychology and education have conducted experimental studies, factorial studies, and the building of models in efforts to understand observed differences in the mathematics performance of females and males, to identify psychological variables which moderate effects, suppressing or multiplying the effects of gender, and to prescribe, both within and outside schools, changes in the treatment of girls which might increase mathematics performance. Beginning with the Fennema-Sherman studies of the early 1970s (Fennema and Sherman, 1977), and continuing into the present (e.g., Friedman, 1995) these studies have accumulated into a substantial comprehensive literature (see Fennema and Hart, 1994). Psychological constructs such as state and trait anxiety, internality/ externality, field dependence/ independence, aggression, fear of success, and achievement motivation, among others, contribute to the understanding of relations among the variables studied. At the same time, new constructs such as "math as a male domain" (Fennema and Sherman, 1977) and "autonomous learning behaviors" (Fennema and Peterson, 1985) were identified by these researchers and studied to clarify anomalous findings.

Despite the increasing refinement of studies and findings, however, dissatisfaction if not disillusionment with this line of research has grown among those concerned with gender and math for several reasons. After an initial flurry of concrete findings which suggested concrete actions, the results of this research seem to many to have neither explanatory power comparable to the perceived magnitude of the problem nor prescriptive power sufficient to create fundamental change. Feelings that this line of research might have passed its usefulness were exacerbated by announcements of the welcome findings that the sexes "no longer differ" in mathematical ability or aptitude (Linn and Hyde, 1989) and by accumulating evidence (Linn and Hyde, 1989; Tartre, 1990) that spatial abilities are *not* related to sex differences in mathematical performance. The emergence of these findings both validated the "gut level" beliefs of many gender researchers and served to catalyze a change in direction. In the hotel lobbies at AERA meetings, at WME meetings, and in other places where gender and mathematics researchers come together, a tentative, forbidden thought began to take on the dimensions of a rallying cry: "There's nothing wrong with the women; let's stop trying to fix the women and start to work on fixing the mathematics." One way of interpreting the current agenda for research in mathematics and gender is that the problem under study is to determine exactly what those three words, "fix the mathematics" could possibly mean. Meanwhile, in many other areas of study, including psychology, feminist researchers using different approaches were uncovering interesting findings and propounding interesting theories.

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[1] For a discussion of various strands of feminism, see Jaggar (1983), Donovan (1986), or any introductory text on feminist theory. Black feminist (or womanist) theory has often been ignored in these texts, especially the earlier ones; for a discussion of these theories see Collins (1990) on black feminist theory and Walker (1983) on black womanism. The interested reader may also wish to consult Kramarae and Spender's (1992) anthology of writings on feminist theory and practice in fields ranging from Architecture to Zoology, or Stone's (1994) anthology on feminism and education.

Feminist Psychology

From its earliest development, psychology has been criticized by women within the field (e.g., Woolley, 1903, 1910) who found both the psychological conceptualization and empirical investigation of the "feminine" and its correlates to be without validity in that they were dissonant with the realities of women's lived experience. With the rise of the current wave of feminism, these criticisms were revived and expanded (Weisstein, 1971; Sherif, 1979). Researchers in psychology and education (e.g., Eichler, 1987; Squire, 1989) examined the conduct of experimental research, uncovering evidence of prevalent biases at the levels of problem statement, sampling, instrumentation, treatment, data analysis, and interpretation and reporting of results. A cataloging of these findings is well beyond the scope of this paper, but a few examples are instructive. In a sampling of studies of interactions between parents and their young children, most were conducted using observations of mothers; findings of good interactions were typically reported in the gender-neutral language of parenting while findings of deficient behaviors were uniformly discussed in the female-specific language of mothering and maternal activity. While this failing might be corrected by re-analysis and interpretation of existing data or by new experimentation, there are more fundamental gender-based critiques. Psychological constructs which have prior associations with the masculine (e.g., aggression) tend to be studied using high-status experimental techniques, while those associated with the feminine (e.g., anxiety) are studied using low status Likert-type instruments. As a result, knowledge of "masculine" traits is reported with the certainty of cause and effect, while "feminine" traits emerge as correlational, marginally significant predictors lacking in strength. Thus, by their design, the research tools participate in the very phenomenon and problem that gender researchers have sought (and are seeking) to address and redress. In this context, the wisdom of black feminist Audre Lorde (1984, p. 110) is evident: "the master's tools will never dismantle the master's house."

From the perspective of women, another major failing in psychological research was (and is) the development, refinement and application of comprehensive theories of "human" development based entirely on male data. Kohlberg's theory of moral development has been particularly troublesome because when this four stage theory is applied to both sexes women are most frequently found to be in a state of arrested development (stage 3) while men proceed to the higher (fourth) stage. In a series of studies of women's moral development, Carol Gilligan (1979, 1982) developed an alternate theory around ideas of care and responsibility. Briefly, in her research and theories, a conception of the self as connected and in relation to others, together with a theory of knowledge as connected,[4] supports an ethic based on responsibility and care, while a view of the self as autonomous and in separation from others leads to an ethic of rights and justice. While women's beliefs and actions were in the spirit of the former, men believed and acted in relation to the latter. Rather than a stage theory, Gilligan's is a theory of socialized differentiation highly related to gender socialization.[5, 6]

Gilligan's work has had profound influence on feminist research and theory, and on applications of feminist theory to the (female dominated) "helping professions". Of particular interest here, this work influenced feminist psychological study in relation to another developmental stage theory which was based entirely on data gathered from males:[7] William Perry's (1970) *Forms of Intellectual and Ethical Development*. Basing their queries on Perry's stages, Mary Belenkey, Blythe Clinchy, Nancy Goldberger and Jill Tarule studied 135 women in various sites of post-secondary education. The resulting book, *Women's Ways of Knowing*, (Belenkey, et al., 1986, henceforth referred to as WWK) outlines six phases (not stages in the usual sense) in women's acquisition and organization of knowledge, and examines implications for the transformation of teaching. Since it's publication, *Women's Ways of Knowing* has been very influential on the study of gender and mathematics, as will be discussed below.

Before turning to that discussion, the recent work of two other psychologists merits attention; Sandra Bem (1993) and Meredith Kimball (in press) have both made extensive study of the massive volume of scholarly literature produced by feminist scholars and theorists (in philosophy of science, epistemology, sociology, history, cultural studies, literary criticism, media criticism, law, and other areas) since the early 1980s, and have incorporated major ideas from these literatures into their work. Sandra Bem has been engaged in gender research since the 1960's, and is perhaps best known as co-developer (with Daryl Bem, her husband) of the Bem Sex Role Inventory. Her current work might be called a kind of (qualitative) meta-analysis of earlier work by herself and others in her generation of psychological researchers. Here, she argues that throughout this research sex/gender has functioned as a series of transparent, but distorting, lenses through which science looks at women and men, notably lenses of androcentrism, gender polarization, and biological essentialism. Examining the lenses in detail, she uses feminist epistemology and related literatures to expose the distortion, and argues that, especially if gender is to be depolarized, a revolution in psychology is needed.[8] Meredith Kimball may be a harbinger of that revolution; the importance of her work lies in her deconstruction of binary pairs which are basic to current psychological study: male/female, ethic of care/ethic of justice, connected/separate, gender similarity/gender difference, among others.[9]

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[4] Connected knowing is described briefly below; a more extended but still brief discussion can be found in Becker (1995).

[5] The work of Nel Noddings (1984, 1992) on a caring ethic and care in schools is rooted in philosophy, not psychology, but is important to any discussion of care in schooling. The ethic of care has been adopted or adapted by many feminist ethicists, and seriously criticized by others; a brief synopsis of the critiques can be found in Damarin (1994c).

[6] The distinction between ethics of care and ethics of justice is related to the distinctions between an approach to education of diverse students based on fairness in dealing and an approach based on equity. Thus, this paper is within the tradition of educational thought inspired by Gilligan's work.

[7] Perry's subjects were Harvard students of the 1960s, that is, young (17-22), white, upper middle class and upper class males.

Considering Race and Ethnicity II

Because contemporary science in general, and psychology in particular, have developed in a Euro-American tradition, and because feminist critiques are based in the effects of absences and biases at every level, these critiques invite analogues with respect to persons of non-white races and non-European cultures. Moreover, the constructions, within the dominant Euro-American discourses, of blacks (and all people of color) throughout the history of the human sciences since the time of Darwin is a history of the "mis-measure of man" (Gould, 1981) in the service of social agendas of white progress and supremacy (see Gould, 1981; Harding, 1991; Lewontin, 1992; and many studies cited therein). The contributions of Africans and African-Americans in particular, and of all people of color, to the development of science and technology have been denied, ignored and erased from the public record. The fields of Black Studies, Native American Studies, Latino Studies, and other fields of cultural study, including ethnomathematics, have emerged in recent decades in an effort by scholars from those cultures (primarily) to reclaim and correct some of this history and to reclaim for people of color not only the traditions of scholarship and science consistent with their life experiences but also recognition of their accomplishments throughout history.

The importance of recognizing and meeting through instruction the culturally specific ways in which students understand the world and their relation to it is central to much discussion of multicultural education (Sleeter and Grant, 1991; Banks, 1993; Secada, 1990; Delpit, 1988, and numerous others). Like other lenses through which white educators peer, psychological theories reflect the world views and epistemologies of their developers. A part of the substantial literature of black psychology, Afrocentric psychology is one alternative to Eurocentric theories.

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[8] Bem's work provides for psychology findings comparable to those of work in biology (e.g., Fausto-Sterling, 1985) and primatology (Haraway, 1991).

[9] Within her work, Kimball pays particular attention to mathematics. Also, see Kimball (1989).

Afrocentric psychology, like other articulations of Afrocentric theory, is based in the study of the lives and history of Africans and African Americans. The centrality of faith, belief, and ethics to Afrocentric "Optimal Psychology" reflects the importance of these to the African experience and describes conceptual systems grounded in the spiritual as opposed to the material grounding of Euro-American psychological traditions. Importantly to mathematics education, this leads to an epistemology based in self-knowledge through symbolic imagery and rhythm in contrast to knowledge of the external world gained through scientific observation, measurement, and counting. In this approach, all things are seen as interrelated and knowledge is connected, not compartmentalized (Asante, 1987; Myers, 1988).

Study of optimal psychology can help white mathematics educators become open to the construction of new ideas about how black (and perhaps other students) organize and use knowledge. Building on the relations between optimal and Euro-American psychological theories (discussed in Myers, 1988), we can all gain different and fuller understandings of findings such as those of Stiff and Harvey (1988) that black students benefit from mathematics instruction based on field dependence. Optimal psychology invites us to view the "field" in more complex ways and to reach a different understanding of figure/ground interrelations and field dependence. With this new understanding, any biases which portray field dependent thinkers as mathematically dull should disappear, and we should be able to design more interesting and effective learning activities for field dependent thinkers.

Gender and Math Informed by Feminist Psychology

The epistemological model explicated in *Women's Ways of Knowing* describes six ways of knowing exhibited by the women studied; although they are listed and analyzed in WWK in an order that reflects growth from total reliance on others to self-reliance and autonomous knowing, the authors emphasize that these are not stages in the usual sense. Women may know differently dependent upon the knowledge domain, for example, and some women may be "boundary riders", mixing elements from two phases for long periods of time. Overall, women grow intellectually from one way of knowing to the next; the authors do not address knowing prior to adulthood, and therefore questions such as whether all female knowing begins with silence are not addressed. Briefly described, the ways of knowing are:

(1) *Silence*, characterized by belief that authorities are all-powerful, inability to form mental representations, absence of expectations of understanding

(2) *Received Knowing*, learning by listening, accepts authority in a rote manner ("whatever you say, doc")

(3) *Subjective Knowing*, knowing "in my gut"; "it's only my opinion, but my gut tells me ..."; assumes there are right answers

Procedural Knowing in two forms

(4) *Separate Knowing*, impersonal, propositional reasoning

(5) *Connected Knowing*, seeking explanations for perceptions, interested in the thoughts of others

(6) *Constructed Knowing*, effort to integrate knowledge, appreciates complexity

This model describes women's lives, in particular and as gleaned from the WWK data, in several ways. Silence is seen as an effect of generations of women's socialization to acquiesce to male authority. Subjective knowing identifies and valorizes what has traditionally been denigrated as "women's intuition." And, connected knowing identifies a kind of procedural knowing which is different from "masculine rationality," but qualifies as reasoning. Connected knowing is described (Gilligan 1982) as involving intuition, creativity, hypothesizing, relativism, induction, incompleteness; based on experience, it is contextual.

Researchers on gender and mathematics have used WWK in several ways. In the most direct applications of the six ways of knowing (and the transitions between them) to mathematics classrooms, researchers and teachers interpret them in relation to selection and/or design of representations of mathematical concepts and in relation to planning events of instruction. Joanne Rossi Becker and Judith Jacobs have focused on the representational problem (Becker and Jacobs, 1989; Jacobs, 1994). Discussing the theorem "The sum of any two odd numbers is even", Jacobs offers a representation of whole numbers by arrangements of squares in two horizontal (contiguous) rows and compares this representation with other which are common (e.g., 2n, 2n+1). The Jacobs representation is perceptual and generalizable by connected knowers (in theory, at least), allowing students to accumulate instances and develop "gut level" subjective knowledge of the odds and evens, and later the theorem itself. At issue in distinguishing this representation from others are the accessibility of the concept to subjective knowers and the (perceptual) attributes which invite reasoning (connected knowing). The role of visual perception in models such as this is especially interesting (and research worthy) because only a few years ago it was thought that women were demonstrably inferior with respect to visuo-spatial skills. Other direct approaches to the articulation of WWK into the classroom involve the development of pedagogies for connected learning (Becker, 1995); many of the studies sited below were conducted in classrooms which use such an approach, as does the SummerMath program for high school girls (Morrow and Morrow, 1995).

Feminist Pedagogy is an approach to teaching developed first in Women's Studies which decenters the authority of the teacher and conscientiously seeks to bring previously marginalized students into the mainstream of classroom activity and discussion (Culley and Portuges, 1985; Disch and Thompson, 1990) With their emphases on voice, Gilligan's work and WWK provide both rationale and direction for this style of teaching. Several mathematics educators have experimented with feminist pedagogy, documenting the classroom events and using the ideas of voice, care, connectedness, and others in their analyses (Buerk, 1985, 1995, and others).

The importance of voice in all of this research is extended by some researchers to include writing, and the mathematics autobiography (usually written, sometimes oral) has gained an important place as a pedagogical and research tool. Dorothy Buerk (Buerk and Szablewski, 1993; Kalinowski and Buerk, in press) has extended the autobiography to include journals in which math students regularly write about their reactions, attitudes, and feelings in relation to mathematics and reflect upon themselves as knowers of mathematics. For researchers such as Buerk, these writings have become both research and pedagogical tools because of their demonstrated usefulness to the student. Extending the idea of bringing the margins into the mainstream, some researchers (e.g., Erchick, in press) are including in their conceptual frames (and/or mathematics classes) published writings about mathematics (and about women as knowers) authored by women who are not, by any of the usual definitions, mathematicians.

Reading across several of these studies, one is struck by the regularity with which women (including young teenagers) reveal themselves as currently or recently silent knowers with respect to mathematics, and, as importantly, that these women almost invariably report a salient critical event in which a statement or action by a teacher (or less frequently a family member) led to a resolve to be silent in the face of mathematics. Some of the events reported would make all of us cringe, but others are "standard fare" in the mathematics classroom.

> At about 8 or 9 I had a totally intimidating teacher (the headmaster) for maths, for one term. He taught us times-tables in a militaristic type of way; chanting out a times table, pointing at you and expecting you to fire back an answer within a second. If unable to answer some fate worse than death would be waiting. That is how it seemed when I was a completely powerless, timid 8-year-old. From then on started a slippery slope downhill. Although I had some good and encouraging teachers along the way, I had come to associate maths with fear and panic.
>
> (Isaacson, 1990, p. 23)

The writing and interviews of many women in these studies reflect their past, and often current, beliefs that math is an area in which one must learn from authorities. Frequently this belief remains a reified "fact" (gut level knowledge?) within a larger, more sophisticated way of knowing. In the following example, women in a group interview reveal how they moved beyond received knowing, rejecting the authority of the math teacher concerning their future mathematical needs while still holding the mathematics, itself, to be knowledge which is gained from authority.

> Well, they just kind of came along and gave you sets of rules, didn't they? That's how I was taught, anyway.
>
> And little books to look them up in.
>
> Yeah, there's your rules. Off you go and use them ...
>
> If you've got something, and you don't know what you're ever going to use it for, you don't bother learning it.
>
> That's right.
>
> (Isaacson, 1990, p.25)

Across these studies, many women reflect on how the opportunity to talk or write about themselves as learners and doers of mathematics has helped them to establish a new relation with and understanding of the subject matter. In relation to this finding, some researchers (Fullerton, 1995) have examined mathematics register (see Pimm, 1987), finding that many women have no words with which to talk about mathematics. Through writing, interviews, and group work, they "came to voice" in mathematics.[10]

Examining these studies, it is interesting to see emerge in the data repeated mention by the participants of the very ideas that were captured in the original Fennema-Sherman scales and studies (Fennema and Sherman, 1977). These women often perceive math as a male domain, taught by "father-figures" (Rogers, 1990) and learned best by boys (Isaacson, 1990, among others). Reports on discouragement from the study of mathematics by mothers, fathers, teachers, and/or peers are a consistent presence across nearly all the studies (though not all women interviewed). Fear and panic (Isaacson, 1990) and other expressions of anxiety in the face of mathematics are reported frequently, as are lack of confidence and expectations of incompetence (Fullerton, 1995).

Thus, these studies provide a kind of retrospective construct validity for the Fennema-Sherman scales which *do* (still) capture salient aspects of women's expressions of their experiences, attitudes and feelings with regard to mathematics; these have not changed.[11] What the current research does, however, is provide a way of seeing the constructs measured as *effects, not causes.* Early (and some current) research was designed on the assumption that negative experiences would cause students to do poorly in and/or leave mathematics. The studies discussed here reveal that, for many women, succeeding and staying in mathematics has the effect of creating increased opportunity to experience the negative phenomena captured in the scales. Many of the women in the current studies are high school and college mathematics students; some are school teachers (Erchick, 1995) and some are college professors (Taylor, 1990). They have endured, and sometimes learned, a lot of mathematics.

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[10] The reader schooled and practiced in quantitative research methodologies may be thinking that none of this has the feel of "good science" and "hard data", so a few comments are in order. First, the research itself reflects a search for connected and constructed knowing. Where field data are gathered, the studies generally meet rigorous standards for qualitative research. Further, working from a base in feminist theory, the researchers are within a tradition that includes a serious critique of mainstrean science and which asks whether there can be a feminist science (see Damarin, 1994b, 1995a). While there is no final agreement on this question, certain characteristics emerge as essential for any candidate for feminist science: (1) the theorizing of gender as a variable of consequence, (2) the valuing of women;s experience as a scientific resource, and (3) the positioning of the researcher in the same critical plane as the researched. (Harding, 1987). In conducting their studies, the researchers clearly meet these criteria.

[11] The Fennema- Sherman scales were designed in an effort to predict success vs. failure and continuing vs. dropping enrollment by girls and young women in mathematics. These scales initiated research directed toward building explanatory models. Although the scales had less predictive power than hoped, they precipitated much research; a few of these scales (Math as a Male Domain, Anxiety, and perhaps others) are still used in model building studies.

A note toward the future. The work of psychologist Valerie Walkerdine (1987, 1989, 1990) provides an important parallel to the studies influenced by *Women's Ways of Knowing*. Perhaps because she analyzes her data using the constructs and language of Marxist feminist theory, and more recently postmodern Foucauldian theory, her work is not very well-known to U.S. mathematics educators and gender researchers. In her work she addresses both classrooms in general (often at the elementary level) and mathematics classrooms, in particular. Borrowing analytic tools from Michel Foucault (1977), her most recent book analyzes the education of girls as the creation of "docile bodies", a term used by Foucault to examine the ways that persons become (are made) controlled, self-regulating, obedient subjects. Arguably, docile bodies are received, if not silent, knowers. A full mapping of the relations between WWK and Walkerdine's work is beyond the scope of this paper (and surely such a mapping would fail to be an isomorphism), but there is a commonality in methods, content of data, and some interpretations. Because a substantial amount of U.S., European, and Australian feminist theorizing and sociological study has "taken the postmodern turn," Walkerdine's work and similar efforts are likely to become more important to the study of gender and mathematics in the future.

Concerning Race and Ethnicity III

Belenky, Clinchy, Goldberger and Tarule were careful to include diverse women in their study: black, white, and Latina women ranging widely in age, and involved in educational settings ranging from a parenting skills workshop for welfare mothers to an elite women's college. This diversity notwithstanding, the conceptual roots of their work are clearly Euro-American, Gilligan's theoretical frame is based in psychoanalytic object relations theory, which assumes an autonomous self as central. This theory is appropriate to understanding knowledge building within a culture of individualism, but not (necessarily) within a culture which holds community as central, and/or values community over individual. Therefore, direct transfer of the theory from women to other mathematically marginal groups would violate "fairness in dealing."

Nonetheless, elements of this work would seem to have some relevance to race and ethnicity. First, the finding of the totalizing effects of silencing on students is not new to the literature, but replicates findings in relation to blacks, latinas, Native Americans and students for whom English is a second language. Secondly, the methods of these studies probably are transportable to work with other mathematically marginal populations. Feminist pedagogy has roots in the work of Paolo Freire (1970); and, black feminist bell hooks' (1994) book *Teaching to Transgress* is, in part, an explication and expansion of this method. Mathematical autobiographies of African American, Latino/Latina and other students would undoubtedly provide enlightening information. Moreover, given the importance attached to self-knowledge in optimal psychology, assigning mathematical autobiographies to black students might be both useful to these students and a start in mathematics educators' learning about and coming to appreciate their knowledge systems and values.

Third, half of all students of color are female, and are effected by the social constructions of women both within their ethnic cultures and in the dominant society. Although many black women state that race is primary to gender in defining their life experiences, African American historian of science Evelynn Hammonds (Sands, 1991) details how in collegiate and graduate work in physics, her sex was the major source of her oppression. Mathematics and mathematics classes may operate in a way similar to physics.

Finally, and perhaps most importantly, connected knowing is a central aspect of both Afrocentric epistemology and contemporary discussions in the literature on the education of black children. Connected Knowing, as described in WWK and elaborated in the studies is resonant with these discussions and suggests another area in which common approaches to educational change might be sought by women and blacks. Indeed, a careful look at the literatures of multicultural education and of ethnomathematics would surely reveal important insights which could be transported to research on the education of women (reversing the direction of the analysis and flow of inference in this paper).

Fixing the Mathematics

In the studies discussed above, the areas which emerge as in need of fixing include some in the category of teaching techniques, with some advice (but more questions) on how to select representations and organize instruction to teach toward subjective and connected ways of knowing. Some of this advice (not reiterated in this paper, but available in the studies referred to) has a familiar ring: teachers/readers are advised to use cooperative learning groups, teach to the individual's way of learning, adopt apprenticeship models and other aspects of situated learning theory, teach for cognitive construction of knowledge using constructivist methods, and so on.

The reader might ask, "isn't this just good teaching?" But, the question misses the major point of the authors which is the necessity to engage in good teaching *with specific attention to girls and women*.[12] These studies provide evidence that because the larger society (including many of their teachers, parents, and peers) does not construct women and girls as competent in mathematics, young women must (re)construct themselves as (other than silent) knowers of mathematics. There is also evidence in these studies that writing and speaking about their experiences, attitudes, and feelings can contribute importantly to that self-construction.

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[12] Specific attention to girls does not mean that boys are to be ignored. For years, the literature on gender and education has shown that girls get considerably less attention in classrooms (Maccoby and Jacklin, 1974, followed by numerous others). Unless teachers make it a specific effort to attend to girls, this will continue.

WWK, with it's discussions of silence and received knowing, demonstrates that persons in these conditions of knowing cannot construct knowledge because they are rule-bound creatures who believe knowledge "just is" out there with someone, but not them (silent knowers); or, like special treats, hall passes, and the family car, knowledge is in the hands of the authorities who dole it out when they deem it to be appropriate for use in the designated situation (received knowers). Teaching these students "for constructivism" means changing their epistemologies (and self-concepts) and *then* teaching what we typically think of as "the mathematics."

"Fixing the mathematics" in the context of studies surrounding WWK, means bringing these issues of epistemology and the self as knower into the classroom as a part of the content of the curriculum and instructional activities. The mathematics autobiography, reflective journal, and related classroom discussion are offered as tools which have proven useful in this repair. But, we are warned that some students will need to learn how to use these tools, that is they will have to be taught a language with which to write and speak about mathematics and their reciprocal relation to it.

Although the research cited here provides a compelling rationale, the general idea is not entirely new. The *Standards* call for incorporating writing into mathematics classrooms; Dorothy Buerk has been doing so for a decade at least, and perhaps other teachers have as well. In *Caring*, Nel Noddings (1984) discusses the importance of having students who hate mathematics reflect on the meaning that withdrawal from math will have on their lives. Elsewhere (Damarin, 1990), I have argued that some of the messages about women and math that circulate in the press should be brought into classroom for discussion. In his recent book about mathematics and popular culture, Peter Appelbaum (1995) urges us to consider that all the messages about mathematics that we receive through the media (and he argues that there are many) are part of mathematics and must be brought into the classroom.[13]

If these seem like radical demands, there are other ways in which feminists are studying the question of "fixing the mathematics" which can make the use of mathematics autobiographies and teaching for connected knowing seem like "math class as usual". The effectiveness (for girls) of single sex mathematics classes has been amply demonstrated at SummerMath and other sites, and there appears to be an emergent movement in support of offering high school girls this option. The movement toward Afrocentric Magnet Schools across all grade levels springs from comparable concerns.

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[13] Applebaum's book has much to say about gender and mathematics and is important reading in this area.

Perhaps even more radical, there is a growing number of feminist researchers, both mathematicians and mathematics educators, who are examining the question of fixing the subject matter of mathematics itself. Leone Burton (1995) is working on a redefinition and reorganization of major strands of mathematics. In current work, I examine the ways in which fractions (in press) and probability (in preparation) reflect and contradict gender specific experience of the world by males and females respectively. A current issue of a women's studies journal includes a feminist critique of statistics (Hughes, 1995). And, at a recent conference on The Women, Gender, and Science Question, Ram Mahalingham (1995) and Bonnie Shulman (1995) each presented a critique of the foundations of mathematics based on feminist philosophy. For at least a decade, feminist philosophy and critique of science has invited this activity and provided some direction for it (see Damarin, 1995a, b). Moreover, the "new philosophers of mathematics" are revealing mathematics as a social-cultural-historical construction (Hersh, 1994); in this context, the patriarchal character of the social, the cultural, and the historical, as uncovered by feminist scholars in these areas, invites increased work in these directions.

Feminist epistemologists (e.g., Harding, 1993) and other feminist philosophers and sociologists encourage examination of mathematical concepts in relation to women's subjective experiences of the world. More interesting in the current context of psychology and mathematics education, French feminist philosophers with training and intellectual roots in Freudian and Lacanian psychoanalytic theory (e.g., Irigaray, 1985 1987, 1993; Wittig, 1992) argue that all science, including mathematics (and indeed all knowledge) is rooted in the "male imaginary" (e.g., phallic imagery, imagery of separation from the mother, the law of the father, etc.); the mathematics and science which result are (in their analyses) based on ideas of strict separation, boundaries, closure, duality, and related ideas. In the views of Irigaray and Wittig, true equality for women requires the grounding of (some) knowledge in a "feminine imaginary" based upon women's experience of their sexed (and gendered) bodies. In Irigaray's (1987) explication, a female-grounded mathematics would be based on ideas of connection, partial closure, in betweenness, and semi-permeable boundaries, among others. In the absence of a definition of what, exactly, mathematics is it is hard to examine the validity of these claims .... but, we do, indeed, live in interesting times.

Considering Race and Ethnicity IV

The centrality of Eurocentric thinking in this current work is evident in the references to psychoanalytic thinking. But, Afrocentric philosophy and psychology might also yield similar approaches to the creation of new mathematics (and may have done so already). Native American understandings of the world, Asian philosophies, and root belief systems of other cultures might also eventuate in some set of concepts and procedures which is arguably mathematics. As a stimulus to think about the possibility of misfit between our current mathematics and Afrocentric epistemologies, consider the following event.

> Some years ago a black psychologist gave a workshop on optimal psychology to a group of women recovering from alcohol, drug, and other dependencies. She opened her presentation by holding a pen firmly between her thumb and fingers, horizontal with its length visible to the audience of whom she asked. "Is this pen moving or is it still?" Within the next few minutes, she had elicited from the audience numerous answers involving the rotation of the earth, the movement of molecules, the relation of the pen to its molecules, the meaning of "still", her own (in)ability to be perfectly still, and many other issues. Assuring the audience that their answers were good, she went on to observe that the pen was both moving and still — not either moving or still, but *both moving and still.* What's more, all of our knowledge of the Universe comes together in the pen to make it the unique object that it is. She proceeded to describe a way of knowing the world (and themselves within it) in all its complexity and multiplicity, describing to the women a philosophy and a psychology in which either/or yields to both/and, and in which all things come together in each person. Choosing to focus on some things and not others is possible, and sometimes necessary, but focus is different from truth.

I do not know the mathematics to describe the simultaneous motion and stillness of the pen in the world, or the mathematics of both/and logic. But I do know that this simple introduction to one woman's understanding of Afrocentric thinking enriched my life and my understanding that there are indeed rich and valuable ways of knowing that I did not learn in school.

Concluding remarks

This paper maps a whirlwind journey from the question of everyday fairness in mathematics teaching to the psychological and epistemological underpinnings of mathematical thought. The short version of the paper is this:

> The question of fairness — or equity — in mathematics education is important, interesting, and deep. It is as deep and as difficult as any theorem of mathematics or theory of learning and education, and, in my view at least, more important. Partial answers to the question of how to deal fairly can be found at all levels, and fairness requires that we bring those answers into the mathematics classroom, not as final solutions, but as steps in a continuing process.

The journey sketched here is mine. Your journey, should you choose to make it, must be your own. Perhaps you will begin, not with feminist theory, but with theories of situated learning and the multi-cultural studies which support them. If so, your path will lead you through the work of Burke and McLellan (1995) to the great black educational theorists of education: Booker T. Washington, who admonished black students to "cast down your buckets where you are" situating learning in current reality, and W. E. B. DuBois who disagreed and debated with him. Once in the domain of black history, you will likely happen upon Bob Moses and his work on Algebra as the new civil right (1995) and Frederick Douglass' analysis of racism as diseased imagination. Or, you may begin with a guided tour through some part of the literature of ethnomathematics. Stopping for rest, you will find yourself refreshed by a new-found understanding of the diversity of ways of interpreting the world and a profound respect for the abilities of all peoples to think, to understand, and to construct their own knowledge of the world and of themselves.

A Postmodern Deconstructive Afterword

Postmodern philosophies, afrocentric epistemology, and some feminist epistemology have in common the rejection of binary thinking. Rather than seeing a contradiction between A and not-A, they seek and embrace the simultaneous truth of both. Referred to as diunital logic in Afrocentric theory, this is the basis of the postmodern method of deconstruction.[14] Deconstructing an argument or the constellation of arguments which come together as a construct or theory is accomplished through stating many reversals of and exceptions to all implications, joining all of the new statements to their originals, and making whatever sense can be made of the totality. Deconstruction is reserved, in postmodern analysis for important constructs and texts; the process is lengthy and revealing (see Cherryholmes, 1988; Rosenau, 1991; Usher and Edwards, 1994).

Important arguments in the discourse of mathematics education are those which link race and gender with mathematical ability. The barrage of NAEP-type data, together with texts such as *The Bell Curve* (Herrnstein and Murray, 1994) are presented as "proof" that race, socio-economic status (class) and (to a lesser extent) gender predict mathematics ability and performance. What is not mentioned in these texts is that *gender, race, mathematics, ability*, and *performance* are all social constructions which operate in the construction of each other. Deconstructive readings of these data and texts make equally plausible a constellation of related statements including "mathematical performance predicts race and gender." One interpretation of this statement is that mathematical performance is a critical factor in defining roles of race and gender; that is, we learn how to perform in mathematics classes as a part of our learning of how to perform our roles in society as raced and gendered individuals. In this view, race and gender are not attributes we have; they are activities we do. The critical questions become, how do you perform your gender? How do you perform your race?

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[14] In a series of discussions with Umesh Thakkar (an educational technologist from India, currently at the University of Illinois), he has argued that deconstruction and most other elements and ideas of postmodern philosophies and practice have been taken from Asian and African cultures and renamed in order to deny credit to their originators. For Thakkar, postmodernism is a new site of the continuing white intellectual exploitation of people of color.

In recent related qualitative sociological research on gender and technology (Grint and Gill, 1995) evidence was found to support the assertion that establishing and maintaining certain relations to (computer) technology is a critical factor in how individuals perform their masculinity or femininity. Relatedly, based on his analysis of popular discourse (TV shows, publicity about award winning teachers and about studies of girls and mathematics), Peter Appelbaum (1995) examines the ways mathematics contributes to the representation of gender in the popular culture, and thus to the socio-cultural construction of gender. TV, movies and the press, he finds, give their viewers gender specific directions on how to "do math." Also recently, qualitative researchers are reporting that young African Americans see their peers who excel in mathematics and science as "acting white" (Lattimore, 1995). All of this points to the need to examine ways in which mathematics is implicated both in students' construction of themselves as raced and gendered and in their performance of their gender and race.

Put another way, the mathematics classroom is a theater in which students perform their identities, choosing from the roles and scripts, and using the props available. Until we (the playwrights, producers, directors, and stage hands) produce some new characters, costumes, lines, and scenery this long running play is likely to go on…and on.

References

Appelbaum, Peter. (1995). Popular culture, educational discourse, and mathematics Albany: SUNY Press.

Asante, Molefi Kete. (1987). The Afrocentric idea. Philadelphia: Temple University Press.

Banks, James A. (June-July, 1993). The canon debate, knowledge construction, and multicultural education. Educational Researcher, 22, 4-14.

Becker, Joanne Rossi. (1995). Women's ways of knowing in mathematics. In Pat Rogers & Gabriele Kaiser (Eds.), Equity in mathematics education: Influences of feminism and culture (pp. 163-174). London: The Falmer Press.

Becker, Joanne Rossi, & Jacobs, Judith E. (1989) Communication in the WME Newsletter.

Belenky, Mary, Clinchy, Blythe M. , Goldberger, Nancy R. , & Tarule, Jill M. (1986). Women's ways of knowing: The development of self, voice, and mind. New York: Basic Books.

Bem, Sandra Lilsitz. (1993) The lenses of gender. New Haven, CT: Yale University Press.

Buerk, Dorothy. (1985). The voices of women making meaning in mathematics. Journal of Education 167, (3), 59-70.

Buerk, Dorothy. (1995). A response to the AERA research on gender and mathematics and new directions discussion. unpublished manuscript, mimeo.

Buerk, Dorothy, & Szablewski, Jackie. (1993). Getting beneath the mask: Moving out of silence. In Alvin White (Ed.), Essays in humanistic mathematics. Washington, DC: Mathematical Association of America.

Burke, Garfield, & McLellan, Hilary. (1995). The algebra project: Situated learning inspired by the Civil Rights Movement. In Hilary McLellan (Ed.), Situated learning perspectives. Englewood Cliffs, NJ: Educational Technology Publications.

Burton, Leone. (1995). Moving towards a feminist epistemology of mathematics. In Pat Rogers, & Gabriele Kaiser (Eds.), Equity in mathematics education: Influences of feminism and culture. London: The Falmer Press, 209-225.

Cherryholmes, Cleo H. (1988). Power and criticism: Poststructural investigations in education. New York: Teachers College Press.

Cohen, Patricia Cline. (1982). A calculating people: The spread of numeracy in early America. Chicago, IL: University of Chicago Press.

Collins, Patricia Hill. (1990). Black feminist thought: Knowledge, consciousness, and the politics of empowerment. Boston: Unwin Hyman.

Culley, Margo, & Portuges, Catherine (Eds.). (1985). Gendered subjects. New York: Routledge.

Damarin, Suzanne K. (in press). Thoughts on gender, fractions and toys. Focus on Learning Problems in Mathematics.

Damarin, Suzanne K. (1995a). Gender and mathematics from a feminist standpoint. In Walter Secada, Elizabeth Fennema, & Lisa Bird (Eds.), New approaches to equity in mathematics education (pp. 242-257). New York: Cambridge University Press.

Damarin, Suzanne K. (1995b). Experience, abstraction, and "algebra for all": Some thoughts on situations, algebra, and feminist research. In Carole, Lacampagne, William Blair, & Jim Kaput, (Eds.,), The Algebra Initiative Colloquium, (Vol. 2, pp. 127-136). Washington, DC: U.S. Department of Education.

Damarin, Suzanne K. (1994a). The emancipatory potential of situated learning. Educational Technology 34 (8). 16-22. (Also in Hilary McLellan (Ed.), Perspectives of situated learning, Englewood Cliffs: Educational Technology Publications. (in press)).

Damarin, Suzanne K. (1994b). Unthinking educational technology: A feminist imperative. Holistic Education Review 7 (1), 50-57.

Damarin, Suzanne K. (1994c). Equity, caring, and beyond: Can feminist ethics inform educational technology? Educational Technology 34 (2), 34-39.

Damarin, Suzanne K. (1993a). Technologies of the individual: Women and subjectivity in the age of information. Research in Technology and Philosophy, 13, 185-200.

Damarin, Suzanne K. (1993b). Schooling and situated knowledge: Travel or tourism? Educational Technology, 33, 27-32. (Also in Hilary McLellan (Ed.), Perspectives of Situated Learning, Englewood Cliffs: Educational Technology Publications (in press).)

Damarin, Suzanne K. (1990). Teaching mathematics: A feminist perspective. In Thomas J. Cooney (Ed.), Mathematics Teaching and Learning in the 1990's (pp. 144-151). Washington, DC: National Council of Teachers of Mathematics (1990 Yearbook).

Delpit, Lisa D. (1988). The silenced dialogue: Power and pedagogy in teaching other people's children. Harvard Educational Review 58 (3), 280-298.

Disch, Estelle, & Thompson, Becky. (1990). Teaching and learning from the heart. NWSA Journal 2, 68-78.

Donavan, Josephine. (1986). Feminist theory: The intellectual traditions of American feminism. New York: Ungar.

Eichler, Margrit. (1987). Nonsexist research methods: A practical guide. Boston: Allen and Unwin.

Erchick, Diana. (in press). Women's voices and the experience of mathematics. Focus on Learning Problems in Mathematics.

Erchick, Diana. (April 1995). Will the real authority please stand up? Teacher autobiographies areflection on practice. Paper presented at Research Pre-session, NCTM, Boston.

Fausto-Sterling, Anne. (1985). Myths of gender: Biological theories about women and men. New York: Basic Books.

Fennema, Elizabeth. (1993). Mathematics, gender, and research. Paper presented at the ICMI Conference on Gender and Mathematics, Hoor, Sweden.

Fennema, Elizabeth. (1990). Justice, equity, and mathematics education. In Elizabeth Fennema, & Gilah Leder (Eds.), Mathematics and gender (pp. 1-9). New York, Teachers College Press.

Fennema, Elizabeth, & Hart, Laurie E. (1994). Gender and the JRME. Journal for Research in Mathematics Education, 25, 648-659.

Fennema, Elizabeth & Peterson, Penny L. (1985). Autonomous learning behavior: A possible explanation of gender-related differences in mathematics, In Louise C. Wilkinson & Cora B. Marrett (Eds.), Gender-related differences in classroom interactions (pp.17-35). New York: Academic Press.

Fennema, Elizabeth, & Sherman, Julia A. (1977). Sex-related differences in mathematics achievement, spatial visualization, and affective factors. American Educational Research Journal 14 (3), 15-71.

Foucault, Michel. (1977). Discipline and punish: The birth of the prison. London: Penguin.

Frankenstein, Marilyn. (1995). Equity in mathematics education: Class in thew world outside the class. In Walter Secada, Elizabeth Fennema, & Lisa Bird (Eds.), New approaches to equity in mathematics education (pp. 165-190). New York: Cambridge University Press.

Frankenstein, Marilyn. (1987). Critical mathematics education: An application of Paolo Freire's epistemology. In Ira Shor (Ed.), Freire for the Classroom: A sourcebook for liberatory teaching (pp. 180-210). Portsmouth, NH: Heinemann.

Frazier, Nancy, & Sadker, Myra. (1973). Sexism in school and society. New York: Harper and Row.

Freire, Paolo. (1970). The pedagogy of the oppressed. New York: Continuum.

Friedman, Lynn. (Spring 1995). The space factor in mathematics: Gender differences. Review of Educational Research 65 (1), 22-50.

Fullerton, Olive. (1995). Who wants to feel stupid all the time? In Pat Rogers & Gabriele Kaiser (Eds.), Equity in mathematics education: Influences of feminism and culture (p. 37-48). London: The Falmer Press.

Gilligan, Carol. (1979). Woman's place in man's life cycle. Harvard Educational Review 47, 481-517.

Gilligan, Carol. (1982). In a different voice: Psychological theory and women's development. Cambridge, MA: Harvard University Press.

Gould, Stephen Jay. (1981). The Mismeasure of Man. New York: W.W. Norton.

Grint, Keith, & Gill, Rosalind, (Eds.). (1995). The gender-technology relation: Contemporary theory and research. London and Bristol, PA: Taylor and Francis.

Haraway, Donna. (1991). Primate visions: Gender, race, and nature in the world of modern science. New York: Routledge.

Harding, Sandra. (1993). Rethinking standpoint epistemology: What is "strong objectivity"? In Linda Alcoff & Elizabeth Potter (Eds.), Feminist epistemologies (p. 40-82). New York: Routledge.

Harding, Sandra (1991) The "Racial" Economy of Science. New York: Routledge.

Harding, Sandra. (1987). The method question. Hypatia: A Journal of Feminist Philosophy, 2 (3), 19-36.

Herrnstein, Richard, & Murray, Charles. (1994). The bell curve: Intelligence and class structure in American life. New York: Free Press.

hooks, bell. (1994). Teaching to transgress: Education as the practice of freedom. New York: Routledge.

Hughes, Donna M. (1995). Significant differences: the construction of knowledge, objectivity, and dominance. Women's Studies International Forum 18 (4), 395-406.

Irigaray, Luce. (1993). je, tu, nous: Toward a culture of difference. New York: Routledge.

Irigaray, Luce. (1987). Is the subject of science sexed? Hypatia: A Journal of Feminist Philosophy, 2, 32-44.

Irigaray, Luce. (1985). The sex which is not one. Ithaca, NY: Cornell University Press.

Isaacson, Zelda. (1990). "They look at you in absolute horror": Women writing and talking about mathematics. In Leone Burton (Ed.), Gender and mathematics: An international perspective (pp. 21-28). London: Cassell.

Jacobs, Judith. (1994). Feminist pedagogy and mathematics. Zentralblatt fur Didaktik der Mathematik, 94 (1), 12-17.

Jaggar, Allison M. (1983) Feminist politics and human nature. Totowa, NJ: Rowman and Allanheld.

Kaiser, Gabriele, & Rogers, Pat. (1995). Introduction: Equity in mathematics education. In Pat Rogers & Gabriele Kaiser (Eds.), Equity in mathematics education: Influences of feminism and culture (pp 1-10). London: The Falmer Press.

Kalinowski, Janet, & Buerk, Dorothy. (in press). Enhancing women's mathematical competence: An epistemological Analysis. NWSA Journal.

Kimball, Meredith M. (1989). A new perspective on women's math achievement. Psychological Bulletin, 105, 198-214.

Kimball, Meredith M. (in press). The Worlds We Live In: Gender Similarities and Differences. book manuscript.

Kramarae, Cheris, & Spender, Dale. (1992). The knowledge explosion: Generations of feminist scholarship. New York: Teachers College Press.

Ladson-Billings, Gloria. (1995). Making mathematics meaningful in multicultural contexts. In Walter Secada, Elizabeth Fennema, & Lisa Byrd Adajian (Eds.), New directions for equity in mathematics education (pp. 126-145). Cambridge and New York: Cambridge University Press.

Ladson-Billings, Gloria, & Tate, William F., IV (in press). Toward a critical race theory of education. Teachers College Record.

Lattimore, Randy. (1995). Assessing the competence of African American tenth graders in preparation for a mathematics proficiency test. (Dissertation proposal) The Ohio State University, mimeo.

Lewontin, R. C. (1992). Biology as Ideology. New York: Harper.

Linn, Marcia C., & Hyde, Janet L. (1989). Gender, mathematics, and science. Educational Researcher, 18, 22-27.

Lorde, Audre. (1984). Sister outsider. Freedom, CA: Crossing Press.

Maccoby, Eleanor E., & Jacklin, Carol N. (1974). The psychology of sex differences. Palo Alto, CA: Stanford University Press.

Mahalingham, Ram. (May 1995). Feminist mathematics: Is it possible? Implications for a multicultural mathematics education. Paper presented at the invited conference on The Women, Gender, and Science Question. University of Minnesota, Minneapolis.

Mellin-Olsen, Stieg. (1987). The politics of mathematics education. Dordrecht, Holland: D. Reidel.

Morrow, Charlene, & Morrow, James. (1995). Connecting women with mathematics. In Pat Rogers & Gabriele Kaiser (Eds.), Equity in mathematics education: Influences of feminism and culture (pp. 13-26). London: The Falmer Press.

Moses, Robert. (1995). Algebra, the new civil right. In Carole, Lacampagne, William Blair, & Jim Kaput, (Eds.), The Algebra Initiative Colloquium, (Vol. 2, pp. 53-67). Washington, DC: U.S. Department of Education.

Myers, Linda James. (1988). Understanding an Afrocentric world view: Introduction to an optimal psychology. Dubuque, Iowa: Kendall/Hunt.

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.

Noddings, Nel. (1993). Mathematics, politics, and pedagogy. In Sal Restivo, Jean Paul Van Bendegem, & Roland Fischer (Eds.), Math worlds: Philosophical and social studies of mathematics and mathematics education (pp. 150-161). Albany, NY: SUNY Press.

Noddings, Nel. (1992). The challenge to care in schools: An alternative approach to education. New York: Teachers College Press.

Noddings, Nel. (1984). Caring: A feminine approach to ethics and moral education. Berkeley: University of California Press.

Orenstein, Peggy. (1994). SchoolGirls: Young women, self-esteem and the confidence gap. New York: Doubleday.

Perry, William G. (1970). Forms of intellectual and ethical development in the college years. New York: Holt, Rinehart, and Winston.

Pimm, David. (1987). Speaking mathematically. London and New York: Routledge.

Rogers, Pat. (1990). Thoughts on power and pedagogy. In Leone Burton (Ed.), Gender and mathematics: An international perspective (pp. 38-46). London: Cassell.

Rogers, Pat, & Kaiser, Gabriele. (1995). Equity in mathematics education: Influences of feminism and culture. London and Washington, DC: The Falmer Press.

Rosenau, Pauline Marie. (1991). Post-modernism and the social sciences: Insights, inroads, and intrusions. Princeton, NJ: Princeton University Press.

Sadker, Myra, & Sadker, David. (1994). Failing at fairness: How our schools cheat girls. New York: Simon and Schuster.

Sands, Aimee. (1991). Never meant to survive: A black woman's journey. In Sandra Harding (Ed.), The "racial" economy of science (pp. 239-248). New York: Routledge.

Secada, Walter G. (1995). Social and critical dimensions for equity in mathematics education. In Walter Secada, Elizabeth Fennema, & Lisa Byrd Adajian (Eds.), New directions for equity in mathematics education (pp. 146-164). Cambridge and New York: Cambridge University Press.

Secada, Walter G. (November 1990). Needed: An agenda for equity in mathematics education. Journal for Research in Mathematics Education, 21 (5), 354-355.

Sherif, Carolyn Wood. (1979). Bias in psychology. In Julia A.Sherman and Evelyn Thornton Beck (Eds.), The prism of sex: Essays in the sociology of knowledge (pp. 93-133). Madison, WI: University of Wisconsin Press.

Shulman, Bonnie. (May 1995). What if we change our axioms? A feminist inquiry into the foundations of mathematics. Paper presented at the invited conference on The Women, Gender, and Science Question. University of Minnesota, Minneapolis.

Sleeter, Christine E., & Grant, Carl A. (1991). Mapping terrains of power: Student cultural knowledge versus classroom knowledge. In Christine E. Sleeter (Ed.), Empowerment through multicultural education (pp. 49-68). Albany, NY: SUNY Press.

Sleeter, Christine E., & Carl A. Grant. (1987). An analysis of multicultural education in the United States. Harvard Educational Review, 37 (4), 421-444.

Squire, Corrinne. (1989). Significant differences: Feminism in psychology. London and New York: Routledge.

Stiff, Lee V., & Harvey, William B. (1988). On the education of black children in mathematics. Journal of Black Studies, 19 (2), 190-203.

Stone, Lynda. (1994). The Education feminism reader. New York: Routledge.

Tartre, Lindsay. (1990). Spatial skills, gender, and mathematics. In Elizabeth Fennema & Gilah Leder (Eds.), Mathematics and gender (pp. 27-59). New York, Teachers College Press.

Taylor, Lyn. (1990). American female and male university professors' mathematical attitudes and life histories, In Leone Burton (Ed.), Gender and mathematics: An international perspective (pp. 47-59). London: Cassell.

Thompson, Helen B. (Woolley) (1903). The mental traits of sex. Chicago: University of Chicago Press.

Usher, Robin, & Edwards, Richard. (1994). Postmodernism and education. New York: Routledge.

Walker, Alice. (1983). In search of our mothers' gardens. New York: Harcourt, Brace, Jovanovich.

Walkerdine, Valerie. (1990). Schoolgirl fictions. London: Verso.

Walkerdine, Valerie, (Ed.). (1989). Counting girls out. London: Virago Press.

Walkerdine, Valerie. (1987). The mastery of reason: cognitive development and the production of rationality. London and New York: Routledge.

Weisstein, Naomi. (1971). Psychology constructs the female: or, The fantasy life of psychologists. In Michelle H. Garskof (Ed.), Roles women play: readings towards women's liberation (pp. 68-83). Belmont, CA: Freedom Press.

Wellesley Center for Research on Women. (1992). How schools shortchange girls: The AAUW Report. Washington, DC: American Association for University Women Educational Foundation.

Wittig, Monique. (1992). The straight mind. Boston: Beacon Press.

Woolley, Helen. (1903). see Thompson.

Woolley, Helen B. Thompson. (1910). Psychological literature: A review of the recent literature on the psychological of sex. Psychological Bulletin, 7, 335-342.

A REACTION TO FAIRNESS IN DEALING: DIVERSITY, PSYCHOLOGY AND MATHEMATICS EDUCATION

Ruth Cossey, Mills College

In her plenary paper, Suzanne Damarin initiates a conversation about restructuring mathematics education in order to serve fairly all students. She employs lenses developed through research and practices of feminisms and cultural studies to conceptualize excellence in mathematics education as constituting and as constituted by equity for individuals of differing gender and ethnicity. In Damarin's cogent analyses, she journeys through important, uneven, and sometimes dangerous terrain. In these comments, while referring throughout to the benefits of linking critical perspectives on research, I comment principally on two aspects of Damarin's paper: language usage and the paper within, concerning race and ethnicity.

Language

> language is not a neutral medium that passes freely and easily into the private property of the speaker's intentions; it is populated—overpopulated—with the intentions of others. Expropriating it, forcing it to submit to one's own intentions and accents, is a difficult and complicated process.
>
> Bakhtin (1981, p. 294)

We have insufficient language tools to handle adequately the shifting meanings of constructs at hand: equity, the nature of mathematics, the nature of school mathematics, authentic assessment, culturally sensitive pedagogies, feminist pedagogies, ethnicity, race, gender, class and feminisms. Damarin helps us bridge discursive traditions by providing language with which we can navigate difficult points of intersection. Much of Damarin's language-work is explicit, such as the repackaging of "equity" into the construct of "fairness in dealing." Some of the work is less obvious, such as the careful phrasing when speaking of individuals who identify or are identified as members of marginalized groups. In some places the language she uses uncovers important areas that might have been otherwise overlooked, in other places cloudiness in Damarin's language reflects the difficulties inherent in the task she has undertaken. I will revisit Damarin's discussion of the term *equity*, and her account of the contributions of the liberal feminist traditions. Both are sites of clarity and ambiguity.

Fairness in Dealing

Damarin makes a bid to discard the baggage of other people's intentions with which the word *equity* is laden. One would be hard put to find a mathematics educator or researcher who does not subscribe to both *equity* and *excellence*. But there is little consensus about their meanings either in isolation or in relationship to the other. The fact that divergent and sometimes conflicting beliefs are held by proponents of equity creates the problem that Damarin would mitigate by changing the terms of the debate from the narrower views of equity to broader considerations of fairness. As she indicates, Damarin's fairness terminology dampens arguments for equity inspired by economic determinism by shifting the emphasis to individuals and the communities in which they reside rather than the needs of the economy. It is also easier to consider the ways in which people who have succeeded in mathematics have been subject to unfair treatment using a *fairness in dealing* construct rather than *equity*, as it is harder to see the non-achievement of equity when there are equal outcomes in educational attainment or career accomplishments.

However, *fairness in dealing* does not escape some of the more debilitating characteristics of *equity*. Damarin states, for example, that equity is measure dependent - so, of course, is fairness in dealing. Further, I believe that the fairness terminology carries baggage of its own that will create barriers to its actualization. Until there is a real shift in the general perception of intelligence and its distribution, "fairness" is a dangerous term in the heads and hands of those who find comfort and merit in the arguments presented in *The Bell Curve* (Herrnstein & Murray, 1994). While it may not seem equitable to place some youngsters in classrooms which feature non-challenging, minds-off, rote, procedural-driven mathematics; some may argue that such placement is fair for those students who lack the capacity to engage in powerful mathematics along with their more able peers.

The current mathematical reform movement is built upon the twin pillars of excellence and equity. Indeed the entire national school reform movement is linked to discussions of equity and excellence (Goals 2000 for example). Political struggles have been fought and will continue to rage over whose meaning of equity will prevail. An increasing number of studies and policy developments arising from lines of inquiry generated by scholarship in the traditions of mathematics educational research have explicitly or tacitly adopted standpoints which assume the inseparability of excellence and equity. (e.g. see Keynes 1995; Ladson-Billings 1995; Secada 1995; Cuevas 1995; Hilliard 1995; Silver, Smith, & 1995). I think it beneficial to continue to populate *equity* with these intentions.

The Role of Liberal Feminism

Damarin argues that gender equity narrowly defined as equal male/female outcomes failed as an analytical tool for liberal feminist because its use did not yield results. She indicates that the line of research emanating from the liberal feminist from about 1970 has provided a lack of guidance for the achievement of equity and that there have been no fundamental changes in its attainment. These claims appear to be in contrast to Damarin's own description of the scope of the work of liberal feminist and her report of their findings. According to Damarin, a goal of liberal feminist researchers was to prescribe changes in the treatment of girls within and outside of the classroom which could increase their performance in mathematics. I believe they did just that. I think the changes in course enrollment were fundamental changes which resulted from enactment of educational policies generated from this line of inquiry. I think the research of this genre produced strategies for use by educators and other care givers that make it possible for white middle- class girls to perform just about as well in mathematics as their white middle-class brothers.

Looking back towards 1970, I see a variety of fundamental changes in the position of girls in relationship to boys in school mathematics. I think teachers have available to them tools such as GESA (Gender Expectations/ Student Achievement) training[1] that can make substantial differences in the mathematical achievement of girls in their classrooms. Still, it may be that my differences with Damarin are more semantic than substantive. Like Damarin, I agree that in concert with the "victory" of near equal achievement and declared equal ability/aptitude between white girls and boys,[2] many eyes have shifted to a different prize. (I will return to this victory in the next section) Many no longer see as desirable, for any student, the achievement of the mindless proficiency with symbolic manipulation that was equated with excellence in mathematics only thirty years ago. Many want the mathematics classroom to cease being a place of silence and/or fear for the majority of students. The lot of all students, but especially students underserved by traditional mathematics programs, will be improved greatly as we find ways to rewrite the scripts that currently tell students that doing, enjoying, and succeeding in mathematics is not an appropriate enactment of their various identities along lines of gender, race, ethnicity, class, etc. The shifts in focus for many communities of color and feminist communities away from a notion of equity that meant equal opportunity to digest American-European male-centric curriculums and pedagogies have resulted in promising research and theory building. Further the traditions of mathematics research can benefit from these efforts.

Race and Ethnicity

Damarin weaves a thread of race/ethnicity throughout her paper. She speaks of diversity in mathematics education using gender fairness as a case. For me, the race/ethnicity sections failed even though I am in total agreement that there are important and deep parallels between the work being done in feminist discourse communities and the work being done in other research communities that bring a critical/cultural perspective to their research. It is important to examine critically the absences and biases of European-American psychology and to seek the culturally specific ways that students understand the world. A curriculum and pedagogy that provides for connected knowing of connected knowledge and processes seems beneficial to all students. Educators should strive to help students find their voice in mathematics. It is imperative that we establish educational policies that acknowledge and support different ways of knowing and demonstrating competencies in mathematics. These all seem derivable from findings or hunches within a mulutude of research paradigms. But, given my many agreements with Damarin, I still am uncomfortable with the physical structure of her analogous reasoning and I am cautious about possible inferences that could be made from the content of some of her arguments.

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[1] Information about these materials is available from GESA, Graymill Foundation, Route 1, Box 45, Earlham, IA, 50072

[2] Certainly there have also been reports of equal or better achievement of girls compared to boys within ethnic groups of color. However, since the intersection of possession of skin with high melanin content and attendance at grossly underfunded schools is so large, the achievement difference between most of these groups and white children is a more significant factor than within group gender equality.

Physically Structured Marginalization

While it may appear to white middle-class able-bodied females that gender is the major cultural vehicle that the world uses to encourage and constrain them, it is true that other factors, race, class, health, etc. are also used in the communities within which they interact to organize their life possibilities. Simply put, those from dominant groups do not see their dominant features as salient in the construction of social arrangements from which they benefit. Damarin does not make this mistake, but the organization of her paper marginalizes females and others from non-dominant backgrounds in the United States. Given the physical separation of gender and ethnicity, and given the fact that most of the gender work is done in relationship to white females here and in Europe and Australia, it is difficult not to read part of the paper - the main part- as pertaining to white middle-class women and the other part as pertaining to how other folk are or are not like white middle-class women. A further marginalization occurs as blacks are taken to be the modal ethnic/race group. Almost all extended examples are taken from this group giving what might seem like honorable mention to members of other groups.

Exclusion of Class

Quite understandably given space constraints, Damarin made a decision to exclude discussions of class. I have come to believe that particularly for people of color in the United States that considerations of ethnicity, gender, and class in isolation from each other are less useful than considerations of individuals and groups in more complex constellations which illuminate multiple overlapping identities. I confess an inability to envision a paper that would have meaningfully dealt fairly with members of all major intersecting equity-groups. Perhaps we could write the series of papers, suggested by Damarin, of equity journeys emanating from different perceptual starting points.

Victory of Equal Ability/Attainment

The decisions of feminists to shift to a search for women's ways of knowing mathematics and to find ways to "fix school mathematics" come from a position of power. Feminists have evidence that girls are equal to boys in both mathematical ability and aptitude. They also now have evidence of near equal attainment in course taking, undergraduate degrees, and high school grades for white boys and girls. Clearly, if girls want to do school mathematics they can. Many ethnic Americans of color do not have the advantage of interacting in schools where others will no longer assume that they are intellectually inferior to white Americans.[5] Although existence proofs abound that this ethnic student, that class of poor children, a specific school or district showed mathematical excellence by traditional measures; those examples too often function as the exception that proves the rule.

There is no doubt in my mind that the entire mathematical education enterprise needs reforming for all students and teachers. It is also clear to me that the attainment difference between white middle-class females and oppressed people of color makes the call for culturally relevant teaching (especially of "fixed mathematics") qualitatively different than the call for gender relevant teaching for white middle class females. As an educator/ researcher in favor of such changes, I am fearful that teachers and other care takers of students from backgrounds that are not white and middle class will provide educational experiences that may equip students for the 21st century but will make them ill prepared for next year's high stakes non-state-of-the-art-exam. Practitioners and students are caught in the shifting sands of the appropriate nature of school mathematics and assessment. The changes in curriculum, pedagogy, and assessment in school mathematics should be systemic for all communities, but they must be systemic (at least locally) to provide safety from blatantly unfair individual consequences for students who are members of non- dominate ethnic groups. (See Delpit, 1988) for a discussion of the dilemma of teachers of students from non-dominant culture/classes). All of the gender/ethnicity analogy sections proceeded from this point of power difference. The magnitude of the difference jeopardizes the validity of analogies across the gender/ethnicity sections of the paper.

Voice in the Mathematics Classroom

> Now that the audience for feminist writing and speaking has become more diverse, it is evident that we must change conventional ways of thinking about language, creating spaces where diverse voices can speak in words other than English or in broken, vernacular speech. This means that at a lecture or even in written work there will be fragments of speech that may not be accessible to every individual. Shifting how we think about language and how we use it necessarily alters how we know what we know. . . . I suggest that we do not necessarily need to hear and know what is stated in its entirety, that we do nor need to "master" or conquer the narrative as a whole, that we may know in fragments. I suggest that we may learn from the spaces of silence as well as spaces of speech, that in the patient act of listening to another tongue we may subvert that culture of impe-

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[5] I certainly do not mean to imply that middle class white girls are not now subject to terrible and invidious myths of female mathematical incompetence. I am only indicating that the research arsenal available to help middle class white girls in their battles are vastly greater than those available to many American ethnic groups of color.

> rialism that suggests that one is worthy of being heard only if one speaks in standard English.
>
> bell hooks (1994, p. 174)

I applaud the NCTM's emphasis on mathematical communication and Damarin's endorsement of mathematical autobiographies as means for students to bring voice to their study of mathematics. It is a device that I have found useful in my teaching of both school aged children and adult teacher candidates. Still, I caution us from rushing to embrace written communication about mathematical understanding or feelings about mathematics without paying attention to cultural issues. Narrow criteria of "proper" discourses that ignore whether or not an idea was clearly communicated to a reasonable audience of peers and teachers are inappropriate building blocks of modern mathematical communication classroom environments and assessments. Hopefully, emerging rules of legitimate discourse will not artificially discriminate against users of non-standard Mathematical English terminology or grammar. Valid criteria is more responsive to the quality of the ideas communicated, the logical coherence of the presentation, and the clarity of the communication than to hegemonic syntactical, grammatical forms of decontextualized mathematical discourse typically found in traditional high school and college mathematics textbooks. Again, nothing in Damarin's paper suggests that she would be guilty of silencing students who are slow to warm to communicating through journals or in standard English.

Summary

Near the end of her paper, Damarin treats us to a description of a black psychologist opening presentation of optimal psychology. The fuzzy logic image of the still and moving pen will always be with me. Damarin's' paper is an example of "both/and" reasoning. She manages to walk a path both around the edges and through perspectives so diverse that only a "both/and" thinker could manage to carry others along on such a journey.

References

Bakhtin, Mikhail. (1981). *The Dialogic Imagination: Four Essays.* Austin: University of Texas Press.

Cuevas, Gilbert J. (1995). Empowering all students to learn mathematics. In Iris M. Carl (Ed.), *Prospects for School Mathematics* (pp. 21-42). Reston, VA: National Council of Teachers of Mathematics.

Delpit, Lisa. (1988). The silenced dialogue: Power and pedagogy in educating other people's children." *Harvard Educational Review 58,*(3), 280-298.

Herrnstein, Richard, & Murray, Charles. (1994). *The bell curve: Intelligence and class structure in American life.* New York: Free Press.

Hilliard, Asa G. (1995). Mathematics excellence for cultural "minority" students: What is the problem?" In Iris M. Carl (Ed.), *Prospects for School Mathematics* (pp. 99-114). Reston, VA: National Council of Teachers of Mathematics.

hooks, bell. (1994).*Teaching to tangress: Education as the practice of freedom*. New York: Routledge.

Keynes, Harvey B. (1995). Can equity thrive in a culture of mathematical excellence? In Walter Secada, Elizabeth Fennema, & Lisa Byrd Adajian (Eds.), *New directions for equity in mathematics education*. (pp. 57-92). Cambridge & New York: Cambridge University Press.

Ladson-Billings, Gloria. (1995). Making mathematics meaningful in multicultural contexts. In Walter Secada, Elizabeth Fennema, & Lisa Byrd Adajian (Eds.), *New directions for equity in mathematics education*. (pp. 126-145). Cambridge & New York: Cambridge University Press.

Secada, Walter G. (1995). Social and critical dimensions for equity in mathematics education. In Walter Secada, Elizabeth Fennema, & Lisa Byrd Adajian (Eds.), *New directions for equity in mathematics education*. (pp. 146-164). Cambridge & New York: Cambridge University Press.

Silver, Edward A., Smith, Margaret S., & Nelson, Barbara S. The QUASAR Project: Equity concerns meet mathematics education reform in the middle school. In Walter Secada, Elizabeth Fennema, & Lisa Byrd Adajian (Eds.), *New directions for equity in mathematics education*. (pp. 9-56). Cambridge & New York: Cambridge University Press.

SHUFFLING THE DECK TO ENSURE FAIRNESS IN DEALING: A COMMENTARY ON SOME ISSUES OF EQUITY AND MATHEMATICS EDUCATION FROM THE PERSPECTIVE OF THE QUASAR PROJECT[1]

Edward A. Silver, University of Pittsburgh[2]

In her plenary paper for this conference, Suzanne Damarin takes us on a "whirlwind journey" (in her words) through the field of feminist research related to mathematics education, with some attention also to critical race theory and postmodernism. Throughout her paper, she reminds us that the relationship between gender and mathematics is an important issue worthy of our attention from a variety of perspectives, and she also attempts to tie gender concerns to those of race and ethnicity. This sampler of feminist and other views can enrich both the study and the practice of mathematics education, and it is likely to make a valuable contribution to a small but growing literature generally concerned with the theme of "mathematics for all."

I am not an expert in the areas of feminist research, critical race theory or postmodernist perspectives, but it appears to me that the expanse of intellectual terrain covered in Damarin's paper is impressive. However, as is often the case when a broad range of topics and perspectives are addressed, the attention to breadth rather than depth results in a paper that suffers in many places from a lack of detail. Moreover, the non-feminist perspectives, such as postmodernism, are offered in a generally uncritical, unanalyzed manner that limits the contribution that the paper might have made in helping researchers in mathematics education understand the power and limitations of these less familiar perspectives. Nevertheless, the paper succeeds in addressing a large number of important issues that are worthy of serious consideration in mathematics education research and practice. I was particularly struck by her characterization of equity as "fairness in dealing," and I was able to find connections between this notion and several issues embedded in work that my colleagues and I have been doing in the QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) project.

The QUASAR Project

QUASAR is a mathematics education reform project which has been supporting and studying the design and implementation of innovative instructional programs in middle schools serving economically disadvantaged communities. The project was designed to address the persistent historical association of poverty and low achievement in mathematics by providing students in schools in low income communities with access to mathematics instruction that heavily emphasized understanding, reasoning, and problem solving rather than memorization and imitation.

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[1] Preparation of this paper has been supported by a grant from the Ford Foundation for the QUASAR project. The opinions expressed herein are those of the author and do not necessarily reflect the views of the Foundation.

[2] I wish to acknowledge the contributions of Catherine Brown, Ellice Forman, Margaret Schwan Smith, and Mary Kay Stein, each of whom shared valuable insights with me as I prepared this paper, thereby enriching my understanding of the issues discussed herein. I am also grateful to Barbara Grover, Suzanne Lane, and Maria Magone for their comments.

Located in urban school districts, project schools serve a culturally and linguistically diverse set of students. Aggregated across all QUASAR schools, about half the students are African-American, about one-third Latino/Latina, and about one-eighth Caucasian. The patterns of ethnic distribution within the school population vary across sites, with two schools serving predominantly African-American students, two primarily Latino/Latina students, and the other two having student populations that are internally ethnically diverse. Linguistic diversity is also found in many QUASAR schools. In fact, most schools serve large subgroups of students for whom English is not the primary language spoken at home; in two schools, that group is the majority. Although there is considerable diversity with respect to ethnicity and language, there is very little variance with respect to another demographic characteristic: the vast majority of students who attend each QUASAR school live in poverty.

At each QUASAR school, the mathematics teachers and school administrators have been working with "resource partners" — usually mathematics educators from a local university — to enhance the school's mathematics instructional program. Each site team has operated independently to design and implement its plan for curriculum, staff development, and other aspects of the program, so there is diversity across the schools with respect to curricula and forms of support provided to students and teachers, but there are also many similar features that characterize mathematics instruction in QUASAR schools.

Shuffling the Deck: Some Aspects of QUASAR's Pedagogy of Fair Dealing

Three aspects of the instruction found in QUASAR schools are discussed here as they relate to themes developed in Damarin's paper. The notion of intentional focus is discussed first, as it relates to the contrast between QUASAR's focus on a diverse composite (the poor) and Damarin's focus on gender, race or ethnicity subgroups. Next, the repertoire of instructional practices suggested by Damarin to "fix the mathematics" is expanded by looking at instruction in QUASAR classrooms. Finally, the role that such instruction can play in helping students see themselves as knowers and doers of mathematics is examined.

Intentionality

In discussing lines of feminist research and theory that have addressed "fixing the mathematics" rather than "fixing the women," Damarin underscores the point that these researchers argue for the necessity of paying specific attention to girls and women. Somewhat in contrast, the instructional reform activity of QUASAR was undertaken with specific attention to the children of poverty, regardless of gender, race, ethnicity, or language. Thus, the QUASAR target group was more diverse than the groups addressed in most feminist research or in interventions designed for one particular ethnic, racial or linguistic subgroup. Nevertheless, some (though not all) of the educational approaches used at project sites were adaptations of work that had been developed with a focus on a particular subgroup. For example, one QUASAR site began with an intention to adapt the approaches used in an innovative enrichment program designed for female high school students. At another QUASAR site, the plan was to use curriculum materials that placed a heavy emphasis on visual models; among the reasons for development of the materials was the successful use of such activities with Native American students.

One point seems clear from this brief glimpse into QUASAR. The realities of many educational settings in this country often do not afford mathematics teachers the "luxury" of focus that many of us have in our research and theory. Students, both males and females, may come in several colors, from diverse cultural heritages, and may speak many different languages. Thus, at least some research attention needs to be devoted to the composite mosaic as well as to its components. Although research and theory generated from a perspective of specific focus on a particular gender or cultural group can aid in addressing broader issues, it is unlikely to be sufficient to address all issues of relevance and import to mathematics teaching and learning in diverse classrooms.

Institutional and Instructional Practices

A first step in increasing equity at most QUASAR schools was the elimination of the academic tracking practices that were in place prior to the beginning of the project in 1989. In these schools, as in many similar schools across the country, it had been common for students to be placed in different classes on the basis of test scores and presumed ability. In these different tracks, students either pursued different curricula or studied the same curriculum at different speeds. In general, this practice led to unequal opportunity for students in the lower tracks to pursue courses with higher-level goals and objectives, especially since instruction in lower-track courses tended to omit challenging material (Oakes, 1990). When QUASAR began, the practice of academic tracking was essentially ended at the project sites. As a consequence, all students in the school — including those in bilingual or other "special" mathematics classes — generally received similar instruction.

Once tracking was eliminated, the mathematics classes became more diverse than had been the case prior to the project, and teachers were challenged to develop new instructional approaches that would accommodate more diverse groups of students. Damarin points to a few instructional practices that have been identified as addressing the need to "fix the mathematics" and to connect instruction to the "women's ways of knowing" provided by Belenky, Clinchy, Goldberger and Tarule (1986). In particular, she mentions journal writing as a means of community building and giving students "voice," and several references are made to studies that have suggested the efficacy of cooperative learning for females. In general, in the QUASAR project we have noted the efficacy of these practices and others for diverse groups of students.

Drawing on examples from QUASAR classrooms, Silver, Smith and Nelson (1995) describe the efforts of teachers to develop collaborative discourse communities in their classrooms by using cooperative group work to foster communication and collaboration and by providing mathematics problems that can be represented and solved in multiple ways in order to give students multiple entry points into problem solving. Silver et al. also demonstrate how QUASAR teachers encourage students to engage in and then communicate their own thinking; how teachers support students as they learn to examine each other's reasoning, while at the same time learning both to value different perspectives and to maintain respect for each other as people; and how teachers enhance the "relevance" of mathematics by tying it to students' life experiences, interests, and cultural heritage.

These practices can be seen as related to the notions of "connected knowing" and "constructed knowing" described originally by Belenky et al. (1986) and elaborated more recently by Becker (1995). Some research suggests that these features of what Becker calls "connected teaching" are quite likely to also support the learning of culturally diverse students. In fact, a review of educational practices used successfully with linguistically and culturally diverse student populations (Garcia, 1991) reported that collaboration and communication were key elements of effective instructional practice, especially when the curriculum blends challenging and basic academic content, as is done in QUASAR mathematics classrooms. In order to develop mathematical proficiency in a wider range of students, it is critical to focus not only on alternative modes of instruction but also on appropriate challenging tasks that have the potential to develop students' understandings and capacities for mathematical problem solving and reasoning.

There is evidence that the instruction provided in QUASAR classrooms not only encourages connected knowing in the ways described above but also engages students with challenging mathematical tasks. In particular, an analysis of a representative sample of nearly 150 instructional tasks used in project classrooms over three years, Stein, Grover, and Henningsen (in press) found that about three-fourths of the instructional episodes involved mathematical tasks intended to provoke students to engage in conceptual understanding, reasoning or problem solving. These tasks encouraged students to use mathematical thinking and reasoning—either in connecting procedures to underlying concepts and meaning, or in tackling complex mathematical problems in novel ways. Only about 20% of the tasks were set up and implemented to involve computation or memorization of information without some overt connection to developing understanding. Thus, instruction in QUASAR mathematics classrooms is oriented toward understanding, reasoning, problem solving, and communication to a much greater extent than is found in conventional mathematics classrooms.[3]

It is important to note, however, that within these overall findings there are inter-school and inter-teacher variations that may be important. Not all small group work is likely to be efficacious, nor will all journal writing be enriching. We need to understand even more about which instructional practices work for diverse groups of students and under what conditions they may be ineffective. For example, using a portion of the classroom observation scheme developed and used in QUASAR (Stein, Grover & Silver, 1991), Murrell (1995) observed several middle school classrooms and concluded that "reform-oriented instruction" often worked in ways that were detrimental to the learning of low-achieving African-American males. Secada (1992) has also provided an account of some ways in which students with limited proficiency in English can be "left out" of the discourse in reform-oriented mathematics classrooms, even when the lesson is being conducted by an exceptional teacher. Thus, Murrell's findings and Secada's analysis, along with the admonition of Delpit (1988) not to repeat the failure of earlier "process-oriented" school reforms, suggest that we need to resist premature declarations of efficacy.

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[3] For conventional mathematics instruction, Stodolsky (1988) reported that 97% of the classes she observed dealt with low-level cognitive objectives.

Another reason for caution is the recognition that even the most effective classroom instruction cannot by itself completely overcome the institutionalized prejudice encountered by students in and out of school. Although Suzanne Damarin has chosen to frame equity issues in ways that de-emphasize the power of social class and economic forces in determining much of what happens to students in and out of school, there is no question that we must recognize the power of these structural relations even as we resist the pull toward reductionism and essentialism. In QUASAR schools, students often miss large numbers of instructional days as they and their families struggle with the ponderous forces that act on the urban poor — inadequate health care, housing, transportation, and economic or personal security. And these stresses often result in tremendous instability in the lives of students. Within the group of students completing grade 8 at project schools, only about half are students who have attended the school since grade 6. Furthermore, when QUASAR graduates are denied access to educational opportunities because students from "those schools" are not expected to be "ready" to take "that course," and are subjected instead to mind-numbing instruction devoid of intellectual substance and challenge, the forces of institutionalized racism and class prejudice are clear. In a recent essay, Anyon provided the distilled essence: "Educational reforms cannot compensate for the ravages of society" (1995, p. 88). As we develop theory, research and educational interventions related to equity and mathematics education, we must deal with these forces — the extent to which the "deck is stacked" against fairness in dealing. However, even as we do this, we need to keep in mind individuals as well as institutions.

Identity Development

In her plenary paper, Damarin argues that effective mathematics instruction would allow women to (re)construct themselves as (other than silent) knowers of mathematics. In fact, it is likely that many features of instruction in QUASAR classrooms support a student in developing an identity as a knower and doer of mathematics.

Forman (in press) examined instruction in a QUASAR classroom from the perspective of sociocultural theory, such as Lave and Wenger's (1991) notion of legitimate peripheral participation within a community of practice. She observed that, in contrast to traditional classrooms, QUASAR classrooms offer students a variety of participation structures, including whole-class discussions, group work, student demonstrations, and individual student-teacher interactions. These varied activity settings, when coupled with the other characteristics of tasks and instruction identified above, are likely to allow students multiple opportunities to participate in knowing and doing mathematics since they can "find their voice" by connecting to one or more of these participation structures. Connectedness is also encouraged in many project classrooms through student writing about their experiences, attitudes, and feelings about mathematics. In most project classrooms, students are asked to write reflections on the understandings and confusions associated with selected lessons or assignments.

Forman (in press) also notes that the forms of discourse encountered in QUASAR classrooms deviate from the familiar "recitation script" associated with conventional mathematics instruction. She points to a variety of ways in which teachers support students to become full participants in a classroom mathematical community by assisting them to learn the linguistic practices expected of a full participant, such as explaining their thinking, providing rationales for solutions or approaches, and coming to understand each other's thinking. As students participate in this kind of scaffolded classroom discourse, they can gradually come to see themselves as members of a community of *knowers and doers* of mathematics (Lave & Wenger, 1991). This contrasts sharply with conventional instructional settings in which students instead learn to view themselves as individuals who are *receivers* of mathematical knowledge created by others who are unknown and unavailable.

The importance of identity formation has also been discussed in related research conducted in out-of-school settings. Heath and McLaughlin (1993) studied community-based youth organizations and the ways in which they help inner-city youth develop a strong sense of self, of empowerment, and of persistence. Heath and McLaughlin concluded that, in order to understand the impact of community-based organizational practices on youth, it is important to consider two frames of reference — objective (or outsider) perspectives and subjective (or insider) perspectives — and they argued that personal identity may be at least as important as matters of race or ethnicity in the lives of children: "Ethnicity seemed, from the youth perspective, to be more often a label assigned to them by outsiders than an indication of their real sense of self" (p. 6).

Although much of the work on mathematics and gender, including *Women's Ways of Knowing*, has benefited from a subjective perspective, when policy prescriptions or research agendas have been derived from this work there has been a tendency to lose the individuality of students in favor of assigning each person group membership and identification. Given the importance of individual identity in human intellectual and social development, it seems critical for us to balance the outsider and insider perspectives in our research as we examine the conditions under which diverse groups of students develop views of themselves as being competent knowers and doers of mathematics.

Is It Really a Fair Game?: Assessing Outcomes Responsibly

The impact of instruction in QUASAR classrooms has been examined by measuring changes in students' mathematical performance over time. Damarin refers to the inadequacy of traditional mathematics tests (e.g., commercial standardized tests) to detect mathematical proficiency in gender-sensitive and race-sensitive ways. Traditional measures of mathematics performance are also generally viewed as inadequate to measure the kinds of high-level cognitive outcomes that were intended to be a special instructional focus at project sites. Thus, the project developed the QUASAR Cognitive Assessment Instrument (QCAI) to assess students' mathematical understanding, problem solving, reasoning, and communication (Lane 1993; Silver & Lane, 1993). The QCAI was developed with attention to various equity considerations, such as potential gender, racial, or ethnic bias in task formats, scoring rubrics and test administration (Lane & Silver, 1995).

A first-order question for the QUASAR project was whether or not students were benefiting in the intended ways. An analysis of QCAI results from the first three project years provided clear evidence that students had increased their capacity for mathematical reasoning, problem solving and communication during that time period (Lane & Silver, 1994). Evidence of changes in students' mathematical understanding, thinking and reasoning over time came from an aggregation of holistic judgments of student performance on a QCAI tasks administered across the years at all three grade levels. In particular, the number of students providing responses judged to be at the two highest score levels more than doubled (from 18% to 40%) between Fall 1990 and Spring 1993. Further evidence was obtained from a detailed examination of responses to a subset of QCAI tasks to reveal growth in students' mathematical understanding, in their use of appropriate strategies, and in the quality of their mathematical justifications.

Is it the case that QUASAR students, regardless of gender, race, or primary language benefit in equitable ways? To examine "fairness in dealing" in the project, a series of analyses have been conducted. Lane, Wang and Magone (1995) examined the performance on all QCAI tasks by male and female students in grades 6 and 7 in two different years, and they found no significant gender difference for 30 of the 36 tasks, thereby suggesting that QUASAR instruction was supporting the learning of male and female students equally well. Males did significantly better than females on only two tasks, and females did significantly better on four tasks. Another analysis of QUASAR data revealed that the gains made by various racial/ethnic or linguistic subgroups of students were generally quite similar to each other and to those found for the total student population (Lane, Silver & Wang, 1995). In particular, at the two schools with samples of African-American and Caucasian students sufficiently large enough to permit examination of annual performance gains for longitudinal cohorts of students, it was reported that (a) the total performance gains were similar for three of the four cohorts, and (b) the gap between Caucasian and African-American students, which was quite large at the beginning of grade 6, decreased significantly for three of the four cohorts and remained essentially constant for the fourth. Similarly, at one school which had a population that permitted such an analysis, the performance of two longitudinal cohorts of Spanish-speaking students receiving bilingual Spanish-English instruction was compared with that of non-Spanish-speaking students receiving monolingual English instruction, and it was found that students in the bilingual classes had performed less well when they entered the program but that the performance gap was substantially reduced or eliminated by the end of grade 8.

A further question might be asked: "Does 'fairness in dealing' actually result from 'fixing the mathematics'?" An answer to this question can be found in another project analysis examining the relationship between instructional processes and student learning outcomes in QUASAR classrooms. Stein and Lane (1995) found that student learning gains were especially positive in classrooms in which instructional tasks consistently encouraged high-level thinking and reasoning and involved multiple solution strategies, multiple connected representations, or mathematical explanations, and that student performance gains were small in classrooms using instructional tasks that were procedural in nature and required only one solution strategy or representation, and little or no mathematical communication.

Although we still need to examine the entire corpus of project data collected over five years, the findings of our analyses to date are encouraging for those of us interested in equity and mathematics education reform. Collectively, these analyses support a conclusion that the features of instruction generally being called for by mathematics education reformers and generally being utilized in QUASAR classrooms can support all students in diverse school populations to improve their mathematical understanding, reasoning and problem solving. The findings are especially meaningful because they relate to both an instructional approach oriented toward *dealing fairly* with diverse groups of students as they learn a *fair deal of mathematics* and an assessment measure designed to be equitable, sensitive to change, and reflective of important mathematical learning outcomes.

Coda

Suzanne Damarin has challenged us to examine how gender and race are both predictive of and predicted by mathematical achievement. I have pointed to the equal importance of considering larger and smaller units of inquiry (i.e., the children of poverty and individual student identities) in examining equity issues in mathematics education. If equity issues are examined from multiple perspectives, then we are more likely to achieve "fairness in dealing" with ALL students, regardless of gender, race, ethnicity, language or social class.

Just as it is important for students to develop an identity as *knowers and doers of mathematics*, it is equally important for mathematics educators to develop an identity as *knowers and doers of equity*. Too little research in mathematics education has focused squarely on equity issues, and even less has focused elsewhere while keeping equity concerns in mind. If progress is to be made, then it seems clear that the entire field, including its research, needs to become more self-consciously interested in matters of equity. At this critical juncture in the history of mathematics education, research can contribute in fundamental ways to understanding and accomplishing the agenda of "mathematics for all." We are fortunate that Suzanne Damarin has provided us with a valuable set of resources for our work.

References

Anyon, J. (1995). Race, social class, and educational reform in an inner-city school. *Teachers College Record*, 97(1), 69-94.

Becker, J. R. (1995). Women's ways of knowing in mathematics. In P. Rogers & G. Kaiser (Eds.), *Equity in mathematics education: Influences of feminism and culture* (pp. 163-174). London: Falmer Press.

Belenky, M., Clinchy, B. M., Goldberger, N. R., and Tarule, J. M. (1986). *Women's ways of knowing*. New York: Basic Books.

Delpit, L. D. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard Education Review*, 58, 280-298.

Forman, E. (In press). Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform. In L. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Garcia, E. (1991). *Education of linguistically and culturally diverse students: Effective instructional practices* (Educational Practice Report 1). Santa Cruz, CA: University of California, Santa Cruz, National Center for Research on Cultural Diversity and Second Language Learning.

Heath, S. B., & McLaughlin, M. W. (Eds.). (1993). *Identity and inner-city youth: Beyond ethnicity and gender*. New York: Teachers College Press.

Lane, S. (1993). The conceptual framework for the development of a mathematics performance assessment instrument for the QUASAR project. *Educational Measurement: Issues and Practice*, *12*, 16-23.

Lane, S., & Silver, E. A. (1994, April). *Examining students' capacities for mathematical thinking and reasoning in the QUASAR project*. Paper presented at the AERA annual meeting, New Orleans.

Lane, S., & Silver, E. A. (1995). Equity and validity considerations in the design and implementation of a mathematics performance assessment: The experience of the QUASAR project. In M. Nettles & A. L. Nettles (Eds.), *Equity and excellence in educational testing and assessment* (pp. 185-219). Boston: Kluwer.

Lane, S., Silver, E. A., & Wang, N. (1995, April). *An examination of the performance gains of culturally and linguistically diverse students on a mathematics performance assessment within the QUASAR project*. Paper presented at the AERA annual meeting, San Francisco.

Lane, S., Wang, N., & Magone, M. (1995, April). *Gender-related differential item functioning on a middle school mathematics performance assessment*. Paper presented at the AERA annual meeting, San Francisco.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York: Cambridge University Press.

Murrell, P. C., Jr. (1994). In search of responsive teaching for African American males: An investigation of students' experiences of middle school mathematics curriculum. *Journal of Negro Education, 63*(4), 556-569.

Oakes, J. (1990). Opportunities, achievement, and choice: Women and minority students in science and mathematics. In C. B. Cazden (Ed.), *Review of research in education, Volume 16* (pp. 153 - 222). Washington, DC: American Educational Research Association.

Secada, W. G. (1992). Evaluating the mathematics education of LEP students in a time of educational change. In *Proceedings of the National Research Symposium on Limited English Proficient Student Issues: Focus on Evaluation and Measurement, Vol. 2* (pp. 209-256). Washington, DC: Office of Bilingual Education and Minority Languages Affairs, United States. Department of Education.

Silver, E. A., & Lane, S. (1993). Assessment in the context of mathematics instruction reform: The design of assessment in the QUASAR project. In M. Niss (Ed.), *Assessment in mathematics education and its effects* (pp. 59-70). London: Kluwer.

Silver, E. A., Smith, M. S., & Nelson, B. S. (1995). The QUASAR project: Equity concerns meet mathematics education reform in the middle school. In E. Fennema, W. Secada, & L. Byrd Adajian (Eds.), *New directions for equity in mathematics education* (pp. 9-56). New York: Cambridge University Press.

Stein, M. K., Grover, B., & Henningsen M. (In press). Enhanced instruction as a means of building student capacity for mathematical thinking and reasoning. *American Educational Research Journal.*Stein, M. K., Grover, B., & Silver, E. A. (1991). Changing instructional practice: A conceptual framework for capturing the details. In R. G Underhill (Ed.), *Proceedings of the Thirteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA)*, (Vol. I., pp. 36-42). Blacksburg, VA: Virginia Polytechnic Institute and State University.

Stein, M. K., & Lane, S. (1995, April). *Instructional tasks and student learning: An analysis of the relationship between teaching and learning in the QUASAR project.* Paper presented at the AERA annual meeting, San Francisco.

Stodolsky, S. (1988). *The subject matters: Classroom activity in mathematics and social studies.* Chicago: University of Chicago Press.

A RESEARCH BASE SUPPORTING LONG TERM ALGEBRA REFORM?[1]

James J. Kaput, University of Massachusetts - Dartmouth

1. Defining and Situating Algebra Reform[2]

Before discussing any research base supporting algebra reform, we must address some prior questions:

> What kinds of reform, what kinds of algebra, and reform on what time scales?

But even before discussing what kinds of reform and algebra, we should acknowledge why algebra reform is so widely called for. Where are we coming from?

1.1. Recent and Current Practice: The Base-Line

School algebra in the U.S. is institutionalized as two or more highly redundant courses, isolated from other subject matter, introduced abruptly to post-pubescent students, and often repeated at great cost as remedial mathematics at the post secondary level. Their content has evolved historically into the manipulation of strings of alphanumeric characters guided by various syntactical principles and conventions, occasionally interrupted by "applications" in the form of short problems presented in brief chunks of highly stylized text. All these are carefully organized into small categories of very similar activities that are rehearsed by category before introduction of the next category, when the process is repeated. The net effect is a tragic alienation from mathematics for those who survive this filter and an even more tragic loss of life-opportunity for those who don't.

It would be easy to mistake this cryptic description for a deliberately harsh and cartoonish denigration of actual practice, but, unfortunately, it is reasonably accurate for the great majority of students studying algebra in the U.S. today, *especially as experienced by those students*. (Watch them, listen to them, and examine their errors. What is the race or income of those whose lives are most likely to be damaged?) Some of these activities might be described by teachers or other adults as, say, "expression simplifying," "equation solving," "or problem solving." Some others might describe them as "function rewriting," "function comparisons," or "modeling," respectively. Others might describe them as operations in and applications of rational or algebraic functions over the rationals or reals. But most students see little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test. Most arithmetic and calculus are experienced similarly, while teachers burn out by the tens of thousands annually trying to teach the unwilling the unwanted. Well-meaning policy makers are now requiring the algebra medicine for all students, since, at least when viewed from a distance, it seemed to have a salubrious effect on some students. The widely appreciated political rhetoric "We can't afford to waste a single person" is now colliding with a curriculum that, in fact, wastes millions. Algebra has been transformed in the national consciousness from a joke to a catastrophe.

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[1] The preparation of this paper was partially supported by OERI Grant (#R117G10002) funding the National Center for Research in Mathematical Sciences Education. The views expressed, however, are those of the author and may not represent those of the funder. The author wishes to thank Sigrid Wagner, David Slavitt, John Mason and Mary Spence for a critical reading of all or parts of a first draft.

[2] This paper will focus on school algebra in the United States, on the assumption that its larger features are shared by our PME-NA neighbors and that reform efforts in the US are of interest to our immediate neighbors.

1.2 Three Phases of Reform

A potential for confusion exists regarding the kinds of reform possible or desirable over different time scales. What may seem radical as a proposal for immediate implementation appears less so in the context of a longer term picture. Hence we will discuss three overlapping phases of reform, short, intermediate, and long term. Short term, over the next two or three years, involves carrying out ongoing changes in existing curricula - the use of graphing calculators in existing Algebra I and II courses, for example. Intermediate term, covering the period from the late 1990s through the first few years of the next century, involves implementing the larger middle school and other reforms currently nearing completion of their first editions. The third, long term, phase begins during the early years of the next decade and involves deep restructuring of the curriculum that makes room for important new content and flexibility, especially at the secondary level. Indexing phases of reform temporally ignores the fact that change moves unevenly across the land, so that one phase may be well underway in one location while its predecessor is in full swing elsewhere. My comments will focus more on the longer rather than the shorter term - the genuine and significant influence of research on practice is inevitably long term. Short term connections between research and practice are usually closely related to evaluation of one or another innovation or theoretical perspective.

1.3 Three Dimensions of Reform

To clarify the nature of the reforms to be discussed, and implicitly predicted, I offer three dimensions in which to measure change:

(1) *Breadth* - breadth of conceptions of algebra coherently implemented;

(2) *Integration* - curricular integration of algebra with other subject matter; and

(3) *Pedagogy* - movement towards a more active, exploratory pedagogy, particularly exploiting electronic technologies.

The Breadth dimension refers to the many forms of algebra and algebraic reasoning and the ways that they cohere: algebra as generalizing, abstracting and representing; algebra as the syntactically defined manipulation of formal objects; algebra as the study of structures abstracted from computations; algebra as a modeling language or as a cluster of related languages; algebra as the study of functions, relations, and joint variation; algebra as means of controlling physical or cybernetic events, including simulations. These will be elaborated below. The Integration dimension, curricular integration of algebra strands with other subject matter, is meant to include both mathematical and non-mathematical subject matter. Taken together, Breadth and Integration enable a large scale restructuring of the curriculum that removes algebra as a costly pair of high school courses, and when coupled with restructuring of other subject matter into more longitudinally coherent strands, make space in the secondary school curriculum for the new mathematics needed by students of the next century - space also needed for curricular innovation and exploration that is absolutely impossible today.

The Pedagogy dimension has relatively little directly to do with algebra in strictly mathematical terms as a received cultural artifact, but *everything to do with the way that algebra is experienced by students.* Without improvement in this dimension along the lines described in the NCTM *Professional Teaching Standards*, (NCTM, 1991) for example, change in the other dimensions will be meaningless.

"Reform" in the usual modern sense, perhaps deriving from the 19th century notion of "progress," implies improvement relative to some value-norms, and I take the three dimensions to be ordered in some sense appropriate to each: more Breadth and more Integration are presumed to be better, as is a more student-active-reflective Pedagogy. There is no clean separation among the phases of reform to be described, and most reform efforts vary in their progress across dimensions. Furthermore, the dimensions themselves are not entirely independent - increased Breadth serves Integration, and vice-versa, while improved Pedagogy serves both. Lastly, different implementations of the "same" reform can vary, especially in the Pedagogy dimension (Romberg, 1981; Romberg, 1983). Folks seeking non-intersecting categories, orthogonal dimensions and linear orderings will not find them in realistic appraisals of educational change in such a sprawling domain as algebra - at least not in this paper.

2. Three Dimensions of Algebra Reform

2.1. The Breadth of Algebra: Five Aspects of Algebra

Despite the fact that we all use one word "algebra," there is no one algebra, no monolith. Instead, we need to make sense of a richly interwoven tapestry of constructs and processes that both serve and constitute mathematics. The analysis offered here is somewhat finer than that used in the NCTM Algebra Document (in preparation), but, I believe, consistent with it - where the NCTM Document refers to "themes," we refer to "aspects" - although when attending to how they develop in students' minds or appear in curricula, we also refer to them as "strands."

Talk about mathematics often slips between mathematics as implicitly shared cultural artifacts— objects, procedures, relations independent of any individual— as when we talk about learning functions, polynomials, factoring, ring theory, linear algebra, and so on— and mathematics as ways of thinking— generalizing, specializing, abstracting, computing, analogizing, justifying, and so on. To describe algebra requires mixing both types of talk. Finally, characterizing algebraic reasoning in terms of the types of mathematical objects involved is inadequate - students may be working with matrices or with integers mod 7 in clock arithmetic (Picciotto, in preparation), for example, in entirely concrete, arithmetic ways rather than algebraically. On the other hand, they might be reasoning quite abstractly while using specific numbers, perhaps only orally, with no writing (Bastable & Schifter, in preparation).

The first two aspects of algebra embody "kernel" features of algebraic reasoning that infuse all the others, the middle two amount to centrally important mathematical topics, and the last addresses algebra as a web of languages. All the aspects should be regarded as loosely spun and richly interwoven—they are by no means separate. And each has different roots in human cognitive and linguistic powers and draws on different kinds of experience, particularly in its primitive and emergent forms among younger children.

2.1.1. [Kernel] Algebra as Generalizing and Formalizing Patterns & Constraints, especially, but not exclusively Algebra as Generalized Arithmetic Reasoning and Algebra as Generalized Quantitative Reasoning

Generalization and formalization are an intrinsic feature of much mathematical activity, and the mathematical systems and situational contexts in which generalization and formalization can be done are unlimited. I suggest that there are two sources of generalization and formalization: reasoning in mathematics proper, and reasoning in situations based outside mathematics, but subject to mathematization. The particular forms described below, arithmetic and quantitative, differ in exactly this fundamental way: generalizing in arithmetic (numerical patterns, arithmagons, etc.) begins within a mathematical system, (often) the system of integers, their properties and operations, where understanding of the mathematical structures plays the core constraining role; quantitative reasoning is based in mathematizing situations and offers a different basis for generalizing and formalizing, where understanding of the semantics of the situation plays the core constraining role.

Both the means and the goal of generalizing is to establish some formal symbolic objects that are intended to represent what is generalized and render the generalizations subject to further reasoning, perhaps aided by computation - where the computations are at least temporarily guided by syntax and patterns associated with the formal system rather than what is formalized. Acts of generalization and gradual formalization of the constructed generality must precede work with formalisms - otherwise the formalisms have no source in student experience. The current wholesale failure of school algebra has shown the inadequacy of attempts to tie the formalisms to students' experience after they have been introduced. It seems that, "once meaningless, always meaningless." We now turn to the two prime candidate sources for generalization and formalization in school mathematics.

Algebra as Generalized Arithmetic Reasoning. An enduring theme in algebra education, with roots in 18th-19th century views of the subject (Pycior, 1981; Sfard, 1995), regards algebra as a language that encodes the general rules of arithmetic, particularly rules concerning the operations. It has proven itself to be attractive as a factor in curriculum design because it explicitly builds on what students presumably know (arithmetic), helps generalize that knowledge, helps build a more general ability to generalize in the process, and exploits the rich intrinsic structure of the integers as a context for pattern development, formalization and argument - for example, how many reasonable conjectures might one make concerning combinations of consecutive integers? Linchevski (1995) put it thus: "Algebra with Numbers and Arithmetic with Letters: A Definition of Pre-Algebra" (Summary Report to the ICME-7 Working Group on Algebra, 1995). Work along the same lines by Bastable and Schifter (in preparation) offers rich examples of second to fourth grade students generalizing and discussing generalizations of arithmetic relations based in specific cases, where formal representations are not used, but where generality is at the heart of the activity and discussion. This is one set of examples that points the way to building depth in arithmetic, serving the Integration dimension of reform. Other types of activity involving arithmagons and numerical patterns, as examples among many possible, provide contexts for extending this strand of algebra towards simultaneous equations and beyond (Bell, 1995; Romberg & Spence, 1995; van Reeuwijk, in preparation). It forms the major part of some recent attempts to begin the study of algebra in the early middle grades (Curcio, 1994).

Algebra as Generalized Quantitative Reasoning. As defined by Thompson (1993; 1995), Thompson & Smith (in preparation) a person is thinking of a quantity when he/she is thinking of a quality of some aspect of a situation that he/she regards as measurable (or countable)— length, density, mass, age, velocity, numbers of red marbles, area, rate of inflation, and so on. Such conceptual acts may or may not involve the actual assignment of numerical values to the quality involved via the use of some unit of measure or counting. Quantitative reasoning might also involve abstract quantities, such as in determining "how many 3's in 15" (where the quality is simply "size") by, for example, counting how many units of 3 need to added together to yield 15. Thus this aspect of algebra can be thought of as encompassing the Generalized Arithmetic aspect. I argued (Kaput, 1995), and Thompson & Smith (in preparation) argued that quantitative reasoning is superior to arithmetic in opportunities to build algebraic reasoning. It draws more fully on different forms of experience, including growth and change, can be more oriented towards the expression of relationships for purposes of inference rather than merely towards computations of values of quantities, and, unlike arithmetic-based activity, it involves a more direct link to physical and cultural experience. Indeed, a

Subject Matter

As we all know, and for many good reasons both cognitive and practical, the NCTM *Curriculum and Evaluation Standards* (NCTM, 1989) put a premium on "connections." Integration and connections can take place at several different levels:

> Within-mathematics connections between different representations of given mathematical objects such as functions, or between different areas of mathematics, as between algebra and geometry involving, for example, traditional analytic geometry or connections between matrices and transformations of the plane.
>
> Connections between mathematics and subject matter from other mathematical sciences such as computer science, probability, or statistics.
>
> More distant connections usually involve mathematics in modeling situations developed within the structures and from the perspectives of other disciplines, in the physical, life and social sciences, as well as in business, medicine and engineering.

Pedagogical Power. To the extent that algebra can be learned while learning other subject matter, not only is its power appreciated, but its power is learned. And importantly, learning of the other subject matter is enhanced - how much "science" is learned in grades K-8 as vocabulary, or, more recently, as collections of interesting phenomena, without any quantitative content (AAAS, 1994).

Curricular Efficiency. We can no longer afford to teach academic subjects one at a time, end-to-end. We need to exploit the compounding effect of connecting algebra with other subject matter: the algebraic languages reveal the common structures across domains. Building algebra in different domains can reveal the similarities of the underlying ideas while simultaneously strengthening understanding of the structures, exercising the associated procedural skills, and enhancing appreciation of mathematics' power.

Curricular Depth. But perhaps even more importantly, this last observation applies to much of the mathematics that now appears in K-8: *How much could the mathematics of the pre-high school grades be enriched, deepened and made more coherent if, at every turn, questions of generality and extension were raised and pursued* (Bastable & Schifter, in preparation)? To raise such questions inevitably invites algebra as a means for expressing generality and abstraction, and for reasoning within these expressions.

Longitudinal Coherence - From Layercake Filter to Coherent Strands. Algebra is not only a powerful filter of students, but it is also a barrier preventing access to powerful ideas. As now structured, algebra courses lie between elementary mathematics and calculus - the mathematics of change - and all the fields that use calculus. Historically, only a small minority of students cross this barrier, but recent work in the SimCalc Project indicates that the mathematics of change may be an ideal site for the learning of algebra, a notion implicit in the growth and change theme of the NCTM Algebra Document (in preparation). Integration, coupled with Breadth, are critically important dimensions of reform.

2.3. Changes in Pedagogy

The distinction between curriculum and pedagogy is a slippery one, especially when one departs from a description of mathematics as a received cultural artifact as represented in books and other media, and instead discusses mathematics as constructed or experienced by individuals. Nonetheless, for analytic purposes it is useful to distinguish between descriptions of acts of teaching and their surrounding circumstances on one hand, from the shared objects, procedures, relations, forms of reasoning, and notation systems that we expect students to learn on the other. Desirable pedagogies have been set forth in the NCTM *Professional Teaching Standards*, (NCTM, 1991) for example, and, for brevity's sake will not be repeated here except to note that it is possible to achieve surface forms of valued pedagogies while failing entirely to engage students with significant mathematics. We often hear that changes in curriculum without changes in pedagogy are empty changes. But the reverse is at least as true, perhaps because it may be easier, especially at the lower grade levels, where teachers are often more equipped to grow pedagogically than they are to grow mathematically. An implication is that growth in pedagogy and growth in mathematical power need to be intimately linked in the kinds of teacher education that will move practice along the three reform dimensions.

3. Research Supporting Algebra Reform

3.1. Research Associated with the Breadth Dimension: Mapping Algebraic Thinking in Its Full Diversity

Obviously, the aspects, especially when thought of as strands, interweave complexly. Mapping these connections, especially how they grow in students' minds under various instructional approaches, is an important research agenda for long term algebra reform. Acknowledging the real complexity and breadth of algebra in our research and how algebra may emerge in students' own language and action, particularly in diverse forms, is an important step towards research of relevance to long term reform that respects the diversity of both the students who need to learn algebra and the many ways they will use it (Confrey, 1995, Dennis & Confrey , 1995). Steps in this direction are necessarily made by the large curriculum development projects in outlining curricula, and these can serve as starting points, e.g., (Romberg & Spence, 1995).

3.2. Traditional Research Supporting and Informing Current Practice

Research and curriculum are, as parts of a larger integrated social and cultural system, intimately, albeit complexly, connected. And, as noted, the forces now at work pushing reform of algebra emanate at least as strongly from the larger society as they do from education researchers, a fact not uncommon historically (Howson, et al., 1981, chapter 1). To the extent that they share a common vision of school mathematics, curriculum and research each helps define the other. This has been especially true in the case of the deficit model "disaster literature," where student shortfalls in learning, "misconceptions," and so on (e.g., Kaput & Sims-Knight, 1983; Kuchemann, 1981; Kuchemann, 1984; Matz, 1982; Sleeman, Kelly, Martinak, Ward & Moore, 1989) are in large part a measure of the impact of the current or recent curriculum, although this is seldom suggested in the research reports, which seemed to take for granted the basic shape of existing curricula. On the positive side, in studies of what students can learn, researchers' visions of school algebra have extended well beyond what typically appears in current courses. However, some researchers have studied the learning of symbol manipulation (Davis, Jockusch & McKnight, 1978), especially learning within computer environments (Chaiklin, 1989; Feurzeig, 1986; McArthur, Stasz & Zmunidzinas, 1990; Sleeman, 1982; Sleeman, 1984; Sleeman et al., 1989), where the subject matter fits reasonably well with the formal side of today's curriculum, although the organizations offered by researchers tend to be much more principled than those embodied in the textbooks.

Prior research also tended to treat algebra one aspect at a time. A significant amount of earlier research, particularly research emanating from other countries (Soviet Studies in Mathematics Education, 1976), was directed towards algebra as generalization, especially generalized arithmetic (Bell, 1995), or formal argumentation (Davydov, 1975, 1990). Some research viewed algebra as a modeling language (de Lange, 1987). Another line of research has investigated students' development of understanding of concepts of function (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Dreyfus & Eisenberg, 1984; Dubinsky & Harel, 1990; Eisenberg & Dreyfus, 1994; Thompson, 1994) and the different representations of functions (Goldenberg, 1988; Romberg, Carpenter & Fennema, 1993; Yerushalmy, 1991). Again, it is worth emphasizing that this research did not strongly affect practice in the U.S., which has been tightly defined by commercial textbook series for "Algebra I & II" dominated by a few major publishers.

Integration has traditionally taken the form of algebra applications in the form of "word problems" rather than in the larger senses described above. And, since these researchers by and large shared the curricular assumption that ability to use algebra is reflected in ability to solve such problems, much research, far too extensive to be cited here and extending well into the psychological sciences, focused on learning how to solve word problems of various types. This research helps only indirectly in the current reform effort, because the current reform no longer shares this curricular assumption. Research centered on pedagogy is perhaps best exemplified by Rachlin (1981), who shows how far one can move along the pedagogy dimension with the current content.

3.3. The First Phase of Reform: Short Term

First attempts at reform leave the larger course structures in place, but can be characterized as significant enrichments, inevitably using electronic technology, of existing courses. These enrichments give much more prominence to and encompass a wider set of applications; utilize the production, comparison and manipulation of functions in linked numerical, graphical and symbolic forms; and usually engage students in conjecture and exploration using the interactive technology. I would judge these attempts as relatively low in the Integration dimension since they share the feature of reforming algebra where it already appears in the grade 8-12 curriculum, leaving the algebra as isolated from other subject matter except as it may be incorporated into problem-applications. In terms of the Breadth dimension it is a significant move towards inclusion of a functions-view of algebra, forced in part by the input-expectations of the electronic devices used. These same devices support multiple, linked representations of these functions—largely defined symbolically, of course—and hence support within-mathematics progress in the integration dimension . Also, depending on the case at hand, generalization and the expression of generality play an increased role in the Breadth of algebraic experiences offered.

Much of this work is the product of innovation by individual teachers or the use of slightly modified texts or supplementary materials (usually associated with graphing calculators). Obviously, much variation is embedded in this category, especially in the Pedagogical dimension. Nonetheless, especially as the technology supports exploration and active learning, significant movement along the Pedagogical dimension tends to occur. However, movement in all these dimensions is limited by the presence of the traditional constraints of the courses in which the innovation is taking place.

3.4. Research Supporting and Informing the First Phase of Reform

A very revealing dissertation study of a short term Algebra II reform effort led by an individual teacher at a progressive private school has been provided by Slavit (1994). The teacher was extremely competent by all standard measures, the students were committed to learning, and the classroom circumstances were near-optimal for use of graphing calculators. We would rate him "high" on the Pedagogy dimension (he was a Presidential Award winner). Many teachers and mathematics educators would envy this teacher's situation and applaud his and his students' achievements, which were considerable. However, his students were afflicted with most of the limitations of concept image of function reported by Vinner (1983; Vinner & Dreyfus, 1989), particularly as revealed by problems involving functions that were not described in algebraically closed form. What of typical students and teachers working under sub-optimal conditions? While the teacher's efforts and achievements were impressive, certain key elements of the curriculum remained unchanged; for example, functions were almost always described in algebraically closed-form (except on a revealing assessment), the course was sandwiched in a rather traditional sequence, and the problems and activities were usually textbook-brief (with a few exceptions) and made relatively little use of real data (physical or otherwise), not unlike findings from another pair of dissertation studies (Rich, 1990; Teles, 1989) and well known work by Heid (Heid & Kunkle, 1988) and others. It is important, both for fairness and for our analysis, to note that where algebra started: a review of the historical "algebra" problems dating back to Arabic algebra reveals them to be quantitative reasoning problems, not arithmetic problems (Katz, 1995). Nonetheless, and despite their concreteness, they served as bases on which general, more algebraic solutions could be (and were) built.

In thinking of algebra both as generalized arithmetic and as generalized quantitative reasoning, it is important to keep in mind that the generalizing does not start with elementary school mathematics and end there, leading to algebra. Generalizing is a continuing activity that can occur at the most sophisticated levels of mathematics (e.g., in algebraic number theory or advanced mathematical modeling) where the qualities being defined and measured might be subtle economic constructs, such as elasticity of demand or the impact of the Fed's interest rate on fluidity of capital.

2.1.2. [Kernel] Algebra as Syntactically-Guided Manipulation of (Opaque) Formalisms

The tremendous power of formalisms is behind the prodigious development of modern science and technology (Bochner, 1966). For example, when one computes the derivative of $(3x^2+2)^{1/2}$ using the Chain Rule, one is exploiting the formalisms developed by Leibniz (Edwards, 1979). Indeed, the word "calculus" refers precisely to this feature - applying rules to calculate with symbols without regard to what they might refer to. When dealing with formalisms, whether they be traditional algebraic ones or more exotic ones, the attention is on the symbols and syntactical rules for "manipulating" them (changing their form). However, it is possible to act on formalisms semantically, where one's actions are guided by what you believe the symbols to stand for. To clarify, consider two ways of solving the equation $3x-2=10$: One way is semantically guided— in this case by reasoning within the numerical conceptual system represented by the formal equation. It is usually approached as an inverting process. One thinks something like "If I take 2 away from 3 times a number, I get 10. So 3 times the number must be 12, so the number must be 4." The syntactically guided approach treats the symbols as objective entities in themselves, and the conceptual system of rules applies to the system of symbols, not what they might stand for. In this case, one applies a rule for adding 2 to both sides of the equation, to get $3x=12$, and then one divides both sides by 3 to get $x=4$. And often these rules come to be thought of as applying to the symbols as physical objects "move the -2 to the right hand side and change its sign."

As noted, much of the traditional power of algebra arises from the internally consistent, referent-free operations that it affords. For an historical discussion of the loosening of referential constraints, see (Kaput, 1994, pp. 101-103). Many (e.g., Cuoco, in preparation) take syntactically guided computations on formalisms to be the essence of algebra. However, as already noted, neither the formalisms nor the actions on them can be viably learned without some semantic starting point where the formalisms are initially taken to represent something in the student's

2.1.3. [Mathematical Topic] Algebra as the Study of Structures Abstracted from Computations and Relations

Acts of generalization and abstraction give rise to formalisms that support syntactic computations that, in turn, can be examined for structures of their own, usually based in their concrete origins. This aspect has some roots in the 19th century British idea of algebra as universalized arithmetic (Kline, 1972) but also can draw on structures arising elsewhere in students' mathematical experience—for example, in matrix representations of motions of the plane, in symmetries of geometric figures, and in manipulations of letters in words. These structures seem to have three purposes, (1) to enrich understanding of the systems that they are abstracted from, (2) to provide intrinsically useful structures for computations freed of the particulars that they once were tied to, and (3) to provide the base for yet higher levels of abstraction and formalization. While this aspect in the past has been reserved for elite students at the college level, some now call for earlier introduction for the majority of students (Cuoco, in preparation; Picciotto, in preparation; Picciotto & Wah, 1993, March).

2.1.4. [Mathematical Topic] Algebra as the Study of Functions, Relations and Joint Variation

Fey (1984) recalls the long history of attempts to use the idea of function as an organizing principle for the mathematics curriculum, including and especially algebra. Schwartz (Schwartz & Yersulshamy, 1990) and Yerushalmy & Schwartz, (1993) have offered an analysis of how studying the idea of function and its several standard representations can simplify and organize the confusing algebra curriculum confronted by today's students and teachers, while Dubinsky and colleagues (Breidenbach, et al., 1993) and Thompson have analyzed its conceptual growth in individuals. As a product of generalization, the idea of function has roots in causality, and joint variation (Freudenthal, 1982; van Reeuwijk, in preparation) and hence permeates the sciences. Examples of young students developing this idea have been offered by Tierney & Monk (in preparation), and middle school curriculum materials embodying this point of view have been produced by Connected Math Project, TIMS. On the other hand, functions used in the context of less temporally mediated phenomena, such as occurring in arguments involving divisibility of products of consecutive even integers (where the underlying variable works to carry generality more than it works to carry covariation), the idea of covariation may be less salient, and attention focuses on the generality of the patterns being expressed. When coupled with the ideas of iteration and recursion in computational media, functions feed into the idea of dynamical system (Devaney, 1989; Sandefur, 1990). This strand grows out of and intertwines with the General-and Quantitative Reasoning strand.

[illegible] Phenomena-Controlling Languages

Modeling Languages. Some would argue (e.g., Freudenthal, 1983) that modeling is the primary reason for studying algebra. The generalized quantitative reasoning aspect can be regarded as part of a larger modeling aspect that extends to include the rapidly widening collection of notation systems that are used to represent and visualize phenomena of all sorts. These support the new forms of visualization and reasoning associated with dynamical systems, deterministic chaos, and generally the modeling of nonlinear phenomena. Of special interest is how "algebraic" are the various notation systems? One way to approach this question is to ask how do they express generality, and how do they support syntactically guided manipulation? Some are pictoric, some are coordinate-based, while others are character-based. The computer medium now supports operations on virtually any notation system; for example, one can systematically adjust the color scales of a color-coded temperature map to help reveal patterns, or one can overlay such [illegible] map with a topographic map, etc. Is this modeling in the classic sense that [illegible] ping a differential equation for describing the motion of a falling body [illegible] ? Of course, many, perhaps most models have functions at [illegible] ticular and cognitive strand it weaves intimately with th[illegible]

[illegible] Languages that Create and Control Physical and [illegible] [illegible] a. In modeling, we begin with phenomena and attempt to [illegible] But computers now enable us to reverse this referential relationship in [illegible] ways - by creating simulation phenomena within the computer medium ([illegible] 1994) and by driving physical devices (Nemirovsky, 1994). In these cases, one usually cycles repeatedly between the phenomena, wherever they happen to be located, and the notations that give rise to them. In recent work (Kaput, in preparation) we are also able to import phenomena into the computer via standard MBL systems and compare them with algebraically generated phenomena. For example, one can "walk" a certain velocity graph that controls the motion of a character in a simulation, and then create algebraic functions that control another character whose motion can be compared with your motion as they "walk" side by side. In these sorts of environments new relationships between algebra and physical phenomena are possible. Lastly, computer languages, beginning with FORTRAN, then BASIC and more recently Logo (Grant, Faflick & Feurzeig, 1971; Noss, Hoyles & Sutherland, 1993; Papert, 1980) and ISETL (Dubinsky, 1991; Dubinsky & Leron, 1994) amount to algebraic formalisms within which one can create or experience explorable and extensible mathematical environments. Nor do these languages need to be alphanumeric, e.g., Function Machines (Feurzeig, 1993). As has been noted (Kaput, 1986; Noss et al., 1993), these computer environments change in fundamental ways the relations between the particular and the general, and hence the nature of mathematical experience available to students, including and especially means of argument and justification.

none of these factors was within the teacher's (or researchers') direct control. They await the next phase of reform, and, in fact, define the boundary between Phases 1 and 2.

3.5. The Second Phase of Reform: Intermediate Term

The second phase of algebra reform centers on the integration of algebra into the middle school very much in the spirit of the first level of reform, but with two important differences: (1) the algebra is integrated into a larger curriculum, and (2) as middle school mathematics, it is intended (by its authors) to be offered to *all* students. Again, considerable variation exists in this category, particularly in the role and types of applications. Generally, however, the curriculum and the activities tend to be structured in larger pieces than Phase 1, and the algebra tends to emerge from the activity and contexts in which students work. Furthermore, materials are usually structured according to topic strands, with algebra used to express generalizations and abstractions within these strand topics (Romberg & Spence, 1995). Thus considerable movement along the Integration dimension is achieved in Phase 2.

Algebra as a means of modeling and generalization is increased, the place of functions and their multiple representations is preserved—if not increased to include non-traditional diagrammatic and pictorial notations (Romberg et. al, 1995)—and some of the materials broaden the subject to include some formal, structural aspects of algebra as arise in the contexts of matrices and clock arithmetic. Hence further movement along the Breadth dimension is achieved.

In the Pedagogical dimension, even more movement occurs, since much material is open-ended by design, involves students working in groups, and in some cases involves students designing and producing artifacts (Goldman, 1994). The level of pedagogical change has, in some reports, reached the limits of traditionally educated teachers' ability to adapt.

Most of this work is connected to ongoing curriculum development projects that will not be widely available until 1996 or 1997, with the exception of UCSMP, whose newer editions began to appear in the mid 1990s, and which is distinguished by its K-12 comprehensiveness. Phase 2 seems likely to dominate the end of this decade and the early part of the next. Because of the shift of the focus of these innovations to middle school, many of the constraints of existing secondary school structures are loosened. However, the resulting changes at the secondary school are unclear, except that much of the Phase 1 activity will be inappropriate for those students who will have progressed through Phase 2 materials in middle school. Hence Phase 2 reform is more clearly defined at the middle school level than it is at the secondary school level, a fact that is likely to yield considerable difficulty in transition between Phase 1 and Phase 2.

3.6. Research Supporting and Informing the Second Phase of Reform

Most of the research about Phase 2 has taken the style of research-based formative evaluation of curriculum materials and the school-based implementation process because the innovators are either researchers themselves, or are affiliated with researchers.

3.7. The Third Phase of Reform - Long Term

This phase of reform has not yet begun in the U.S. (to my knowledge), although, as argued below, the ingredients needed to begin are available. It involves full integration of the development of the many forms of algebraic reasoning across all grades with the learning of important mathematics. In this phase algebra is treated less as a subject in its own right (with exceptions noted shortly), and more as a general, ubiquitous means for creating, expressing and operating on generalizations and abstractions, as a medium for modeling, and as a set of computer based languages to create as well as model phenomena. It serves a wide variety of purposes, making sense of the quantifiable and structural aspects of experience in the context of modeling and in other mathematics. It is also a medium for creating new mathematics and reorganizing old mathematics (including concepts of number and operations on numbers). Algebraic reasoning, and the various notational systems, conventional and otherwise, grow organically and gradually, developing as they are needed, with technology likewise introduced gradually as needed. At certain junctures, however, consolidation and some practice are required, perhaps as long as a few months, but not a full course. The exception could be mathematical electives at the secondary level, where particular aspects of algebra may be explored more fully, (e.g., linear algebra, or algebraic structures) (Cuoco, et al., 1995). Computer technology supports just-in-time learning that enables students to learn specific skills when they are needed. In this phase of reform, algebra enhances and provides coherence to the learning of other subject matter strands—the mathematics of number and quantity, of space and dimension, of data and uncertainty, of growth and change (including growth and change in other sciences such as physics and biology), of data structures, and so on. Algebra disappears both as a set of isolated courses and as a set of intellectual tools, in the sense that for the carpenter, when in use the hammer becomes an extension of the arm (Polanyi, 1958). The different aspects of algebra become habits of mind, ways of seeing and acting mathematically—in particular, ways of generalizing, abstracting and formalizing across the mathematics and science curricula, including curricula leading to the world of work. The new freedom from the constraints of the historic high school mathematics curriculum is exploited to include mathematical electives such as dynamical systems and nonlinear modeling (Sandefur, 1992), combinatorics, number theory, non-Euclidean geometry, and so on, studies not currently present in school curricula. A market for innovation is incentivised and mediated by telecommunication technologies that enable individuals to offer instructional materials to geographically dispersed students on a royalty basis.

Relative to content, this under specified and utopian-appearing scenario is not too far from the approach to algebra taken in certain other countries, (e.g., the Netherlands, Russia, and elsewhere). However, I would suggest that the particulars in the U.S. may very well be substantially different from those that have evolved in other countries, especially given that computer technologies are a powerful ingredient operating in Phase 3 but not strongly present today. Glimpses of details are provided in the new NCTM Algebra Document for Algebra in the K-12 Curriculum (in preparation) where algebra is depicted as a K-12 enterprise touching all aspects of mathematics. While provision is made for practice and consolidation, the implicit pedagogy is student-centered, with active exploration, conjecture, verification and student authorship of mathematics and models emphasized throughout.

3.8. A First Pass at Organizing Research Supporting and Informing the Third Phase of Reform

The research basis of this approach certainly does not exist today, although the issue has been discussed as early as the 1930s (Slàvit, 1994) and thirty years later in the mid 1960s (Davis, 1964 ; 1984). Below I will attempt to point to research that seems to offer promising starting points. This research largely involves younger children since I believe that the early grades will initially and necessarily be the locus of greatest change in algebra instruction, leading to even larger changes at the secondary level later. Secondly, we need to revisit and extend research in the learning of specific subject matter, especially at the foundational levels, in order to find where and how opportunities to generalize and abstract can be exploited, that is, opportunities to learn and use algebra. Thirdly, we also need to look closely at research and development work in other countries where algebra learning has been integrated with other learning, and where the approaches seem to be in line with what seem appropriate for students of this country.

3.8.1. Beginning the Strands in Elementary Mathematics

Early work has taken the form of documenting opportunities for generalizing and formalizing in arithmetic (Bastable & Schifter, in preparation), and in quantitative reasoning (Confrey, 1994; Confrey & Smith, 1995; Thompson, 1994; Thompson, 1995; Tierney & Monk, in preparation). Additional work, based on new curricula, has shown children capable of handling formal symbolism (Romberg, et al., in press), building abstract formal structures in geometry (Lehrer & Dannecker, in preparation), and handling complex interpretation of graphs (Russell, et al., 1995; Ainley, 1995). An important feature of much early work is the subtle and oral rather than written character of children's early attempts to generalize. Since they have not developed symbolism to represent their generalizations, they must use natural language and the many oral strategies for expressing generality developed in daily communication (Mason, in preparation). Hence those who would study these activities as opportunities for the development of algebraic reasoning need a sensitive eye and ear. And furthermore, teachers who would nurture the development of algebra as a means to express generality likewise would need to be sensitized to create as well as identify such opportunities. Fortunately, foundations for such work already exist in the research of those who have studied the development of arithmetic reasoning (e.g., Carpenter, Fennema, & Peterson, 1987;

Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Carpenter & Peterson, 1989; Fuson, 1990; Sandefur, 1990; Steffe & Cobb, 1988) as well as in the study of the associated teacher development (Cobb, Wood, & Yackel, 1990; Cobb, Wood & Yackel, in press; Fennema et al., 1989; Gravemeijer, 1992; Schifter, 1994).

An important alternative to the oral expression of generality and an accompanying move to formal expression occurs in computer environments, especially in situations such as Logo programming (Harel, 1991; Lehrer, et al., submitted; Noss, in preparation; Noss et al., 1993), where the formal expression is intrinsic to the production of a dually-layered visible artifact—the Logo program and the outputs of that program. Another context involves the control of simulations, where students need to set algebraic parameters as part of the process of exploring the phenomena of the simulation (Kaput, in preparation). For example, when controlling the motion of synchronized swimmers in a pool, the students must determine how to distinguish between a positional and a temporal head start; furthermore, in some circumstances they must deal with as many as 20 coordinated swimmers, each of whom is to be offset in their initial position by a fixed distance from the swimmer to their left, say. In this case, to achieve efficient and systematic control of the swimmers begs parametric thinking, where each swimmer's motion is a particular function of time, but where the functions themselves vary systematically across the swimmers. We are currently developing simulation environments to scaffold this kind of thinking among 5th-7th graders.

3.8.2. Approaches in "Algebraically Successful" Countries

Perhaps the best, and surely the most available example, of a curriculum that approximates the vision sketched above is that developed by the Freudenthal Institute in the Netherlands. This curriculum contains no algebra courses, but is rich in algebra experiences beginning in the early grades. A distinguishing feature is the repeated application of the principle of "progressive formalization," whereby students' productions are gradually shaped into more formal systems over time, all in the context of realistic applications.

Another example of active early development of student algebraic reasoning and argumentation is offered in the work of the Russian mathematics educator Davydov (1990). A comparison study of the rather dramatic impact of Davydov's approach has been made by Morris (1995).

4. How Can Research Lead Practice in New Directions?

4.1. General Strategies: Embed Knowledge in Shared Artifacts

One way the insights of disciplined inquiry find their way into practice is by being embedded within artifacts—curricula, tools, and explicit pedagogies associated with these—just as medical research leads to drugs, apparatus, and therapies. The process of reification of knowledge in widely usable tools and representations is a primary means for the distribution of that knowledge (Latour & Woolgar, 1986; Pea, 1993). This is exactly the approach taken by the Dutch (Gravemeijer, 1992). It seems likely to me that such systemic approaches are likely to have the greatest long term impact, partly due to the changing economics of R&D work (Lesh & Lovitts, 1994), and partly due to the dramatic increases in connectivity afforded by electronic networks that will allow distributed collaborative efforts involving many researchers working together at a distance (Hunter, 1993, Fall; Hunter & Goldberg, 1994). Another traditional way not to be ignored is through policies and vision statements such as the various NCTM standards statements, especially the NCTM Algebra Document (in preparation), and MSEB vision statements.

4.2. Changes in Perspectives on What Constitutes Algebra Research: Switching the Duck for the Rabbit

The foreground/background switch that I have advocated for algebra's place in the school mathematics curriculum needs to be matched with a corresponding switch in the way we approach research in the development of algebraic reasoning. Much of the research will need to be based in the learning of the subject matter that gives rise to the use of algebra. Not only does this imply that we need to study the processes of generalizing and notating that generality in basic arithmetic and quantitative reasoning, but also in the context of other major subject matter strands (the mathematics of space, change, data, and so on). For example, in my own work, we are examining how the mathematics of change, including the basic ideas of calculus, can be the site for algebra learning in the latter elementary and early middle school. This changes deeply the traditional prerequisite relationship between algebra and calculus, and moves it closer to the historical relationship, wherein they co-evolved (Kaput, 1994).

4.3. Methodological Changes: Testbeds for Longitudinal Study

I have already noted the need for larger studies coordinated with materials development and teacher development. In addition, we need long-term studies extending over four or more years, following both a cohort of students and a group of teachers as they evolve under circumstances that differ in major ways from today's practice. This calls for a testbed approach, wherein one or more sites participate in material development, evaluation and research for an extended period in an ecologically authentic context that involves practitioners throughout (Hawkins, 1994). The process of dissemination may also take the form of such a site becoming a specially supported resource on the World Wide Web that can act to support teachers and graduate students at other sites, perhaps sharing clinical data. Just as abrupt, late, isolated algebra may be a curricular strategy that deserves to be abandoned, the same might be said of brief, narrow, and isolated laboratory algebra research, especially as a primary strategy.

References

AAAS. (1994). *Benchmarks for Science Literacy*. Washington, DC: American Association for Advancement of Science.

Ainley, J. (1995) Re-viewing graphing: Traditional and intuitive approaches. *For the Learning of Mathematics, 15*(2) June, 10-16.

Bastable, V., & Schifter, D. (in preparation). Classroom stories: Examples of elementary students engaged in early algebra. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Bell, A. (1995). Purpose in school algebra. *Journal of Mathematical Behavior, 14*, 41-73.

Bochner, S. (1966). *The role of mathematics in the rise of science*. Princeton, NJ: Princeton University Press.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247-285.

Carpenter, T. P., Fennema, E., & Peterson, P. L. (1987). Cognitively guided instruction: The application of cognitive and instructional science to mathematics curriculum development. In I. Wirszup & R. Streit (Eds.), *Developments in school mathematics education around the world*, (pp. 397-417). Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-532.

Chaiklin, S. (1989). Cognitive studies of algebra problem solving and learning. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, (pp. 93-114). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Wood, T., & Yackel, E. (1990). Classroom as learning environments for teachers and researchers. In R. Davis, C. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics*. Journal for Research in Mathematics Education Monograph No. 4, (pp. 125-146). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Wood, T., & Yackel, E. (in press). Discourse, mathematical thinking, and classroom practice. In N. Minick, E. Forman, & A. Stone (Eds.), *Education and mind: Institutional, social, and developmental processes*. New York: Oxford University Press.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning*, (pp. 293-332). Albany, NY: SUNY.

Confrey, J., & Smith, E. (1995). Splitting, covariation and their role in the development of exponential functions. *Journal for Research in Mathematics Education*.

Confrey, J. (1995). Student voice in examining splitting as an approach to ratio, proportions, and fractions. In L. Miera, & D. Carraher (Eds.), *Proceedings of the 19th Annual Conference for the Psychology of Mathematics Education*, Vol. 1, Recife, Brazil, 3-29.

Cuoco, A. (in preparation). Early algebra and the structure of calculations. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Cuoco, A., Goldenberg, E. P., & Mark, J. (1995). Habits of Mond: An organizing principle for mathematics curriculum. *The Journal of Mathematical Behavior.*

Curcio, F. (Ed.). (1994). *Curriculum and evaluation standards for school mathematics: Addenda series, grades 5-8: Patterns and functions.* Reston, VA: National Council of Teachers of Mathematics.

Davis, R. (1964). *Discovery in mathematics: A text for teachers.* Menlo Park, CA: Addison-Wesley.

Davis, R. B., Jockusch, E., McKnight, C. (1978). Cognitive processes in learning algebra. *Journal of Children's Mathematical Behavior, 2*(1), 10-320.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* Norwood, NJ: Ablex Publishing Corporation.

Davydov, V. (1975). The psychological characteristics of the "prenumerical" period of mathematics instruction. In L. P. Steffe (Ed.), *Soviet studies in the psychology of learning and teaching mathematics,* (Vol. 7, pp. 109-206). Chicago, IL: The University of Chicago Press.

Davydov, V. (1990). *Types of generalization in instruction.* Reston, VA: National Council of Teachers of Mathematics.

de Lange, J. (1987). Mathematics, insight and meaning: Teaching, learning and testing of mathematics for the life and social sciences. Unpublished dissertation, Rijksuniversiteit Utrecht, Utrecht, The Netherlands.

Dennis, D., & Confrey, J. (1995) Functions of a curve: Leibniz original notion of functions and its meaning for the parabola. *The College Mathematics Journal, 26*(3) May, 124-131.

Devaney, R. (1989). *An introduction to chaotic dynamical systems.* (Second Ed.). Redwood City, CA: Addison-Wesley.

Dreyfus, T., & Eisenberg, T. (1984). Intuitions on functions. *Journal of Experimental Education, 52,* 77-85.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking,* (pp. 95-123). Dordrecht, The Netherlands: Reidel.

Dubinsky, E., & Harel, G. (1990). *The nature of the process conception of function.* Purdue University, West Lafayette, IN.

Dubinsky, E., & Leron, U. (1994). *Learning abstract algebra with ISETL.* New York, NY: Springer-Verlag.

Edwards, C. (1979). *The historical development of the calculus.* New York, NY: Springer-Verlag.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformation. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education. I,* (Vol. 4, pp. 45-68). Providence, RI: American Mathematical Society.

cognitively guided instruction: A new paradigm for curriculum development. In K. Clements & N. F. Ellerton (Eds.), *School mathematics: The challenge to change*, (pp. 174-187). Geelong, Victoria, Australia: Deakin University Press.

Feurzeig, W. (1986). Algebra slaves and agents in a Logo-base mathematics curriculum. *Instructional Science, 14*, 229-254.

Fey, J. (1984). *Computing and mathematics: The impact on secondary school curricula.* Reston, VA: NCTM.

Freudenthal, H. (1982). Variables and functions. In G. v. Barneveld & H. Krabbendam (Eds.), *Proceedings of Conference on Functions*, (pp. 7-20). Enschede, The Netherlands: National Institute for Curriculum Development.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht, The Netherlands: D. Reidel.

Fuson, K. (1990). Conceptual structures for multi-unit numbers: Implications for learning and teaching multi-digit addition, subtraction, and place value. *Cognition and Instruction, 7*, 343-403.

Goldenberg, E. P. (1988). Mathematics, metaphors and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behavior, 7*(2), 135-173.

Goldman, S. (1994). *Middle school mathematics through applications project (MMAP): Second year progress report to the national science foundation.* Palo Alto, CA: Institute for Research on Learning.

Grant, R., Faflick, P., & Feurzeig, W. (1971). *Programming languages as a conceptual framework for teaching mathematics. Volume 2: Logo teaching sequences on numbers, functions and equations* (Report No. 2165). Cambridge, MA: Bolt, Beranek and Newman.

Gravemeijer, K. (1992). Educational development and developmental research in mathematics education. Unpublished paper, State University of Utrecht, Freudenthal Institute, Utrecht.

Harel, I. (1991). *Children designers. Interdisciplinary constructions for learning and knowing mathematics in a computer-rich school.* Norwood, NJ: Ablex.

Hawkins, J. (1994). Cognitive Science Conference Report. Available from author, Educational Technology Center.

Heid, M. K., & Kunkle, D. (1988). Computer generated tables: Tools for concept development in elementary algebra. In A. Coxford (Ed.), *The ideas of algebra, K-12*, (pp. 170-177). Reston, VA: National Council of Teachers of Mathematics.

Howson, G., Keitel, C., & Kilpatrick, J. (1981). *Curriculum development in mathematics.* Cambridge, England: Cambridge University Press.

Hunter, B. (1993, Fall). Collaborative inquiry in networked communities. *Hands On (TERC Newsletter), 16*(2), 1.

Hunter, B., & Goldberg, B. (1994). *Learning and teaching in 2004: The big dig.* Paper prepared for the Office of Technology Assessment. Available from the first author at Bolt, Beranek and Newman, Inc. Educational Technologies Dept., Cambridge, MA.

representational windows. *Journal of Mathematical Behavior, 5*, 187-207.

Kaput, J. (1994). Democratizing access to calculus: New routes using old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving*, (pp. 77-156). Hillsdale, NJ: Erlbaum.

Kaput, J. (1995). Long-term algebra reform: Democratizing access to big ideas. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initative Colloquium*, (Vol. 1, pp. 33-49). Washington, DC: U.S. Department of Education.

Kaput, J. (in preparation). *Year 2 SimCalc NSF Annual Report*. Available from author.

Kaput, J., & Sims-Knight, J. (1983). Errors in translations to algebraic equations: Roots and implications. In M. Behr & G. Bright (Eds.), *Mathematics learning problems of the post secondary student*, Special Issue of *Focus on Learning Problems in Mathematics*.

Katz, V. (1995). The development of algebra and algebra education. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initiative Colloquium*, (Vol. 1,). Washington, DC: U.S. Department of Education.

Kline, M. (1972). *Mathematical thought from ancient to modern times*. New York, NY: Oxford University Press.

Kuchemann, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics concepts: 11-16P*, (pp. 103-119). Oxford, England: Alden Press.

Kuchemann, D. (1984). Stages in understanding algebra. *Journal of Structural Learning, 8*, 113-124.

Latour, B., & Woolgar, S. (1986) *Laboratory life: The construction of scientific facts*. Princeton, NJ: Princeton University Press.

Lehrer, R., & Danneker, D. (in preparation). Springboards to algebra. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Lehrer, R., Horvath, J., & Schauble, L. (manuscript submitted for publication). Developing model-based reasoning.

Lesh, R., & Lovitts, B. (1994). *Priorities for Research*. National Science Foundation Program for Research in Teaching and Learning, Washington, DC.

Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behavior, 14*, 113-120.

Mason, J. (in preparation). Invoking children's mathematical powers of mathematical thinking. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. H. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems*, (pp. 25-50). New York: Academic Press.

McArthur, D., Stasz, C., & Zmunidzinas, M. (1990). Tutoring techniques in algebra. *Cognition and Instruction, 7*, 197-244.

Morris, A. K. (1995). Development of algebraic reasoning in children and adolescents: Cultural, curricular, and age-related effects. Unpublished doctoral dissertation, The Ohio State University.

[illegible] *standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (in preparation). *Algebra in the K-12 Curriculum* (Discussion Draft). Reston, VA: Author.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and velocity sign. *The Journal of Mathematical Behavior,13,* 389-422.

Noss, R. (in preparation). Programming and mathematics: Equation or inequity? In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Noss, R., Hoyles, C., & Sutherland, R. (1993). *Teachers' characteristics and attitudes as mediating variables in computer-based mathematics learning.* Paper presented at the Proceedings of the Fourteenth PME Conference, Mexico City.

Papert, S. (1980). *Mindstorms.* New York, NY: Basic Books.

Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Salamon (Ed.), *Distributed cognition*, (pp. 47-87). New York: Cambridge University Press.

Picciotto, H. (in preparation). A proposal for new directions in early mathematics: Operation sense, tool-based pedagogy, curricular breath. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Picciotto, H., & Wah, A. (1993, March). New algebra: Tools, themes, concepts. *Journal of Mathematical Behavior, 12*(1).

Polanyi, M. (1958). *Personal knowledge.* Chicago, IL: Chicago University Press.

Pycior, H. M. (1981). George Peacock and the British Origins of Symbolic Algebra. *Historia Mathematica, 8,* 23-45.

Rachlin, S. (1981). Processes used by college students in understanding basic algebra. Doctoral Dissertation, University of Georgia, Athens, GA.

Rich, B. (1990). The effect of the use of graphing calculators on the learning of function concepts in precalculus mathematics. *Dissertation Abstracts International, 52,* 835A.

Romberg, T. (1981). The field-based basis of quantitative procedures for the study of schooling. In B. Tabachnick & T. Popkewitz (Eds.), *The study of schooling: Field-based methodologies in educational research and evaluation.* New York: Praeger.

Romberg, T. (1983). A common curriculum for mathematics. In G. I. Fenstermacher & J. I. Goodlad (Eds.), *Individual differences and the common curriculum,* (pp. 121-159). Chicago, IL: NSSE.

Romberg, T., et al. (in press). *Mathematics in context: A connected curriculum for grades 5-8.* Chicago, IL: Encyclopedia Britannica Educational Corporation.

Romberg, T., Carpenter, T., & Fennema, E. (Eds.). (1993). *Integrating research on the graphical representation of functions.* Hillsdale, NJ: Lawrence Erlbaum Associates.

[illegible] on algebra for the evolving work force. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initiative Colloquium*, (Vol. 2, pp. 177-192). Washington, DC: U.S. Department of Education.

Russell, S., et al. (1995). Investigations 3rd/4th grade interpreting graph units. Palo Alto, CA: Dale Seymour Publications.

Sandefur, J. (1990). *Discrete dynamical systems: Theory and applications*. Oxford: Clarendon Press.

Sandefur, J. (1992). Technology, linear equations, and buying a car. *Math Teacher article*, *85*(7), 562-567.

Schifter, D. (1994). Voicing the new pedagogy: Teachers write about learning and teaching mathematics, *Center for the Development of Teaching Paper Series*. Newton, MA: Education Development Center.

Schwartz, J., & Yerushalmy, M. (1990). *The function supposer*. Pleasantville, NY: Sunburst Communications.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14*, 15-39.

Slavit, D. (1994). Algebra instruction using the graphing calculator and its effect on students' conceptions of functions. Unpublished doctoral dissertation, University of Delaware.

Sleeman, D. (1982). Assessing aspects of competence in basic algebra. In D. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems*, (pp. 185-200). New York: Academic Press.

Sleeman, D. (1984). An attempt to understand students' understanding of basic algebra. *Cognitive Science, 8*, 387-412.

Sleeman, D., Kelly, A. E., Martinak, R., Ward, R., & Moore, J. (1989). Studies of diagnosis and remediation with high school algebra students. *Cognitive Science, 13*(4), 551-568.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Soviet Studies in Mathematics Education, Vol. 6. (1976). Reston, VA: NCTM.

Summary Report to the ICME 7 Working Group. (1995). *Journal of Mathematical Behavior, 14*.

Teles, E. J. (1989). Numerical and graphical presentation of functions in precalculus. *Dissertation Abstracts International, 51*, 777A.

Thompson, P. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, J. Kaput, & A. Schoenfeld (Eds.), *Research in collegiate mathematics education*, (Vol. 1,). Providence, RI: American Mathematical Society.

Thompson, P., & Smith, J. (in preparation). Quantitative reasoning as a foundation for the development of algebraic reasoning. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208.

BEST COPY AVAILABLE

Thompson, P. W. (1995). Quantitative reasoning, complexity and additive structures. *Educational Studies in Mathematics*.

Tierney, C., & Monk, S. (in preparation). Children's reasoning about change over time. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

van Reeuwijk, M. (in preparation). Algebra & realistic mathematics. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14*(3), 293-305.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(5), 356-366.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning, 7*, 42-57.

Yerushalmy, M., & Schwartz, E. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating Research on the Graphical Representation of Functions*, (pp. 41-68). Hillsdale, NJ: Lawrence Erlbaum Associates.

ALGEBRA REFORM, RESEARCH, AND THE CLASSROOM: A REACTION TO A RESEARCH BASE SUPPORTING LONG TERM ALGEBRA REFORM

Gail Burrill, University of Wisconsin-Madison

The Aspects of Algebra

The five aspects of algebra, as generalizing, abstracting, and representing; manipulation of formal objects; as a study of structures; as a language; as a study of functions, relations and joint variation; as a means of controlling physical or cybernetic events, described by Kaput in his paper seem to be consistent with the organizing themes postulated in the draft version of the Algebra Document from the National Council of Teachers of Mathematics (NCTM, in preparation). Both documents begin from the premise that algebra has different faces and is interpreted very differently by different people. The NCTM (in preparation) Algebra Document uses as exemplars four organizing themes: function, modeling, language or representation, and structure. The themes in the NCTM document, which can be almost directly mapped to Kaput's aspects, are not meant to be inclusive but to suggest that there are a variety of ways in which people construct a view of algebra. An important point in the NCTM (in preparation) Algebra Document is that no one theme in itself seems to be sufficient to give students a complete picture of what it means to know and be able to do algebra. A collection of themes provides a teacher with multiple entry points into the ways children think about algebra. It is interesting to note that from the perspective of modeling, the two forms of a linear equation, $y = a + bx$ and $ax + by = c$, seem to students to be unrelated because the physical situations that generate each model are too different. From an organizing theme of structure, however, it is easy to demonstrate that the two are equivalent equations. Whether you think of themes or aspects, the question becomes, How can a curricular sequence be designed to incorporate different perspectives in a way that will give students a coherent and useful understanding of algebra?

The NCTM (in preparation) Algebra Document postulates that thinking and reasoning in algebra must be *about something*, and so suggests that the themes build student understanding through activities embedded in contextual settings, such as growth and change, data, and uncertainty; number, size and shape; or patterns and regularity. These settings provide an opportunity for students to make connections between algebra and other disciplines and to see how algebra helps make sense of patterns in areas such as biology or economics or can be used to describe relations between geometric figures. Kaput's integration dimension might be linked to this notion of setting.

The third dimension, pedagogy is, as Kaput indicates, critical in how students come to perceive algebra, not as a set of magic tricks that can be used effectively only by someone who has the "math gene," but rather as a discipline useful to everyone, that has a logic and beauty of its own. Instruction should move toward an active exploration of algebraic ideas and exploit technology as a vehicle to

enable students to build an understanding of algebra and what it can do. Kaput's caution that "it is possible to achieve surface forms of valued pedagogies while failing entirely to engage students with significant mathematics" needs to be taken seriously. Not all rich investigations are worth doing. Is it really useful to fit a cubic to a piece of a curve generated by data from a CBL? Not all algebra can be constructed through rich investigations; at some point, students do need to have some facility with algebraic ideas (The question of which ideas and how much facility is not at all clear, however.). It is also not the case that one form of delivery should be the dominant form. While discourse, where ideas and interpretations are shared, discussed, and dissected, is a vital part of understanding for most people, real learning also depends on individual reflection and internalization.

Current Practice/Short Term Reform

Current practice has major implications for thinking about future directions. The baseline described by Kaput, essentially confined to two courses, broken into small categories of activities largely consisting of symbol manipulation and "story" problems, however, has a long history in the United States. In the 1909 text, *First Course in Algebra*, designed for a fourteen year old, the authors state in the preface, "A serious effort has been made to utilize the valuable suggestions in which the widespread discussion of the teaching of algebra for the last ten years has been fruitful....The material itself has been selected with the intention of affording the student ample drill in the elementary technic of algebra and a commensurate development of his reasoning power....Especial care has been used in the selection of the exercises in equations, the object being to have as great a variety as possible and yet to give only equations whose roots can be verified with a reasonable amount of labor" (Hawkes, Luby & Touton, 1909, pp. iii-iv). They describe the presence of frequent review exercises in factoring so students will "acquire in the shortest possible time a secure grasp of forms and methods and the careful blend of problem situations with technical work to avoid spending long periods of time on mere technic" (Hawkes, Luby & Touton, 1909, pp. iv). Typical sets of exercises include: 36 distance-time-rate problems, 80 factoring problems, and a chapter review with 66 problems. There are two essential differences between this book and the pre reform texts of the 90s: The content and exercises have become divided into two parts over the years, Algebra I and Algebra II, and the manipulation processes are based on axioms and rules. The historical notes are also different and quite interesting!

Technology possesses the power to change this at-least-one-hundred-year-old picture and to do so in dramatic ways. The graphing calculator has already had a major impact on the secondary mathematics curriculum, changing the focus of instruction and expectations about what is important for students to know. Each year, literally thousands of teachers attend calculator workshops where they learn to use a calculator to teach algebra, trigonometry and precalculus. These calculators are making the questions (solve, simplify, and factor) trivial which teachers used to teach students to answer from 1909 to 1995, and teachers are not sure what new questions should take their place. In 1989 several of the chapters in *Research Issues in the Learning and Teaching of Algebra* (Wagner & Kieran, 1989) raised the issue of the need for research on the impact of technology on the algebra curriculum. Yet six years later there still seem to be very few studies available that can begin to enlighten those in the classroom about the directions they should be taking. Teachers are making changes, and text books are incorporating the changes. The question is, Are these changes contributing to the algebraic (and mathematical) knowledge of students in ways that are important? Is a coherent picture of algebra the driving force? Some of the changes might be seen as driven by what is possible or what seems to be motivating, without care given to what is important in overall understanding. The lag between research and curricular change is placing the mathematics community in a reactive position, rather than a proactive one.

Further, the development of technology promises to continue at an ever increasing rate. Technology should make a difference; it can move students beyond what is currently possible in school algebra and can do so in new ways. Yet, those working on curriculum design must think deeply about a curriculum that capitalizes on technology, but is not out of date by the time it is ready for print. They must also take care that the emphasis on technology does not create a new morass of symbols and procedures.

As Kaput indicates, algebra has already begun to change for a large number of schools and teachers. Many students now begin algebra in grade 8 or earlier. The content of the early algebra courses varies from what used to be considered prealgebra to a traditional ninth grade course to a version of reform algebra. There are currently many different "reform" movements underway, and some of these have been in place long enough to begin to produce results (although what results are desirable is not clear). Are these current changes making a difference and what effect will they have on the long term reform efforts? As an example, some of those who have embraced the graphing calculator have placed functions (explicitly defined) at the heart of algebra, and no longer pay much attention to equations generated to provide an algebraic representation of a plane or an algebraic statement of condition. In fact, there is a trend towards replacing traditional story problems with new versions; out go upstream, downstream, work, money, integer, and mixture problems, and in come the taxi cab, interest, and tossing a ball.

While there are inherent problems with the implementation of algebra for all, from poorly prepared students and enormous failure rates to inappropriate texts, there are also some gains. For some cities, such as Milwaukee, that mandated algebra for all, even though the failure rate is high, more students are passing algebra than before the mandate when only selected students took the course.

Kaput's case study of change regarding a Presidential Award winner's use of graphing calculators and the failure of this approach to produce students who understood function is an inappropriate one to use as evidence. Consider first that change without understanding what is important will not succeed. Consider also, the tenuous nature of what it means to understand function. Students have always had difficulty understanding function, and teachers and texts have, probably unknowingly, contributed to the problem. "These students perceptions of what constitutes a function could be dismissed as just so many misconceptions, that is, errors in their conceptualizations....Indeed, all of these misconceptions are due to ***an overgeneralization of the initial examples used in the introduction of the function concept*** . These overgeneralizations, which occur naturally, become hurdles in the construction of the more global notion of function" (Herscovics, 1989, p. 80, emphasis added). Malik (1980) in a historical review traces the definition of function since Euler and indicates that each new definition of function was the subject of heated debate among the community (and most teachers probably do not know there are different definitions). Herscovics conclusion is that "learner's difficulties with functions are reflections of the history of the evolution of the concept." The bottom line is, a) did the teacher have sufficient knowledge and understanding of function and how to think about it (Many current secondary teachers were taught the set theoretic definition of function, one derided by Freudenthal (1973) and others), and b) what has research provided to help the teacher understand the cognitive difficulties students face in trying to understand the concept? New tools that do the same thing will not solve old problems. Our challenge is to determine how to make these new tools do new things that will solve these problems and, as part of that process, to provide a channel for using the knowledge gained from research to inform teaching.

Phase Two: Intermediate Reform/Backlash

The major curriculum projects are suggesting changes in the way algebra is construed as a school subject and in the way it is presented to students. Information about the success of these projects in building students' understanding of algebra so far seems to be primarily anecdotal, not surprising since, at the middle school level, the materials are just being completed, and few students have experienced a complete program. The high school materials are still in the developmental stage. Although examples of student work from the University of Wisconsin middle school project, *Mathematics in Context*, do support the success of the principle of "progressive formalization" described by van Reeuwijk (in preparation), long term research is needed.

As reform moves into this second phase, the problems are exacerbated by the issues raised above regarding the status quo. The change will not come easily and, in many places, those who advocate reform are already experiencing severe criticism. The tradition of algebra is firmly ensconced in the minds of everyone who had any experience, whether successful or unsuccessful, with algebra. Many teachers themselves are reluctant to abandon what they have done "successfully" for years. They "know" what their students should learn in an algebra course. Those who are willing to change are not sure to what they should change nor how to integrate this with the rest of the mathematics they are teaching. Algebra has been a filter, and parents are convinced that their children should learn what they themselves learned (or didn't learn) in order to make it past the filter. They perceive that changes in algebra will change the rules for success, and they are not about to do this lightly. The results of a survey for the NCTM (in preparation) Algebra

Document indicated a surprising number of educators who do not think everyone can learn algebra. Many mathematicians and scientists are concerned that a new algebra, driven in many cases by technology, is essentially providing a watered down curriculum that will produce students with little if any mathematical knowledge. Finally, many parents do not want their children to be part of an "experiment." The foundation for algebra reform should have some roots in research, and an analysis of such research, in language accessible to teachers and parents, should be used to provide support for the direction of reform.

Phase Three/Long Term Reform

The NCTM (in preparation) Algebra Document is based on three assumptions: A reconceptualization of algebra is needed; all students can learn algebra and should have the opportunity to do so; and algebra should be taught throughout the K-12 curriculum. The long term reform advocated by Kaput implicitly builds from these same three assumptions. Success depends on major shifts by researchers, mathematicians, mathematics educators, and teachers working through the issues together. Researchers (and mathematicians) focus on whether the value of the quadratic formula lies in its ability to complete the analysis of quadratics (Thorpe, 1989) or because its development depends on a very important mathematical tool, completing the square. Teachers, who once taught students just to use the formula, have to make a major shift—what does it mean to complete the analysis of a quadratic; what does a mathematical tool such as completing the square buy for you? Such shifts will help teachers and students focus on the algebraic concepts that are important, not the output of the algorithm. The background and preservice training of K-12 teachers is currently inadequate for such thinking. (Ask any ninth grade algebra teacher to suggest a coherent sequence for the topics in algebra.) Mathematicians need to recognize that the community will be well served if some of the traditional K-12 content is replaced by concepts from discrete topics, from statistics or linear algebra.

Some of the shifts brought by the short and intermediate phase of reform will have a major impact on long term efforts. In some cases manipulatives have begun a life of their own; elementary teachers are reluctant to even think about teaching "algebra;" some new curricula carry reform directions to an extreme (no factoring, no practice); and "tracking" issues are critical. In every case, research can help educators make decisions. Kaput cites many examples of current research efforts. Unfortunately, the channels for communicating the results of that research to those in the classroom and for promoting dialogue among the mathematics community are not in place. Our task is to establish these channels as well as investigate algebraic teaching and learning if we are to have effective long term reform.

References

Freudenthal, H. (1973). Mathematics as an educational task. Dordrecht, The Netherlands: D. Reidel.

Hawkes, H., Luby, W., & Touton, F. (1909). *First course in Algebra* (pp.iii-iv). Boston: Ginn and Company.

Herscovics, N. (1989). Cognitive obstacles in the learning of algebra. In S. Wagner & C. Kieran (Eds.), *Research in the learning and teaching of algebra* (pp. 60-86). Reston, VA: National Council of Teachers of Mathematics.

Malik, M.A. (1980). Historical and pedagogical aspects of the definition of function. *International Journal of Mathematics Education in Science and Technology*, 11, 489-492.

National Council of Teachers of Mathematics. (in preparation). *Algebra in the K-12 Curriculum* (Discussion Draft). Reston, VA: Author.

Thorpe, J. (1989). Algebra: what should we teach and how should we teach it. In S. Wagner & C. Kiernan (Eds.), *Research in the learning and teaching of algebra* (pp. 11-24). Reston, VA: National Council of Teachers of Mathematics.

van Reeuwijk, M. (in preparation). Algebra & realistic mathematics. In J. Kaput (Ed.), *Employing Children's Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics.*

Wagner, S., & Kieran, C. (Eds). (1989). *Research issues in the learning and teaching of algebra.* Reston, VA: National Council of Teachers of Mathematics.

A RESPONSE TO A RESEARCH BASE SUPPORTING LONG-TERM ALGEBRA REFORM

Elizabeth Phillips, Michigan State University

My reactions to Kaput's paper fall into three categories: comments on the Dimensions of algebra reform, a brief discussion of the algebra reform from the view point of a curriculum developer, and finally, some concerns about the three stages of reform.

Some Musings on the Dimensions of Algebra Reform

Kaput offers three dimensions in which to measure change in algebra reform: Breadth, Integration, and Pedagogy. In the discussion of Breadth he describes five aspects of algebra. He claims that the first two aspects, *Algebra as Generalizing and Formalizing Patterns & Constraints, especially, but not exclusively Algebra as Generalized Arithmetic Reasoning and Algebra as Generalized Quantitative Reasoning*, (1), and *Algebra as Syntactically-Guided Manipulation of Formalisms*, (2); give rise to all the others—*Algebra as the Study of Structures Abstracted from Computations and Relations*, (3); *Algebra as the Study of Functions, Relations, and Joint Variation*, (4); *Algebra as a Cluster of Modeling Languages and Phenomena-Controlling Languages*, (5).

There has been a great deal of effort and time devoted to categorizing, describing, or defining school algebra. Kaput's five aspects of algebra are yet another, but not dissimilar, cut on school algebra. Most recently, the NCTM Algebra Working Group (NCTM, in prparation) wrestled with these same questions of school algebra and settled on four themes, *Functions and Relations*, *Modeling*, *Structure*, and *Language and Representations*, around which to organize discussions of "algebra for all" in the K–12 curriculum.

Recent discussions of reform in school algebra have tended to broaden the view of school algebra, which has caused some lively reactions. Some people have argued that function, which is common to both Kaput's and the NCTM Algebra Working Group's descriptions of school algebra, is not algebra, but analysis. Many view school algebra as being closely related to abstract algebra at the college level. For example, at the Algebra Initiative Conference (Lacampagne, Blair & Kaput, 1995) over 60 mathematics educators and mathematicians met for three days to discuss algebra in the K–16 curriculum. Most of the research mathematicians present were algebraists. If school algebra is to be categorized as a study of functions, then shouldn't research analysts be involved with discussions of school algebra? The study of functions was usually allocated to a course called precalculus or analysis—a course for mathematics and science majors. Functions is a recent addition to the school algebra curriculum, in part due to the accessibility and implementation of graphing utilities into the curriculum.

Some people have also argued that modeling cuts across all areas of mathematics as does structure. The NCTM Working Group also proposed that the organizing themes could be developed by studying important ideas in change and growth (analysis), size and shape (geometry), uncertainty (probability), number, data, etc. This led one reviewer to question, "How does algebra differ from the other content areas in mathematics?"

Is there a danger that school algebra is becoming too broad? For whom are these categories helpful? It is important for teachers, curriculum developers and mathematics educators to have a working definition of algebra—even if it is very broad and encompassing. Teachers need to have a sense of the "big picture" of algebra to help them make decisions about the curriculum and student's understandings. Curriculum developers need a vision of algebra to develop a coherent and balanced curriculum. Researchers need a framework around which to organize their research. What view does the general public have of algebra? According to Wheeler (1991, as reported in Romberg & Spence, 1995), "proponents of the current reform movement argue for a particular perspective that is different from that held by diverse individuals, including the perspective of many (if not most) working mathematicians." Romberg & Spence (1995) claim that the current perspectives about algebra from an absolutist perspective are about mathematics in general. Does it make a difference if we all select different themes, strands, or definitions to guide our thinking? Do all roads lead to Rome?

Is there a simpler answer that could help guide these discussions and reform efforts? Romberg & Spence (1995) claims, "For students, algebra should be a way to express real-world phenomena in mathematical language. Their experience of algebra should include many and varied problems from the real world so they will gain understanding of the power and usefulness of algebraic notations and conventions." Romberg & Spence's (1995) claims together with Kaput's strands 1, 2, and 5 suggest that language and representation for expressing generalization and formalization of mathematical ideas could be a main organizing theme or strand of algebra. Kaput devotes a large part of his paper discussing "language and representations" and "generalizations and representations."

If language and representations are the organizing theme of school algebra, then the focus of algebra reform shifts to "why one needs a mathematical language," "which language," and "how one learns the language." This theme would allow a rich and dynamic language (symbols, graphs, tables, pictures, computer languages, simulations, etc.) to develop as students study engaging problems. In turn, the problems would lead to the development of powerful reasoning strategies and understanding of important mathematical ideas in arithmetic, analysis, geometry, statistics, probability, or even abstract algebra. The language becomes the means to represent the ideas and reasoning—or the means to represent the "generalizations and formalizations." With this categorization of algebra, functions and structure are still important—perhaps, more so. The important ideas of functions and structure can emerge on their own. Mathematics education researchers whose interests are the development of functions or structure would continue their research under the umbrella of analysis or abstract algebra (or just functions and structure).

It is the development of a mathematical language that is both brief and general to encode mathematical ideas and reasoning that has been a cornerstone in the development of mathematics. However, this suggestion of school algebra as a language for generalizing and formalizing mathematical ideas is not new and it too will cause controversy—partly because this has been the perceived dominant theme of traditional school algebra. It may be too close to a "drill and kill" curriculum. If "language and representations" and "generalizations and formalizations" are the dominant themes of school algebra, the emphasis should not be on a single course devoted to practicing isolated unrelated skills. Instead, language should be developed along with the mathematical ideas of function, geometry, data, and so on.

While all of Kaput's strands, as well as those offered by the NCTM Algebra Working Group and others, are all important ideas in mathematics, are they school algebra? Are they too broad? Is there a broad view for mathematics educators interested in algebra and another for the general public? The suggestion of "language and representation" and "generalizations and formalizations" as organizing strands for school algebra is offered as a middle ground for the various interpretations of algebra.

Perspectives of Algebra and Algebra Reform from a Curriculum Developer

This section contains a brief description of a curriculum project and the implications of this project for research in school algebra.

Description of CMP

The Connected Mathematics Project (CMP) is a middle school curriculum project funded by the National Science Foundation (Lappan, et al., 1995) that is being developed at Michigan State University (W. Fitzgerald, G. Lappan, and E. Phillips) together in conjunction with the University of Maryland (J. Fey) and the University of North Carolina (S. Friel). The developers of the CMP curriculum believe that observations of patterns and relationships lie at the heart of acquiring deep understanding in mathematics. Therefore, the CMP curriculum is organized around interesting problem settings—real situations, whimsical situations, or interesting mathematical situations. Students solve problems and in so doing they observe patterns, and relationships; they conjecture, test, discuss, verbalize, and generalize these patterns and relationships. The mathematical strands of number, measurement, geometry, probability and statistics, and algebra are developed across the middle grades.

Algebra in the CMP Curriculum

If mathematical concepts are developed from a problem situation or context, then the variables in the situation and how they are related become ideas that permeate all the units. Thus "generalizing and representing" these relationships is part of all the CMP units, including those units designated as algebra. For example, in an early two-dimensional measurement unit a sequence of activities leads to a generalization of a strategy for finding the area of a circle. One of problems in the sequence has students investigating which measures are most closely related to the price change in pizzas—circumference or area or radius or diameter. This problem seeks a relationship between the measures of a circle and the cost of a pizza. Eventually the sequence of activities ends in a generalization about the area of a circle given its radius. In a geometry unit on two dimensional shapes, students investigate the relationship between the angle measure (or the number of sides) on the shape of a plane figure. In the data units students decide which variables and which relationships to investigate, and how to represent these relations. When mathematics flows from the study of problems or contexts, then variables and the manner in which they are related, naturally arise. Furthermore, in such situations there may be more than two variables, and students must decide which variables to study and then discuss possible effects of the other variables.

While variables and patterns are part of each unit, they come to the foreground in a unit called *Variables and Patterns*. The focus is on looking at a variety of situations and more formal ways to represent these situations. Pictures, words, tables and graphs together with some algebraic symbolic representations are studied. Moving freely among the representations takes time to develop and hence is also an important part of all the units. Three other units, *Moving Straight Ahead* (linear functions), *Growing, Growing...* (exponential functions), and *Launching Rockets and Leaping Frogs* (quadratic functions), investigate patterns of regularity among the rate of change between the variables. It is the concept of "rate of change" that helps students identify, represent and reason about linear, exponential, or quadratic functions. Students do some work with symbols, which are temporally free of context, for the purpose of investigating the general characteristics of a specific function. While symbols are used to represent these situations, along with other representations, symbol manipulation is not the focus of the units—modeling and functions are the foci. Another unit, *Say it With Symbols*, looks more closely at ways to represent problem situations, symbolically—particularly those that give rise to different, but equivalent expressions— while another unit, *Thinking with Mathematical Models*, looks more closely at modeling.

Research in the Connected Mathematics Project

Of utmost concern to the CMP curriculum developers are what students will be able to do and know at the end of three years. The research component of CMP consists of several stages: videotapes, student work, interviews, teacher and student surveys, and observations have been conducted throughout the project, primarily to guide the development of the teacher and student materials. To assess students' understanding and reasoning, pretesting and posttesting of both CMP and non-CMP 6th, 7th, and 8th grade classes are currently going on. The tests consist of a pretest and posttest using both the Iowa Test of Basic Skills and a test designed by an outside evaluator that reflects the recommendations of the *NCTM Standards* and three authentic assessment tasks from the Balanced Assessment Project (Schoenfeld 1995) administered at various times during the year. The Iowa Test of Basic Skills was strongly promoted by the CMP Board. They felt that the public, regardless of any other evidence, needed to be convinced that students participating in the program would not do worse on tests of basic skills. Otherwise, these new curriculums may vanish on the vine (see further comments under the following section on the Three Stages of Reform). Since the set of units designated for 7th grade in CMP has a strong proportional reasoning theme running throughout the units, the principal investigators are also conducting research on students proportional reasoning abilities at the end of 7th grade.

The *NCTM Standards* based test does not give a complete picture of students' knowledge or reasoning. It does begin to paint a picture of students' reasoning and problem solving abilities as well as their ability to make connections and communicate. The research described above falls into Kaput's short or intermediate stages of reform. During the long-term phase of reform some of the questions that need to be addressed are:

- Does the CMP curriculum give students more power to solve morecomplex problems?
- Is it possible to build a complete mathematics program based on explorations of interesting problems? What are the strengths and weaknesses? What misconceptions might arise from these curriculum reform efforts?
- How long must students be engaged with an important mathematical idea so that the student carries understanding of the idea into the next grade or level? How many years must a student be involved with a "reformed" classroom to reap the benefits? What are the implications for students who go from a reform based curriculum at one level to a standard curriculum at the next level?
- What transitions, and over what period of time, do students need to make connections? What kind of transfer activities do students need to move from a problem based setting to a symbolic based setting or to otherrepresentational schemes?
- What algebraic reasoning do students develop? What knowledge of algebra (or any other area of mathematics) do they carry over into high school? Does the CMP curriculum provide deeper insights into algebraic forms?
- Will the possible loss of personal manipulative skills be a longer term stumbling block to mathematics development? (This question is perhaps more important as students move to the high school.) How much symbolic skill (arithmetic and algebraic) is necessary for students to model a situation or to manipulate an expression to reveal new information about a situation? (A similar question could be asked about other representational schemes.)
- How much help do teachers need to implement a new curriculum that requires a different view of mathematics and a different peda-

gogy? What kind of support and at what levels do teachers need this support? What mathematics do teachers learn by teaching these new curricula?

- What kind of linkages among teaching, learning, and assessment do these curriculum projects provide?

The CMP curriculum, as well as other curriculum projects, are based on the best available research from mathematics education and the cognitive sciences. However, none of the research to date has been conducted in settings where students have been engaged in significant mathematics in classrooms and have developed their understandings and reasonings as a community of scholars over several years. These new rich curriculum projects provide a unique opportunity to carry on significant research over a long period of time that has not been available since the new math—research on teaching, learning, and assessment. Kaput suggests that the long-term research efforts should begin in the elementary grades. There is enormous potential for developing students' mathematical power with these new curriculum projects in Grades 6–12. Algebra reform should begin on several fronts.

Comments on Kaput's Three Stages of Reform

Kaput's discussion of the three stages of reform, on the surface, appear reasonable. It makes sense to tinker with short-term reform—these efforts could also inform the long-term efforts at reform. Much of the reform in algebra that has been going on since the release of the *NCTM Standards* has been short-term reform. Most of these reform efforts have been "add ons," such as the use of graphing calculators, computer software, or manipulatives, to the existing curriculum.

There is a danger to these short-term efforts. First, most of these efforts ignore the weaknesses and deficiencies of the present curriculum. There is at the same time a tendency to be a bit caviler about the benefits of graphing software packages. To use graphing software utility effectively requires a deeper understanding of functions and relations than is currently in the curriculum. Many students do not understand functions and consequently mimic the procedures needed to use a graphing calculator without understanding what they are doing. There is a danger that we could be replacing abstract symbol manipulation with equally abstract algorithmic techniques on how to use the graphing calculator (or computer). Furthermore, some people interpret statements like, "with graphing calculators and symbolic algebras there is little need for work with symbol manipulations" as meaning no need for skill development. Putting graphing calculators into the hands of students requires very careful reorganization and re-conceptualization of the algebra curriculum and research to support these long-term efforts. How effective these technologies are will have to wait for Kaput's long-term stage of reform.

However, short-term efforts may torpedo future reform efforts. For example, the implementation of graphing calculators into the college algebra courses at one Big Ten university was perceived as weakening the algebraic skills of students going into the calculus class and has since been forbidden at this university. At the heart of this conflict about algebra reform is the role of symbol manipulation. The general public and many mathematicians perceive the mastery of symbolic manipulations as an important part to learning algebra. Students need to model situations using a variety of representations, including algebraic symbols. They need to show that different expressions for the same situation are equivalent. Students need to transform equations or expressions into equivalent forms that can be entered on a computer or graphing calculator. Further reasoning with equivalent symbolic forms can very often reveal information that is not apparent in graphs or tables. Mindless drill and practice has not worked. But what is the role of symbol manipulation? How much understanding and skill with symbolic transformations or manipulations are needed to reason effectively with symbols? These questions need careful research to convince the general public about the needs and benefits of reform in algebra.

K–12 teachers are key players in the algebra reform movement and what they do or do not do is closely tied to public and university approval. Any efforts at reform must help teachers understand the proposed changes. If teachers are convinced that such efforts will lead to greater understanding and reasoning for their students, they will support the reform activities. However, even these teachers ask for help to convince administrators and parents that these changes will not be harmful—and that these changes will help.

Without some fundamental changes that Kaput describes in *Breadth, Integration, and Pedagogy*, these short-term reform efforts will have little effect and in fact may offer a real roadblock to the needed long-term reform. The general public also has a short attention span—there is a tendency in this country for quick solutions. Will they have the patience to withstand the efforts needed to implement long-term reform? The backlash has already begun. The Michigan State Board of Education is recommending that the *Standards* as described in their State Frameworks not be mandatory. In addition, the State Board inserted stronger standards on skills in many of the subject areas, deleted some that appeared to imply value judgment, and only narrowly voted down a proposal to make the teaching of the creation science mandatory for all students. Similar backlashes are occurring in other states. The strengths of short-term reform efforts must be advertised with the promise to look at both the gains and losses of such efforts.

So my questions are: (1) Do the relative benefits of short-term reform out weigh the more dangerous backlash that occurs when the public perceives these efforts as detrimental to student learning (even when there has been no real decline of skills)? (2) If the short-term reform is seen as a failure, what does this say about the intermediate and long-term reform? (3) Will we have time to carry out the reform? (4) What can we as a community of mathematics educators do to provide the time and opportunity for reform to progress?

References

Lacampagne, C., Blair, W., & Kaput, J. (Eds.), (1995). *The Algebra Initiative Colloquium*. Washington, DC: U.S. Department of Education.

Lappan, G., & al, (1995). *The Connected Mathematics Project*. Palo Alto, CA: Dale Seymour Publications.

National Council of Teachers of Mathematics. (in preparation). *Algebra in the K-12 Curriculum* (Discussion Draft). Reston, VA: Author.

Romberg, T., & Spence, M. (1995). Some thoughts on algebra for the evolving work force. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initiative Colloquium*, (Vol. 2, pp. 177–192). Washington, DC: U.S. Department of Education.

Schoenfeld, A. (1995). Balanced Assessment Project. NSF grant # MDR 92529.

Wheeler, D. (1991). What does it mean to know mathematics? Paper presented at the International Commission on Mathematics Instructors' Study Conference on Assessment in Mathematics and its Effects on Instruction. Calonge, Spain.

Advanced Mathematical Thinking

BRIDGING MANIPULATIVE-EXPLORATORY PLAY AND THE DEVELOPMENT OF MATHEMATICAL CONCEPTS IN A TECHNOLOGY-RICH ENVIRONMENT

Sergei Abramovich, University of Georgia

The paper shows that the study of mathematics can be organized as a complex learning enterprise integrating manipulative-exploratory play into a newer software tool environment — a dynamic geometry, a spreadsheet, and a relation grapher, and it reflects work done in a lab setting with preservice and inservice teachers enrolled in contemporary general mathematics and problem solving courses. The psychological aspects of learning mathematical concepts through integrating off and on computer activities and possible implications of the approach for mathematics teacher education are highlighted from Vygotskian perspective.

The role of play in learning abstract structures has received much attention in educational psychology research. Particularly, Dienes (1964) studied children's learning of mathematical concepts from experiences with concrete materials under the assumption that play and the higher cognitive activities are closely connected. With the advent of advanced technology, it has become considered helpful to use suitably designed computer-based simulations of concrete materials in the learning of mathematics (Thompson, 1992; Kaput, 1994; Steffe & Wiegel, 1994). These uses of a computer, however, involve topics not beyond the elementary and middle levels. The appearance of newer software tools with their tremendous potential for promoting the spirit of exploration and discovery in mathematics classrooms makes it possible to extend the use of concrete materials to more advanced levels of mathematics and to consider play associated with both off and on computer activities. Note we consider the notion of play in the spirit of Hoyles and Noss (1992); that is, student engagement into a play within a learning environment implies exploration, experimentation, wondering about, and enjoyment.

The paper suggests that integrating manipulative-exploratory play into a multiple-application environment enhances the study of advanced mathematical concepts and highlights three essential functions of a computer as a learning medium. First, the variety of available colors and shapes places the choice of manipulatives under the control of learners, and this may strengthen their constructive activity and smooth possible differences in the perception and conceptualization of color and shape (Ratner, 1991). Second, the computer environment takes into account indistinct boundaries of manipulative play which may quite imperceptibly move over to an exploration (Dienes, 1964). Manipulative-exploratory play is, in fact, a search for regularities, something that may become an object of manipulation at a higher level. This suggests that the third function of a computer in this setting is to provide the learner with an opportunity of instant transfer from screen images to computing activities and back; that is, the use of appropriate applications integrated into the medium allows the generating of numerical and/or diagrammatic evidence as abstractions from a number of simultaneously scrutinized concrete situations presented by these images. Regularities can then be studied again through play at a higher level of cognitive activity. In addition to these functions, the approach considers learning to be deeply anchored in interactive instruction and emphasizes the role of a teacher in developing students' mathematics knowledge. This role assumes a teacher to be a partner in advancement, one who links small-group explorations and whole class discussion, and mediates the spirit of mathematics learning through a mutually enriching teacher-student dialogue. An equal partnership in such dialogue contributes to the learning of being a reciprocal activity (Confrey, 1995), something that affects both student curiosity and teacher intelligence.

Environment for approaching Fibonacci numbers

The realistic mathematics education argues for instruction to be a process that emphasizes the importance "to recognize a mathematical concept in, or to extract it from, a given concrete situation" (Ahlfors et al., 1962, p.190). A relevant context for accommodating such instructional philosophy is Fibonacci numbers. To approach the concept we suggest to students the following play activities: coloring buildings of different stories, making offspring in the rabbit problem, cutting a square and rearranging the parts in the so-called paradox problem. More specifically, students are engaged in the following phenomenological explorations.

- Exploration 1. Buildings of different numbers of stories are given and one may color them with a fixed color in such a way that no consecutive stories are colored with it. How many different ways of coloring one, two, three, four, etc.-storied buildings are possible?
- Exploration 2. A pair of rabbits is placed in a walled enclosure. Find out how many offspring this pair will produce in the course of a year if each pair of rabbits gives birth to a new pair each month starting from the second month of its life.
- Exploration 3. When you cut a figure and reorder the parts, the shape may change but, the area can not. Consider Figure 1: the square is cut into two congruent triangles and two congruent trapezoids. Can we chose x and y so that the square can be transformed into rectangle as shown?

Within each activity, the same sequence of numbers, known as the Fibonacci sequence, occurs as a result of students' extracting appropriate concepts from manipulative-exploratory play. Once Fibonacci numbers have come into view, they can be explored through spreadsheet modeling; that is, numerical evidence can be used for discovering a number of situations of similar type and extracting an abstraction from these. Moreover, numerical evidence provides a gateway for developing induction proof of the abstraction through visualization with its subsequent symbolization as an important point in the process of learning mathematical concepts (Abramovich, 1995). Thus, the didactical emphasis of the activities is both on conjecturing and developing formal proof rather than on exploring computer-generated patterns "at the expense of discovering their underlying relationships" (Noss, 1994, p.9). Yet the environment accommodates learners of different *zones of proximal development* allowing for the diversity in the pace of activities, in the consuming of teacher-mediated assistance, and in the depth of exploration. Finally, when the mystery of the paradox problem is resolved, the use of a relation grapher enables students to make sense of the concept of the golden ratio. Note that geometric aspects of the paradox problem can be explored both in a traditional setting (paper grid and scissors) and in that of software setting. Comparing off and on computer activities in resolving the problem leads to the following important observation: when manipulating parts of a square within off-computer activities a student uses geometric transformations such as rotation, reflection, and translation almost automatically or spontaneously, yet these capacities lack conscious awareness. Though the student does act consciously in performing transformation, his or her attention is not directed toward the possessing of geometric skill, and its nonvolitional nature is shaped by the structure of the particular situation. On the other hand, the use of a computer allows for the learning of conscious awareness of the same operations while operating software. Therefore one can use this example in order to discriminate an instructional use of manipulatives associated with on and off-computer activities. This distinction is constructed on the lines of Vygotsky (1987) who, using language acquisition as a paradigm case, argued that the role of school instruction in written speech and grammar is to make a child learn "conscious awareness of what he does ... [that is, the child] learns to operate on the foundation of his capacities in a volitional manner" (p. 206). In much the same way, learning to operate dynamic geometry software like *GSP* leads to the mastery of school geometry and plays an important role in this process.

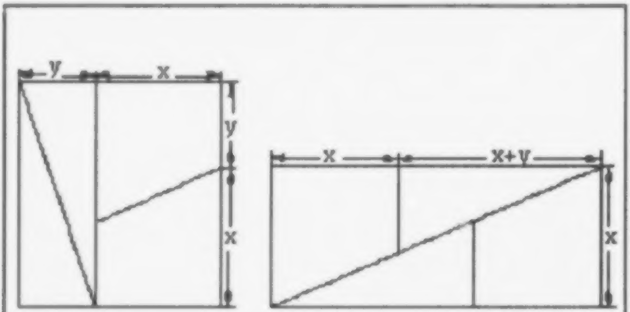


Figure 1. The paradox problem.

♦	♣	♠
♥	♦	♣
♥	♦	♠
♥	♣	♠
$C(4,3)=C(3,3)+C(3,2)$		

Figure 2. Recursive strategy.

Environment for the study of combinatorics

The study of combinatorics can be nicely orchestrated through integrating manipulative-exploratory play and computing activities. For example, consider the following problem: *In how many ways can three suits be selected from four different suits?* Combinatorial reasoning can be better acquired by clearing away the situation through the use of manipulatives. As one in-service teacher noted: "I feel manipulatives are probably one of the best ideas I have ever seen for showing C(n,r) — combinations of *n* things taken *r* at a time. This will be very useful in my upcoming lecture on combinations."

Indeed, in the first stage manipulatives serve as a means for solving the counting task so that abstraction from a number of similar arrangements of manipulatives occur in the form of recursive definition of combinations. In carrying out this task, we first discovered students' involuntary behavior in creating the combinations, something that seems to bring a chaos into the approach as the number of objects increases. Yet, a spontaneous strategy is not a useless experience, but on the contrary, it allows students to reach the threshold in the development of mathematical thinking beyond which conscious awareness of recursive strategy becomes possible. Indeed, when asked to be systematic, students often apply recursive reasoning: when hearts are not in use at all, three remaining suits can be selected in C(3,3) ways; when the heart is in use, two other counterparts can be selected in C(3,2) ways (Figure 2). This strategy possibly lacks conscious awareness of recursive thinking, though, in fact, this is the case of recursion. In the words of Vygotsky "consciousness and control appear only at a late stage in the development of a function, after it has been used and practiced unconsciously and spontaneously. In order to subject a function to intellectual control, we must first possess it" (cited in Bruner, 1985, p. 24). The teacher-mediated link between spontaneous and purposeful problem solving strategies thus becomes crucial for 'good learning,' for it is the link of the zone of actual and proximal development of the learners.

The next step in the study of combinations involves setting up on a spreadsheet boundary conditions for combinations obtained through manipulative-exploratory play and modeling them using a recursive nature of software. It is worth noting that the language of communication with the software might be that of pointing to cells (a kind of the substitution of speech for concrete action) rather than solely formula-based, and this allows students to shift the onus of both symbolization and generalization onto a spreadsheet. In such a way, the software serves as a support system that helps learners to make a non-algebraic leap from empirical data linked by an intuitive guess to the numerical projection of its generalization (modeling data). Once a large pool of combinations come into view the activities focus on making connections among combinations and testing these connections in terms of manipulatives. In other words, the activities deal with creating visual proofs of combinatorial propositions. One may note, for example, that C(4,3)=C(4,1), or C(4,3)=C(3,3)+C(2,2)+C(2,1) and then justify these findings using manipulatives (visual proofs). In doing so, one is engaged into a play on a higher level of cognitive activity using, in fact, the same concrete embodiments that allowed for the reaching of this level. Transferring from a special case of identities involving numbers with combinatorial meaning to their general form results in students involvement in the development of inductive proofs of the identities, mathematical activity stimulated and guided by computer-generated numerical evidence.

Note that although, as observation shows, the task to discover Fibonacci numbers among combinations (both through exploring numerical patterns on a spreadsheet and imparting combinatorial meaning to the coloring task) proves to be a challenge for most of the students, the principle of "raising the ante" of the task (Bruner, 1985) allows for maintaining students' interest in developing mathematical concepts and for demonstration of the endless mathematical explorations through the intertwining of different learning strands.

Assessment through reciprocal teaching

The environment described in this paper may have important implications for an assessment practice that incorporates reciprocal teaching (Palincsar & Brown, 1984). We applied this procedure for final sessions by splitting students enrolled in a problem solving class into equal groups, each of which was assigned to create a task for an associate group. The instructional goal was to demonstrate how the environment allows for students' affluent and seemingly endless performance on a regular task and encourages the development of mathematical ideas that are far beyond the task's original design. The sessions have shown that *all* students may become motivated and challenged by learning mathematics, provided that a classroom environment is conducive to students' pursuing avenues of personal interest and attaining ownership of concepts discovered. We conclude the paper with a hope that computer-enhanced reciprocal teaching embodies N.C.T.M.'s (1995) vision of an assessment as "a dynamic process that informs teachers...and supports each student's continuing growth in mathematical power" (p.6).

References

Abramovich, S. (1995). Technology-motivated teaching of advanced topics in discrete mathematics. *Journal of Computers in Mathematics and Science Teaching*, 14 (3), to appear.

Ahlfors, L.V. et al. (1962). On the mathematics curriculum of the high school. [Memorandum]. *American Mathematical Monthly*, 69 (3), 189-193.

Bruner, J. (1985). Vygotsky: A historical and conceptual perspective. In J.V.Wertsch (Ed.), *Culture, communication, and cognition: Vygotskian perspectives* (pp. 21—34). Cambridge University Press.

Confrey, J. (1995). A theory of intellectual development. *For the Learning of Mathematics*, 15 (2), 36-45.

Dienes, Z.P. (1964). *An experimental study of mathematics learning*. London: Hutchinson Publishers.

Hoyles, C., and Noss, R. (1992). A pedagogy for mathematical microworlds. *Educational Studies in Mathematics*, *23*, 31-57.

Kaput, J. J. (1994). Developing conceptual grouping structures: New software approaches. In D. Kirshner (Ed.), *Proceedings of the 16th annual meeting of the PME-NA*, v.2, p.50. Louisiana State University.

N.C.T.M. (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

Noss, R. (1994). Structure and ideology in the mathematics curriculum. *For the Learning of Mathematics*, 14(1), 2-10.

Ratner, C. (1991). *Vygotsky's sociohistorical psychology and its contemporary applications*. New York: Plenum Press.

Palincsar, A.S., & Brown, A.L. (1984). Reciprocal teaching of comprehension-fostering and comprehension-monitoring activities. *Cognition and Instruction*, 1(2), 117-175.

Steffe, L.P., & Wiegel, H.G. (1994). Cognitive play and mathematical learning in computer microworlds. *Educational Studies in Mathematics*, 26 (2), 11-134.

Thompson, P.W. (1992). Notations, conventions, and constraints: contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education* 23(2), 123-147.

Vygotsky, L.S. (1987). Thinking and speech. In R.W. Rieber and A.S.Carton (Eds.), *The collected works of L.S.Vygotsky*, vol. 1 (pp. 39—285). New York: Plenum Press.

EVIDENCE AND PROOF: EXPLAINING VECTOR RELATIONSHIPS

Helen M. Doerr, Syracuse University

This paper addresses a central issue in secondary school geometry, namely the role of proof. In an integrated mathematics and science curriculum, the role of proof as a process of conjecturing, explaining and justifying within a small group setting is analyzed. Particular attention is given to the students' use of empirical evidence and algebraic representations.

The role of proof in high school mathematics continues to be a topic of debate among educators. Such debate is frequently centered on the development of formal axiomatic systems in the context of teaching and learning geometry. Recent work with alternatives to axiomatic approaches to geometry has led to research on students' learning with computer-based construction programs such as the Geometric Supposer and the Geometer's Sketchpad. These environments have been shown to be effective in supporting students' exploration of geometry and in the making of conjectures and explaining and justifying their ideas (Chazan, 1993). However, many of these studies do not include pencil and paper construction, mechanical devices, or physical experimentation, but rather begin with the use and modification of representations in the computational medium. In this study, we examine a process of conjecturing, explaining, and justifying that begins with physical experimentation and then uses a multi-representational analysis tool with student-generated diagrams to support the students' development of a convincing argument about the relationships among multiple force vectors acting on an object at rest. This study grounds the explanatory role for proof, suggested by Hanna (1990), in a physical experiment with forces.

Description of Study

This paper will address the development of a convincing argument by one small group of students for the relationships among the forces acting on an object at rest on an inclined plane. This study was part of a larger research project on an integrated modeling approach for building student understanding of the concepts of force and motion and enhancing problem-solving skills. In this larger study, we examined a modeling process which integrated three components: the action of building a model from physical phenomena, the use of simulation and multiple representations, and the analysis, refinement and validation of potential solutions. In this paper, we present an analysis of the development of a geometric argument for vector relationships, which includes the formation of multiple conjectures, qualitative reasoning about those conjectures, and the refinement and validation of the conjectures.

Data Sources and Analysis

Each class session of this unit was videotaped, and during small group work, the focus group of this study was videotaped. Written work and computer work done by the group were made available to the researcher for analysis. Extensive field notes were taken by the researcher during the class sessions. The videotapes of class sessions were reviewed and transcribed for more detailed analysis.

Description of the Curricular Activities

The unit began with a simple physical experiment: an object was held suspended just above an inclined plane so that the plane served as a reference frame for the changing angle of inclination (see Figure 1). Using two spring scales, some rope and an object of known weight, the force parallel to the plane and the force perpendicular to the plane were measured for various settings of the angle theta between zero and 90 degrees. The teachers then posed a very open-ended inquiry to the class: how do these forces relate to the weight of the object, the angle of inclination, and/or to each other? In an earlier unit, the students had established that a force acting at angle can be thought of as having a vertical and a horizontal component and that these components are related trigonometrically.

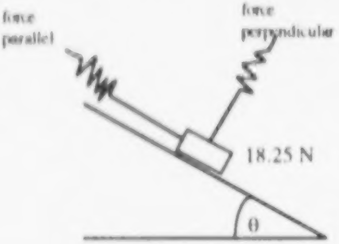
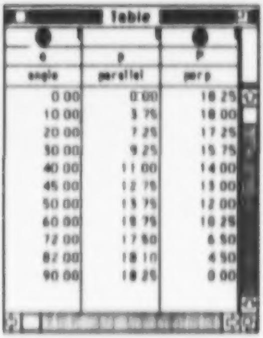


Figure 1. Forces Acting Along an Incline

The students were now faced with the problem of developing one or more conjectures about the relationships between and among the parallel and perpendicular forces, the weight of the object, and the angle of inclination of the plane. Then, in light of these conjectures and the evidence provided by the experimental data for the range of cases between zero and 90 degrees, the students were asked to convince themselves of the validity of one or more of these conjectures.

Results

The focus group of this study began by entering the experimental data into a Function Probe table (see Figure 2). From the table window, Alycia observed that the forces do not add up to a constant and that the forces change at the same rate. She appeared to be observing the symmetry of the covariation of the data for the parallel and perpendicular forces. The students quickly decided to graph the data, placing angle on the x-axis and forces on the y-axis. After graphing the first relationship, Paul observed that the second is symmetrical to the first; Alycia and Jenny confirmed the equality at 45 degrees. After these preliminary observations, they identified that their task was to find a relationship among the variables. At



Table

a	p	P
angle	parallel	perp
0.00	0.00	18.25
10.00	3.75	18.00
20.00	7.25	17.25
30.00	9.25	15.75
40.00	11.00	14.00
45.00	12.75	13.00
50.00	13.75	12.00
60.00	15.75	10.25
72.00	17.50	6.50
82.00	18.10	4.50
90.00	18.25	0.00

Figure 2. Parallel and Perpendicular Forces along an Inclined Plan

this point, they had in front of them, in the graph window, a relationship between the angle of inclination, and the force perpendicular and the force parallel. But nonetheless, for the students, it did *not* answer the question of what is the relationship? The inquiry which follows suggests that the students are looking for an explanation as to why this relationship holds. Paul suggested that they leave the graph window and return to the table, but then Jenny directed the group to the geometry of the situation.

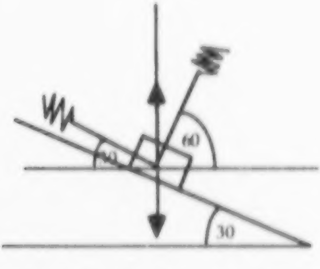


Figure 3. *Paul's Force Diagram*

From their drawings (see Figure 3), they began to clarify the meaning of the force data that they had collected. They began to analyze the role of the vertical and horizontal components of the forces. They continued to explore the symmetry of the geometry: they noted the equality of the forces at 45 degrees, that the forces are opposite at 30 and 60 degrees, and that there are complementary angles within the force diagram. These complementary angles later turned out to be crucial to their final argument. They clearly expressed qualitatively the relationship between the weight of the object and the vertical components of the parallel and perpendicular forces. They then tried to establish an equation for that relationship in the table window:

P: So the vertical component of this [parallel] force is equal to 4.625.

A: And minus that from 18.25 and get the other vertical component.

P: Yeah. So we can make an equation.

A: Let's make another column for vertical.

This reasoning and analysis built on their earlier understanding of the vertical and horizontal components of a force at an angle from the previous sub-unit. They were able to calculate the magnitude of the vertical components of the forces. They knew that the horizontal components could also be computed and that they must be equal. However, their efforts to express the relationship of the weight to the vertical components algebraically in the table window were unsuccessful.

At this point, Jenny restated her earlier lack of confidence in this strategy and suggested that the path they were heading down was one that would lead to a series of relationships (that might in fact be circular) rather than just one:

A: That's what? Round about?

J: Are we gonna, yeah, are we, yeah, totally! I mean that's not going to give us a, that's going to give us a series of relationships. Rather than looking for just one.

P: It's gonna give us, right, well, we can use that whole thing as the relationship between that and that. [the parallel and perpendicular forces]

J: Okay, uh huh.

The group took a short break, but Paul continued to work, going back to the force diagram, repeating his calculations, and attempting to create a relationship in the table window of Function Probe. By the time Jenny and Alycia returned, Paul was not able to create an algebraic relationship to give him the vertical component that he was looking for. At this point, the teacher provided some general instruction to the whole class and gave the groups another data set, so that they could verify that whatever relationships they developed would hold for alternative scenarios. The teacher then joined Jenny, Alycia and Paul. They explained their thinking about the component forces and showed Dave their graphs. Dave focused their attention on the original graph and pointed out that they could read the relationship between the angle and the forces (the original question) directly from the graph. Both Jenny and Paul suggested for the first time that these might be trigonometric curves. They then algebraically fit a sine and a cosine curve to their data and, at the very close of class, established that the magnitude of the perpendicular force is given by the weight of the object times the cosine of the angle of incline, and similarly for the parallel force. As class ended, the group did not bring any closure to the central idea that *they* were working on, namely, that the sum of vertical components of the perpendicular and parallel force must equal the weight of the object.

Class began the next day with a whole class discussion recapping that the relationships which the small groups had found between the parallel and the perpendicular forces, the weight of the object and the angle were given by the equations F^ = Fw * cosq and FII= Fw * sinq and were represented graphically by a pair of curves intersecting at 45 degrees. The teacher pointed out to the students that these relationships were created on the basis of fitting a curve to empirical data and that they didn't have a visual, pictorial argument to support the fact that they came up with sines and cosines. He directed the students to develop a convincing argument for the relationships on the inclined plane based on the geometry of the force diagram. The students moved to an analysis of the geometry in front of them. They had found the two equal angles (see Figure 4) yesterday and now they quickly put that together in a geometric argument to support the empirical curve fit that had been done:

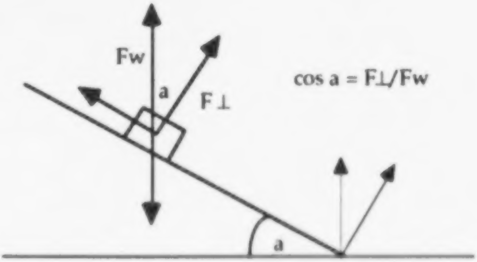


Figure 4. Alycia's Force Diagram

P: No wait, you guys, guys, guys, the reason why this, the, the cosine of this angle, which is also that angle, um (pause).

A: The cosine of this angle is this angle?

P: Yeah! No! The cosine of this angle is the same as the cosine of that angle. It's the same angle.

A: Yeah, hm, hm.

D: Is that the same angle?

P: The cosine of, yeah.

A: Yeah, because it's this minus 90. And these two are the same.

Alycia proceeded to write a two-column proof that argued that if two angles have the same complement then the angles must be equal. They then were convinced that the cosine relationship that they found empirically must also hold from the geometry of the force diagram:

A: So, if these two angles are equal, then the forces...

P: Then, right, then it makes sense that the cosine of this angle times that would equal that force. Cause this...

A: **You can just draw this angle down here. It can be anywhere.**

P: **This force. Right. That force can be anywhere. Cause...**

A: **I just moved it down, so that it's easier to see that they're equal.**

P: **Right.**

D: Hm, hm.

A: And so, of course, two angles that are equal have the same cosine.

They proceeded to make the analogous argument for the parallel force.

Discussion

The focus group had built an understanding of the force relationships involved for an object at rest on a ramp through a process that involved collecting empirical data, graphing the data, examining the table values, constructing a force diagram and then analyzing the vertical and horizontal components of the forces. Their qualitative understanding of the relationship of the vertical components included both the additive relations of the vertical components to the weight of the object and the symmetry between those components. Their analysis showed both an understanding of the relationship between the vertical components of the parallel and perpendicular forces and an argument as to why those components must equal the weight of the object. However, this analysis did not include a symbolic, tabular or graphical representation of the relationship. Directed by the teacher back to the original graph of the empirical data, the students left their model and generated an algebraic relationship from a curve fitted to the data. The students ultimately returned to the geometry of the force diagram, arguing that the empirical trigonometric relationships from the curve fitting made sense in terms of right triangle trigonometry. Thus, while they never returned to their earlier attempts at an algebraic analysis of the vertical and horizontal components of the parallel and perpendicular forces, those attempts led them to a clear and compelling analysis of the overall scenario.

References

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*, 359-387.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6-13.

FORMULATING THE FIBONACCI SEQUENCE: PATHS OR JUMPS IN MATHEMATICAL UNDERSTANDING

Thomas Kieren, University of Alberta
David Reid, University of Alberta
Susan Pirie, University of British Columbia

In the dynamical theory of mathematical understanding (Pirie and Kieren, 1994) understanding is considered to be that of a person (or group) of a topic (or problem) in a situation or setting. In this paper we compare the interactions between the situations and the mathematical understandings of two students by comparing the growth in understanding in a Fibonacci sequence setting in which specific tasks were suggested and interventions made, with that of the same students in a Fibonacci setting in which only a general prompt was offered. In the former, the growing understanding was characterized by jumps, indicating a collection of specific images or patterns. In the second these students exhibited a continuous, non-linear pathway of understanding more governed by epistemological interests and featuring more formulated reasoning.

How Does Mathematical Understanding Grow?

There have been numerous useful ways of thinking about mathematical understanding in terms of types or levels over the past 20 years (e.g. Skemp, 1976; Herscovics and Bergeron, 1993). Under such work mathematical understanding tends to appear as an acquisition or sets of such acquisitions. Following a more phenomenological, constructivist and enactivist view of understanding (von Glasersfeld, 1987; Johnson, 1987), Kieren and Pirie over the last eight years have been building and testing a dynamical theory of the growth of mathematical understanding which views it as a non-linear, non-monotonic, process-in-action. As illustrated in the diagrams below, we observe such change in understanding in action using pathways across eight embedded levels or modes of understanding. Starting from a person's assumed primitive knowing (related to the mathematical situation in which they find themselves) their understanding, if it is not disconnected, grows through three informal modes of action (image making, image having and property noticing) and through three potential formal modes of mathematical activity (formalizing, observing and structuring) and possibly leads to a person developing new diverging mathematical ideas (inventising). The inner, informal modes of understanding are related to more local, image-related knowing and frequently involve unformulated reasoning (Reid, 1995), while the outer levels are more sophisticated and general in nature. But as we have illustrated in a number of studies, less formal understanding is fully implicated in outer, more sophisticated understanding in that students very frequently "fold back" to inner, less formal understanding action. Such folding back typically leads to a broadening of student understanding and appears to be an important if not a necessary condition in its growth.

Research for this paper was supported in part by grants 410 93 0239 and 752 93 3268 from the Social Sciences and Humanites Research Council of Canada.

We are increasingly observing that understanding-in-action is best understood in terms of the inter-actions in which the person engages (e.g. Pirie and Kieren, 1992; Davis, 1994). For example, a teacher may intervene with students in the situation in an attempt to provoke them to move to more general or sophisticated understanding. Such moves are said to have *provocative intent*. Similarly, the teacher may try to have students fold back to inner less formal activity, such moves having an *invocative intent*. We have observed in a number of studies (e.g. Pirie and Kieren, 1992; Kieren, Pirie and Reid, 1994), that it is the subsequent actions of the students which determine the nature of the intervention and not the intent of the teacher.

The Nature of this Study

In a 1994 paper we reported on the understanding of two university students, Stacey and Kerry as they spent approximately an hour investigating a problem situation which grew out of a prompt which asked them to write the recursive rule defining the sequence, if they knew it or could discover it, and to "Look for patterns which relate the index n to the Fibonacci number F_n. For example, is there anything special about F_n when n is a multiple of 3, or a multiple of 4, or prime?" Using a methodology described below their (joint) understanding of the Fibonacci setting was characterized in Diagram 1.

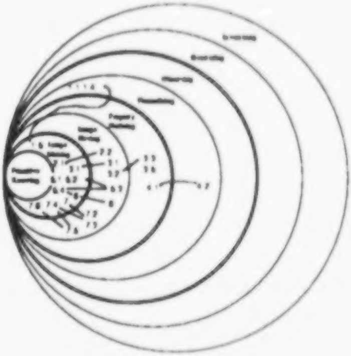


Diagram 1. The first Fibonacci session

Stacey and Kerry's growth in understanding of this Fibonacci situation seems to occur in disjoint jumps rather than as a more continuous pathway. In part this could be attributed to the prompt which in addition to asking for generalizations, gives a series of tasks to be accomplished. In addition the transcripts and mathematical activity trace of the setting reveal that the researcher had responded to requests from Stacey and Kerry with prompts of more things to do with the Fibonacci setting, which they appeared to treat as separate mathematical items. The question for us was, "Is Stacey and Kerry's growing understanding of the Fibonacci sequence inherently like this or should the interaction pattern with the researcher and setting be observed as an important part of Stacey and Kerry's growth in understanding?"

To study this question we provided Stacey and Kerry with an opportunity to respond to an altered Fibonacci prompt, some 15 months after their first session. In the second setting the prompt indicated: "Generalize a property of this sequence" and the researchers said nothing to the pair but simply observed their activities. Because we are attempting to observe understanding as an on-going lived activity in and with an environment, we use a number of inter-related methods both to gather and interpret the data. We term this method "brico-logical." It is a *brico*lage in that the researchers work interactively with various given materials on a piece of a more global problem. At the same time each researcher brings with them a particular *logic* of inquiry, here the Pirie/Kieren model, ideas on reasoning and proving, and the theory of enactivism (Varela, Thompson and Rosch, 1991). Each session was recorded using video tapes, transcripts and observer notes. Three different researchers, the authors, viewed the tapes and interacted and converged on possible conclusions to be drawn about understanding. Mathematical activity traces for both Fibonacci settings were developed in which major episodes of the sessions were identified and characterized. The students themselves were interviewed as to what they observed about their own thinking and researcher observations about it. These deliberations are summarized in the pathways (or jumps) on the Pirie/Kieren model.

Results and Reflections

This research is part of a multi-year study of university student mathematical understanding which itself is part of a larger eight year study of the growth in mathematical understanding in action involving students of many ages. The data gathered and interpretations developed in even these two settings represent a multi-dimensional phenomenon. This is true both because growing mathematical understanding is observed to have a recursive fractal character and because the enactive view which we are taking encourages us to consider many elements in the interaction between the students and the world they are both creating and living in "all-at-once". The report here is limited to only some of the dimensions of the situation, particularly growing understanding and patterns of reasoning as these arise for these two students.

We turn first to the "interventionist" Fibonacci setting growing understanding in which is illustrated in Diagram 1, above. The transcript and subsequent reflections of the researchers indicate that when Stacey and Kerry worked on each of the given tasks (e.g. defining the Fibonacci sequence; describing the character of every third Fibonacci number) mainly by trying a number of numerical cases and looking for a pattern. While in the case of prime indexed Fibonacci numbers the pair developed a numerically based property of the sequence (3.1-3.5 in diagram 1) and also engaged in some disjoint formalizing on the task of finding a pattern in F_{3n}, they did not develop well formulated generalizations about the Fibonacci sequence. In a series of interventions with *provocative intent*, the researchers then offered the pair of students a number of tasks calling for looking for inter-relationships within and between various sub-sequences of the Fibonacci sequence, hoping that the students would develop and justify more formal and sophisticated generalizations. But instead each of these suggestions invoked the students to "fold (or in this case jump) back" to working with specific numerical examples and observing a few local patterns or some numerical inter-relationships (paths 5, 6, 7 on Diagram 1). Thus while the researchers had intended that the students formulate their reasoning about this sequence by finding and explaining generalizations (which they had done in other settings), the students instead tended to do short numerical explorations and their discussion with one another remained at the level of particular numerical examples. This pattern of intervention repeated itself several times and the resulting observed understanding appears as a series of jumps.

Notice that in observing Stacey and Kerry's understanding in this setting as such a series of "jumps," we are not devaluing the knowledge of the Fibonacci sequence developed during this activity. Nor would it be appropriate to suggest that this pattern of understanding would be related to what Skemp would call instrumental understanding—Stacey and Kerry could give local justifications for their numerically based images of the Fibonacci sequence. In fact their understanding in this situation might be described as a collection of independent images and their reasoning could be characterized as exploring to seek local patterns. But we are arguing that the students' growth in understanding here co-emerges with the occasions provided in the situation, particularly the apparently discrete tasks in the initial prompt and the interventions of the researcher.

This is well illustrated when we compare the understanding diagram above with that given below. Remember that in the second setting the prompt was to generalize a property of the given Fibonacci sequence and the researchers made no further active interventions. Overall, one might characterize the growing understanding here as a non-linear growth from re-establishing an image(s) of the Fibonacci sequence to more general formalizing about this sequence. While their understanding in the first setting entailed informal reasoning with numerical examples, in the second setting it is better described as *formulating*. Stacey and Kerry re-established their image of the Fibonacci sequence (including their active re-memberence of one researcher "prohibiting" them from developing a bi-directional sequence) (1.1-1.5 on Diagram 2). They then spent the next 45 minutes elaborating and formulating that image. In particular, they focused on how one defines the "givens" for *the* Fibonacci sequence. At 2.5 Stacey noted a property of *the* Fibonacci sequence: that the rule limits the number of givens needed to two. After characterizing their image of the sequence as possibly being members of the natural numbers or integers (4-6), at Stacey's urging they escaped the boundaries of *the* Fibonacci sequence and folded back (7.1) to exploring and formalizing a bi-directional Fibonacci sequence (7.2-7.3). This latter formalizing activity was clearly connected to their previous thinking; for example, Kerry specifically related in his formalizing about the bi-directional sequence to the discussion they had earlier about the members of the sequence being integers (7.3).

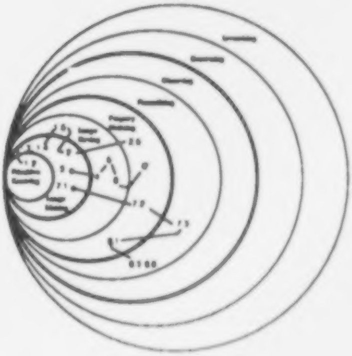


Diagram 2. The second Fibonacci session

Using the bi-directional Fibonacci sequence as a stimulus, Kerry noticed a specific property—one can define the sequence using terms which are separated (F_1=1 and F_{-1} = 1) (8.1, 8.2). They spent the rest of the time formalizing, carefully formulating and re-formulating and justifying a new general definition of *the* Fibonacci sequence (8.3-8.8) as illustrated by this interchange:

Kerry:	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. x_{11}, so, x_1=1 and x_{11}=89. Tada! That's the Fibonacci sequence. I defined it.
Stacey:	Do you always have to start with x_1 = 1?
Kerry:	No, don't have to- I'll get a new x. x_9=34, so- What do we have here? x_{11}=89, x_9=34, that's your Fibonacci sequence. You can not come up with any other sequence if you follow those rules.

It would be easy to conclude that the fact that the researchers made no interventions allowed Stacey and Kerry's understanding to grow in a particular way, but that would be an over-simplification. The researchers' mere presence likely provoked the continuous and more formulated and general formalizing which these two students exhibited. But they also felt constraints that had arisen in the first session. In that session a researcher had noted that *the* Fibonacci sequence had no "zeroth" terms nor any "negative" terms. Although the second setting occurred over a year later, both Stacey and Kerry re-membered and re-constructed this dialogue and this remembrance acted as a constraint on much of their activity, as did their remembered dissatisfaction with their more disjoint previous activity with the problem. Under these constraints they focused on what they called *the* Fibonacci sequence. Their understanding activities can be seen as influenced by what Sierpinska (1994) identifies as epistemological concerns. In fact, their well formulated formalizing (8.1-8.8), and even their whole pathway of understanding centers around generalizing and formalizing conditions underlying *the* Fibonacci sequence.

In the continuous non-linear pathway of growth of understanding in setting two, Stacey and Kerry's reasoning appeared to be governed by a need to explain rather than a need to explore (Reid, 1995). Again we are not saying that such a pathway is evidence of better or more productive understanding than that in the first setting, but that this growing understanding coemerged with the features of the prompt, the (non) actions of the researchers and with the developing epistemological concerns of the students. The students themselves did sense a difference between their understandings in the two settings:

Kerry: Yeah. Cause we- When we walked away from it [setting one] neither one of us felt we'd really climbed a mountain or conquered anything.

Stacey: No.

Kerry: But now, I'm quite happy with it now. The Fibonacci sequence is allowed back in my life.

Stacey: Yeah [laughter].

References

Davis, B. (1994) *Listening to Reason.* An unpublished doctoral thesis at the University of Alberta. Edmonton: University of Alberta, Faculty of Graduate Studies, Department of Secondary Education.

Herscovics, N. and J. Bergeron (1993) An epistemological study of the child's construction of early number. *Alberta Journal of Educational Research* 39 (1) pp. 23-39.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason.* Chicago: University of Chicago Press.

Kieren, T., Pirie, S. & Reid, D. (1994). Mathematical understanding: Always under construction. In J. da Ponte & J. Matos, (Eds.), *Proceedings of the Seventeenth Annual Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 49-56). Lisbon.

Pirie S. and T. Kieren (1992) Creating Constructivist Environments: Constructing Creative mathematics. *Educational Studies in Mathematics.* 23(5) pp. 505-528.

Pirie S. and T. Kieren (1994) Growth in mathematical understanding: how can we characterise it and how can we represent it? *Educational Studies in Mathematics* 26: pp. 165-190.

Reid, D. (1995) *The need to prove.* Unpublished doctoral dissertation, University of Alberta Edmonton: University of Alberta, Faculty of Graduate Studies, Department of Secondary Education.

Sierpinska, A. (1994) *Understanding in mathematics.* London: Falmer.

Skemp, R. (1976) Relational Understanding and Instrumental Understanding. *Mathematics teaching,* No. 77, pp. 20-26.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience.* Cambridge MA: MIT Press.

von Glasersfeld, E. (1987) Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the learning and teaching of mathematics.* (pp. 3-17). Hillsdale NJ: Lawrence Erlbaum Associates.

HOW STUDENTS ESTABLISH THE TRUTH OF THEIR IDEAS IN SCHOOL GEOMETRY

Michael Mikusa, Kent State University

The main focus of the study was to describe how students of various ages established the truth of their ideas in school geometry. Thirty-two students in grades 2, 5, 7, and in a high school geometry class were interviewed. The study found that formal proof was used in less than 1% of the student arguments. Second and 5th graders were most likely to convince themselves or others by using a basic image process or by drawing pictures. High school students and 7th graders were more likely to convince themselves and others using an Intuitive Affirmation. In addition, their arguments were more elaborate and propositional in nature than those arguments given by 2nd and 5th graders.

The NCTM *Curriculum and Evaluation Standards* subscribe to a constructivist view of mathematics learning and teaching in which students learn mathematics meaningfully as they personally construct mental structures and operations that enable them to deal with problematic situations, organize their ideas about the world, and make sense of their interactions with others (National Council of Teachers of Mathematics (NCTM), 1989). The constructive process occurs as students reflect and make sense of their interactions with the world and their peers. In this new view of mathematics learning and teaching, primary responsibility for establishing the truth of mathematical ideas lies with students. Teachers and textbooks are no longer viewed as the providers of mathematical truth. In essence, each student is seen as a mathematician, somebody who is responsible for solving mathematical problems by making conjectures and establishing the validity of those conjectures within the classroom culture. As we place such a heavy responsibility on students, it behooves us to know how they cope with it. We know how mathematicians formally establish the truth of conjectures—they use proofs. But how do students do it? How does the notion of justification evolve in students? Furthermore, if we want to help students learn increasingly sophisticated ways of justifying their mathematical conjectures, we must understand their current ways of justifying ideas.

Procedures

Age and relevant knowledge have been found to be important variables in much of the research on reasoning. Eight second graders, ten fifth graders, eight seventh graders, and six high school students, were randomly selected from a pool of volunteers in two similar school systems. Approximately half of those selected at each grade level were females. To assure that students selected had an adequate knowledge of mathematics, only students with standardized mathematics test scores above the fiftieth percentile were included.

A set of nine problems involving concepts in geometry was selected from a variety of sources so that the problems could be easily understood by students at all age levels in the study (i.e., the problem did not require much formal geometry knowledge or terminology to understand). For example, problem number 7 gave the students a figure of a triangle inscribed in a circle and asked "Is it true that for every triangle that there is a circle that passes through each of the vertices (the three points) of the triangle?"

After solving each problem, students were asked two follow-up questions. The first question was "What makes you think that your answer is correct?" The second was "When I gave this problem to some other students, some of them gave answers different from the one you gave me. If each one of you that gave me a different answer has a chance to convince a group of students (in the same grade) that your own idea is correct, how would you get the others students to believe you?" A tenth interview item explicitly asked students how they establish truth of their ideas in mathematics.

All of the interviews were conducted by the author. The interviews were audio taped, then later transcribed. During the interviews the interviewer wrote on the interview form as the answers were given. In addition to student verbal responses, the interviewer kept track of student drawings used in conjunction with verbal responses.

Coding of Student Justifications

After the students were interviewed and tapes were transcribed, student responses were analyzed. An initial set of the primary components was created from both relevant research and student responses in this study. The first result of the study was an elaboration of the set of primary components in order to accurately describe important lines of reasoning used by the students. Coding began with two people using the set of primary components, each coding a sample of student arguments. These codings were compared and disagreements were discussed. Seventeen primary components were identified and refined (see Table 1 for abbreviated list of primary components) in this study. When this process was completed, a third person, not involved in the development of the primary components, coded a random sample of student responses. The codings were compared with the refined codings, and a rate of 90% agreement was computed. After this process was complete, the primary components were used to complete a final coding of all student responses.

Findings from primary component analysis

Approximately 70% of component usage was accounted for by four components: Draws or Proposes to Draw a diagram (DM), Intuitive Affirmation (IA), Basic Image Process (IS), and Statement of Fact (SF). Furthermore, formal proof (UP) accounted for less than 1% of the 896 arguments given (see Figure 1).

Primary component preference differed somewhat by grade level. The use of Intuitive Affirmation (IA) was greatest among high school students, and the use of both Basic Image Process (IS) and Draws or Proposes to Draw (DM) were most popular with second and fifth graders. The findings also indicated an increased use of Statement of Fact (SF) as grade level increased. Table 2 on the next page

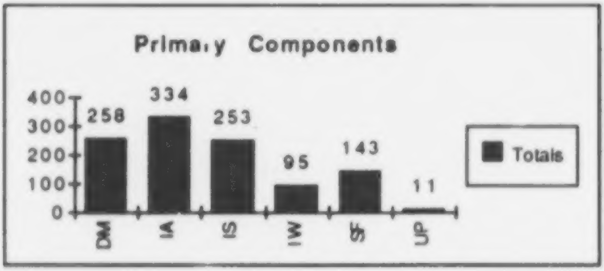


Figure 1. Most frequent primary components used in student responses

displays the percent of use of each component over all student responses by grade level. That is 23% for second grade indicates that of all components used by second graders, 23% of them were Basic Image processes (IS).

Almost 80% of students' argument chains (the string of primary components used in an entire student response to a question) were one or two components in length, with the most popular being the singleton chains Intuitive Affirmation (IA) and Basic Image Process (IS). While brevity is a respected quality of mathemati-

Table 1. *Primary Components: an abbreviated list*

Code	Name	Description of Primary component
DM	Draws, proposes to draw	The student draws or proposes to draw objects or does some kind of manipulation (folding, cutting, or the like) to support his or her argument.
IA	Intuituve affirmation	The student makes a statement that he or she accepts as certain and self evident. No validation of the statement is attempted.
IS	Basic image process	(with direct support of diagrams) - The student draws inference from generating, transforming, or inspecting pictures he or she creates or finds.
IW	Advanced image process	The student draws an inference by generating, transforming, or inspecting images without drawings or existing diagrams.
SF	Statement of (or appeal to) fact	The student refers to facts, definitions, or formulas that he or she assumes is common knowledge.
UP	Uses (or proposes to use) proof	The student presents a set of statements each being justified by theorem, axiom, or definition. In proposing a proof, the student presents an outline or explicit direction of how a proof would be constructed.

Table 2. *Most frequently used Primary Components used by grade level*

Grade	**Primary Components**				
	DM	**IA**	**IS**	**IW**	**SF**
Second	18%	21%	23%	9%	7%
Fifth	22%	18%	21%	7%	7%
Seventh	17%	23%	17%	5%	11%
High School	12%	32%	7%	4%	15%

Table 3. *Number of incorrect components and total components used in incorrect chains*

	DM	IA	IS	IW	SF	UP
Total incorrect components used	1	45	91	35	11	0
Total Components used	122	81	127	41	44	0

cal proof, it is so only when such arguments are sufficiently rigorous. The student chains did not in most cases constitute acceptable rigor. Thus, in this study, the brevity of the arguments seemed to indicate a weakness rather than a strength.

Errors in student arguments

In addition to coding student arguments, each primary component and the argument as a whole were judged to be correct or incorrect. For example a student may have used a Statement of Fact as part of their argument, but stated the fact incorrectly. If this incorrect fact caused the argument as a whole to be incomplete or incorrect, the whole argument was coded incorrect. This data was used to determine what types of errors caused arguments to be faulty.

Fifty percent of all the arguments given by students were incorrect or incomplete. The cause of many of the errors in student arguments was the incorrect or insufficient use of imagery. The most common error that students who were using visual thinking made is that they failed to utilize appropriate propositional knowledge to constrain their thinking or recognize its possible inadequacies. Conclusions from this study indicate that imagery is productively used in problem solving when it is guided and constrained by appropriate propositional knowledge. There is evidence in this study to suggest that as students move up through the van Hiele levels, they don't necessarily stop using visual reasoning. Instead, their visual reasoning becomes more sophisticated, incorporating into it increasingly more sophisticated propositionally stored knowledge. Their visual thinking is different at different van Hiele levels because, at each level, it is constrained by totally different knowledge structures.

Conclusions

I believe this study provides essential information for guiding instructional strategies aimed at promoting and refining students' geometric reasoning because if we wish to help students refine how they reason, we must first understand their current methods of reasoning.

The preference for more verbal rather than visual arguments for the high school geometry students and seventh graders seems consistent with the notion that they were thinking about geometric ideas at van Hiele's second or property-based level. The preference of younger students for visual arguments suggests that the students in the second and fifth grade were thinking about their ideas at van Hiele's first or visual level. Students at the visual level believe in geometric ideas because they "just see it" (van Hiele, 1986) or because of visual transformations (Battista, 1994). Students at higher levels reason based on more elaborate, property-based knowledge. Thus, in addition to visually stored knowledge, students at the higher levels have propositionally stored, property-based knowledge that can be included in their arguments.

The frequent uses of Intuitive Affirmations and Basic Image Processes and the errors in arguments caused by these primary components in this study suggest the need to present students with situations which help them to build and coordinate the use of propositional knowledge with visual knowledge. I believe that exploring geometry using manipulatives or computers, creating conjectures, and then arguing about those conjectures with classmates is essential in helping students developing these mental processes. This study also suggests that having students try to convince others of their mathematical ideas not only forces them to reflect on their ideas, but to elaborate these ideas, making them more mathematically explicit. This finding thus supports constructivist notions of the value of class discussions in which students must argue and support their mathematical ideas. That is, students must be given opportunities to create mathematical ideas, and most importantly, to decide for themselves if these ideas are mathematically sound.

References

Battista, M. T. (1994). On Greeno's environmental/model view of conceptual domains: A spatial/geometric perspective. *Journal for Research in Mathematics Education, 25,* 86-94.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA.: Author.

van Hiele, P. M. (1986). *Structure and insight.* Orlando: Academic Press.

HOW STUDENTS USE THEIR KNOWLEDGE OF CALCULUS IN AN ENGINEERING MECHANICS COURSE

Cheryl Stitt Roddick, The Ohio State University

This study investigated students' conceptual and procedural understanding of calculus within the context of an engineering mechanics course. Four traditional calculus students were compared with three students from one of the calculus reform projects, Calculus & Mathematica. Task-based interviews were conducted with each participant throughout the course of the ten-week quarter. Results from interviews show a distinct difference in approaches to solving engineering mechanics problems that involve calculus. Calculus & Mathematica students, who learned calculus with a conceptual emphasis, were found to be more likely to solve problems from a conceptual viewpoint than were the traditional students, who were more likely to focus on procedures.

The introduction of technology in the calculus classroom has been met with mixed emotions. Many enthusiastic supporters have emerged, yet there have also emerged many critics of the quality of learning that occurs. One major criticism has been that students who learn calculus with the help of technology will not have the skills to be successful in later calculus-dependent courses (Krantz, 1993). Others argue that these students have a stronger conceptual understanding and will have an advantage over students who have taken traditional calculus. Supporters of Calculus & *Mathematica*, a calculus reform project which utilizes the computer algebra system *Mathematica* in students' learning of calculus, believe that the use of technology, together with teaching techniques based on constructivist theory, can encourage student ownership of knowledge and a strong conceptual understanding (Davis, Porta & Uhl, 1994).

This study addressed these issues and investigated students from the Calculus & *Mathematica* sequence as they continued their education beyond calculus. Since calculus is a stepping stone to many other courses, success in future calculus-dependent courses may be determined in part by students' experiences in their calculus courses. The focus of the study is a comparison of Calculus & *Mathematica* students with traditional students on their conceptual and procedural understanding of calculus when applied to different situations. An introductory engineering mechanics course was chosen as the course in which to investigate students' understanding of calculus.

Theoretical Framework

Bell, Costello, and Kuchemann (1983) specify five components of mathematical competence: facts, skills, concepts, general strategies, and appreciation. Two of these components — skills and concepts — are the focus of this study. *Skills* are defined to include "any well-established multi-step procedure, whether it involves symbolic expressions, or geometric figures, or neither" (Bell et al., 1983, p. 78). The essential features of skills include actions or transformations that are connected in a linear fashion. *Conceptual understanding* describes "knowledge that is rich in relationships...[where] all pieces of information are linked to some network" (Hiebert & Lefevre, 1986, p. 3).

The proposed study seeks to explore Calculus & *Mathematica* students' conceptual and procedural understanding of calculus applied to engineering mechanics problems. Shumway (1982) proposes that problem solving can really be investigated by looking at the conceptual and procedural knowledge involved. He observed that the goal of problem solving is to identify a class of problems that can be solved in a similar way. But the process of identifying a class involves conceptual knowledge, whereas determining and carrying out a procedure involves procedural knowledge. So what is really happening during problem solving is that the solver is using conceptual knowledge to reduce a problem to one that can be solved using procedural knowledge. "One could argue that problem solving ends and concept learning begins when one begins looking back, identifying similar problems, and engaging in other post-solution activities" (Shumway, 1982, p. 134).

Silver (1986) believes that it is important not to focus on the distinctions between conceptual and procedural knowledge, but rather to focus on the *relationship* between the two types of knowledge, since problem solving in reasonably complex knowledge domains involves the application of both. Silver suggests we consider the idea that procedural knowledge that is not connected to conceptual knowledge is rather restricted knowledge (Silver, 1981). Thus, a study of the linkages between the two types of knowledge is advised when investigating problem solving.

Mayer and Greeno (1972) found that students who are taught using a conceptual focus produce learning outcomes that are qualitatively different than those produced by students taught with a procedural focus. Their belief is that a conceptual focus encourages the development of a cognitive structure which is more externally connected, or related to other elements in the general structure. This type of cognitive structure would be more useful when faced with problems that may not be familiar to the student. Furthermore, Mayer (1974) examined the resilience of an initial acquired structure and found it to be resistant to change, noting that "an assimilative set is evoked quite early in learning and that content material is structured within the context of the set over the entire course of learning" (p.655). This finding suggests that students who initially learn conceptually will continue to structure new material in the same manner, forming a more externally connected cognitive structure.

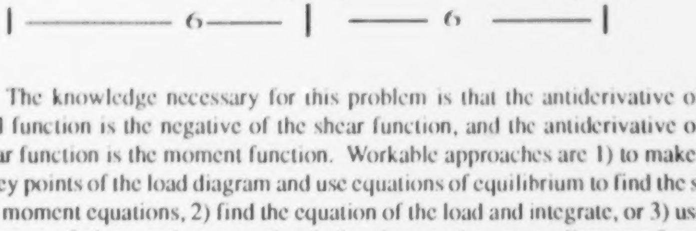
This theory correlates closely with the goals of the Calculus & *Mathematica* sequence, one of which is to promote a conceptual emphasis on the process of problem solving. The framework leads to the hypothesis that Calculus & *Mathematica* students will be better able than traditional calculus students to structure new material from engineering mechanics in a conceptual manner, and will have developed stronger links between conceptual and procedural knowledge.

Methodology

Seven engineering mechanics students were chosen to participate in task-based interviews designed to investigate students' use of calculus in their mechanics course. Three of these students have completed the calculus sequence Calculus & *Mathematica*. Three students have completed the traditional calculus sequence, which employs a lecture-recitation format without the use of technology. The other student has completed an honors section of the traditional course. Two factors were taken into consideration when choosing the engineering mechanics course. Most importantly, the course had to include calculus as one of the prerequisites. This consideration was made to ensure that students had encountered calculus previously and were not learning it for the first time in this course. Secondly, the course had to be one that many students from Calculus & *Mathematica* take. Since a great number of Calculus & *Mathematica* students are engineering majors, the focus was placed on courses that are required for all engineering majors. The engineering mechanics course is an introductory study of statics and mechanics which has a prerequisite of at least three quarters of calculus and one quarter of physics. The main use of calculus in this course is with concepts of differentiation and integration.

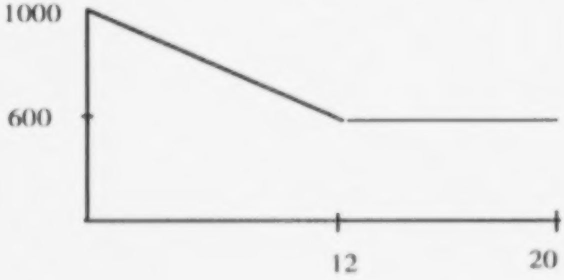
Data Analysis and Results

Data from the task-based interviews were used to investigate how Calculus & *Mathematica* students compare with traditional students on their procedural and conceptual understanding of calculus as evidenced by their ability to solve problems in an engineering mechanics course. Several of the problems presented to the students could be solved using either a procedural or conceptual approach. The greatest differences in approach were found in a problem which asked students to sketch shear and moment diagrams for the following load on a beam:



The knowledge necessary for this problem is that the antiderivative of the load function is the negative of the shear function, and the antiderivative of the shear function is the moment function. Workable approaches are 1) to make cuts at key points of the load diagram and use equations of equilibrium to find the shear and moment equations, 2) find the equation of the load and integrate, or 3) use the concepts of slope and area to sketch the shear and moment diagrams from the given load diagram. The first approach does not require knowledge of calculus, while the other two approaches do. All of the Calculus & *Mathematica* students initially approached this problem in a conceptual manner, using their knowledge of slopes and areas in relation to functions and their antiderivatives. If the need arose, various computations would be used as a supporting method. When asked whether they could confirm their solution in another way, each of the Calculus & *Mathematica* students responded with the procedural approach, finding the equation of the load and integrating. In sharp contrast, the preferred method of the traditional students was procedural. These students employed either the cut method involving equations of equilibrium or the integration method. Some use was made of the concepts of slope and area, but the prevailing method was procedural in nature.

One problem related to shear and moment diagrams was designed to assess students' conceptual knowledge of the relationship between shear and moment. Given the following sketch of the shear diagram and an initial moment value of -15,600, students were asked to sketch the moment diagram. Students were not asked to find the moment functions, but were required to sketch the general shape and to locate the values of the moment for x = 12 and x = 20.



Since the shear function is the derivative of the moment function, students could find the change in moment by determining the area underneath the shear curve. Again, all Calculus & *Mathematica* students approached this problem conceptually and solve it with ease. One student explains his approach: "The shear is the derivative of the moment so it would start at -15,600 and increase to a certain point. The area of this (shear) would be what it (moment) increases to." When asked how he knew the value of the moment was always increasing for x ranging from 0 to 20, he replied, "for it to come back down somewhere this would have to be negative....Since the area is all positive this bottom line will always be increasing." This same student, however, made an initial conjecture that the value of the moment for x = 20 would be 0, because ""it always ends at 0." This belief seemed to be prevalent among the students interviewed. Upon completion of his solution he changed his response.

One of the traditional students (who completed the honors section of calculus) solved the problem conceptually. He explained, " you can just find the areas under the shear diagram and add it to the moments as you go along. So M(12) would be M(0) + Area 1; M(20) = M(0) + Area 1 + Area 2." (Area 1 is the area for x = 0 to 12; Area 2 is the area for x = 12 to 20.) Two traditional students attempted to find the shear functions and integrate, but both made mistakes and were unable to arrive at the correct answer. Of these two students, one made a small integration mistake, and could not suggest any other way to check his work. The other student insisted that the moment always ends at 0 and neglected to consider the value of the moment at $x = 12$. The fourth traditional student made several different attempts yet failed to arrive at a reasonable answer.

Several problems addressed knowledge of specific procedures. One of the problems that addressed procedural knowledge involved finding the x-coordinate of the centroid of the area bounded by $y^2 = 2x$, $x = 3$, and $y = 0$. All four of the traditional students were able to solve this problem. They remembered the formula and were able to perform the integration without assistance. One student, who was having some difficulty responding to some of the earlier questions, expressed his confidence with this particular task. "Oh, yes, this I can do," he said. The three Calculus & *Mathematica* students were also able to solve the problem. One of the students integrated incorrectly, but corrected himself when asked to check his work.

This integration problem is representative of the difficulty level required for the engineering mechanics course. In fact, students were instructed to deal with more challenging integration by using an integral table. None of the students, traditional or Calculus & *Mathematica*, felt uneasy with the differentiation and integration skills required.

Conclusions

Results from interviews show a distinct difference in approaches to solving engineering mechanics problems that involve calculus. Calculus & *Mathematica* students, who learned calculus with a conceptual emphasis, were found to be more likely to solve problems from a conceptual viewpoint than were the traditional students, who were more likely to focus on procedures. These results are consistent with Mayer's (1974) finding that a student's initial cognitive structure is resistant to change. Furthermore, students who expressed the most confidence in their solution were found to have used a combination of conceptual and procedural knowledge. Calculus & *Mathematica* students demonstrated a stronger ability to discuss all aspects of a problem, including both conceptual and procedural issues, while traditional students expressed more uncertainty in their work and were less comfortable in discussions as to how to use other knowledge to check their solutions.

References

Bell, A. W., Costello, J., & Kuchemann, D. E. (1983). *A review of research in mathematical education, Part A: Research on learning and teaching.* Windsor, Berks: NFER-Nelson Publishing.

Davis, B., Porta, H., & Uhl, J. (1994). *Calculus & Mathematica.* Reading, MA: Addison Wesley.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*, (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Krantz, S. G. (1993). *How to Teach Mathematics: A Personal Perspective*. American Mathematical Society, Providence, RI.

Mayer, R. E. (1974). Acquisition processes and resilience under varying testing conditions for structurally different problem-solving procedures. *Journal of Educational Psychology, 66(5)*, 644-656.

Mayer, R. E. & Greeno, J. G. (1972). Structural differences between learning outcomes produced by different instructional methods. *Journal of Educational Psychology, 63*(2), 165-173.

Shumway, R. J. (1982). Problem-solving research: A concept learning perspective. In F. K. Lester & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research* (pp. 131-138). Philadelphia: The Franklin Institute Press.

Silver, E. A. (1981). Recall of mathematical problem information: Solving related problems. *Journal for Research in Mathematics Education, 12*(1), 54-64.

Silver, E. A. (1986). Using conceptual and procedural knowledge: A focus on relationships. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*, (pp. 181-198). Hillsdale, NJ: Lawrence Erlbaum Associates.

CAUSES OF ACADEMIC SUCCESS, PERCEIVED ABILITY, AND UNDERSTANDING OF LIMITS AND CONTINUITY

Umaru A. Saleh, State University of New York at Oswego

This study examined perceived causes of academic success using Weiner's (1979) attribution theory. The Perceived Causes of Success instrument developed by Nicholls (1989) was administered to 75 first semester calculus students, after they had studied the concepts of limits and continuity, to determine the factors (internal or external) responsible for calculus success. Internal factors were ability and hard work and external factors were luck and easy material (Sohn, 1982).

The students were divided into four quartile groups based on their performance on a test designed to assess understanding of limits and continuity concepts developed by the investigator. The test had a Cronbach coefficient alpha of .68. The results showed that an overwhelming majority of the students in each of the performance groups attributed calculus success to internal factors (ability and hard work). Based on the findings of this study, it was clear that irrespective of their performance, students made internal attribution to academic success. The findings of this study were contrary to the view by Simon and Feather (1973) that students attribute their good performance to internal factors and their poor performance to external factors.

The second part of the study investigated the relationship between perceived ability and understanding of limits and continuity. Perceived ability instrument, with a Spearman–Brown alpha of .84, developed by Nicholls (1989) was used to assess students' perceived ability. Student understanding was measured by performance on a test designed to assess understanding of limits and continuity developed by the investigator. The test had a Cronbach coefficient alpha of .68. Simple and Partial correlation analyses, partialing out math ACT scores to control possible initial math ability differences, were used to compare perceived ability and understanding. The results showed a positive correlation between perceived ability and understanding. However, the positive correlation became lower when initial math ability differences, as measured by math ACT, were controlled.

References

Nicholls, J.G. (1989). *Competitive ethos and democratic education.* MA: Harvard University Press

Simon, J. G. & Feather, N. T. (1973). Causal attributions for success and failure at university examinations. *Journal of Educational Psychology.* 64(1). 46-56.

Sohn, D. (1982). Sex differences in achievement self-attributions: An effect size analysis. *Sex Roles, 8(2),* 345-357.

Weiner, B. (1979). A theory of motivation for some classroom experiences. *Journal of Educational Psychology, 71(1).* 3-25.

CHINESE ADULT STUDENTS' INTERPRETATION OF THE MATHEMATICAL SYMBOL ". . ."

Brenda Lee, Wu Feng Institute of Technology and Commerce

The infinite sequence is one of the most important types of the discrete functions. In order to understand infinite sequences, one needs to understand the mathematical symbol". . .". Numerous studies show that students have difficulty accepting the fact that 0.999...=1. It has occurred to the present researcher that part of the difficulty might lie in the representation. What does the "..." symbol mean to the students? Do they have the definite finite view when dealing with the "..." symbol? The mathematical symbol "..." has at least two important meanings: one indicating a definite finite set, such as in (1, 2, 3, 4, ..., 10), and the other an infinite set, such as in F(1,3)=0.333... If one has a finite concept image of the mathematical symbol "...", one would never be able to accept that 0.999... is equal to 1. Thus the researcher was interested in finding out whether the finite concept image affects the understanding of the limit concept. To find that out, the researcher needed to find out students' interpretation of the mathematical symbol "...".

Methods

Two hundred and fifty-seven night school adult students, age 20 or over, from six different calculus classes participated in this study. First, students were formed in small groups to discuss: (1) the meaning of the symbol "...", (2) why we need the symbol "...", and (3) whether there exist other mathematical symbols to represent the meaning of "...". Then, they were asked to give two examples to describe the meaning of "..." in their final examination.

Results

Based on the results of students' examples, the following were the most popular meanings of the symbol "...": infinity, more to come, the sentence was interrupted, continuation, "et cetera", repeating decimals, simplify, omission, the same numbers, or approximation. There were some others like three dots.π, e, $\sqrt{}$, 2, incomplete, and irrational numbers. More than half of the students were unable to provide examples. Among those who did provide examples, more than half of them were unable to describe the meaning of the symbol of "...".

Conclusion

The "..." symbol is used frequently in advanced mathematics, but actually occurs quite early in the usual mathematics curriculum. For example, the equation F(1,3)=0.333...is encountered in learning division in elementary school. Other examples include a definite finite list, such as {1, 2, 3, 4, ..., 10} or infinite sequences, such as 1, 2, 3, 4, ...or 1, F(1,2), f(1,4) , ..., $F(1,2)_0$, ... The students may encounter the "..." symbol again in learning the binomial theorem $(a+b)^n=a^n+na^{n-1}b+\ldots+(^n)a^{n-k}b^k+\ldots+b^n$ in high school. The symbol "..." is used when for some reason we are unwilling or unable to write down all parts of a mathematical statement. It is loosely equivalent to the phrase "et cetera". However, the meaning of "..." differs somewhat in subtle ways in various contexts, as shown by the examples above. Inability to decide whether the symbol "..." indicates a definite finite set or an infinite set is the major cognitive difficulty. The other cognitive difficulty might be the confusing of mathematical terminology with the daily usage of "et cetera".

UNDERSTANDING THE LIMIT: EXPERTS, NOVICES, AND STUDENTS OF THE ANALYSIS SEQUENCE

David E. Meel, University of Pittsburgh

The limit concept is a broad and abstruse subject matter that does not avail itself to easy elucidation of the conceptual framework that comprises the understanding, and, according to Michner (1978), it takes multiple experiences with a topic to develop an understanding. The purpose of this study was to qualitatively examine the understandings of the concept of limit held by four different groups of subjects with differing exposures to the limit concept.

This poster presented the qualitative analysis of student responses to an assessment designed by the researcher and guided by previous research on student understandings of the limit concept. Using the assessment, the understandings of 18 subjects were examined. These 18 subjects were partitioned into 4 groups: those who had not taken a calculus course (NOVICE), those who had completed at least one semester of calculus but had not taken either a Classical Analysis course or Real Variables course (CALC1), those who had taken a Classical Analysis course but not a graduate level Real Variables course (CLASS), and lastly those who had taken both a graduate level Real Variables course and taught calculus (EXPERT).

The assessment instrument focused on the subjects' connections within the concept as well as the external linkages between the limit concept and other mathematical topics. A host of information was garnered from the subjects' responses. For example, the understanding of the terminology associated with the limit concept, "limit", "approaches", "converges", and "tends to", appeared to be differential between groups. In addition, the assessment also examined student acceptance of the various case-restricted definitions delineated in the literature, and the results uncovered between group differences and other interesting results. Many of these case-restricted definitions were documented in the literature and their construction has been attributed to the development of informal definitions of the limit concept based upon an incomplete set of concept images. This poster explicated these differences and indicated that the NOVICE group was considerably more accepting of case-restricted definitions and their implications than the EXPERT group. Additionally, the EXPERT group was found to be accepting of a couple of case-restricted definitions. For example, the case-restricted definition "A number is the limit (*L*) of a sequence if you make better approximations to the end of the sequence you find a value (*L*) to which the terms are stabilizing" was accepted by several subjects of the EXPERT group. Suggestions and recommendations for aiding students in the development of an understanding of the limit concept were also presented.

THE USE OF WORKSHEETS, HINT-SHEETS AND SOLUTION-SHEETS IN THE GUIDED DISCOVERY STYLE TEACHING OF CALCULUS

George L. Emese, Rowan College of New Jersey

Research results on discovery learning are still conflictive and inconclusive. For each experiment showing the discovery approach superior over exposition there is another experiment with the opposite result. A great number of arguments have been given both for and against, a great number of advantages and disadvantages have been listed.

A critical issue in discovery style teaching is the amount of guidance and how it is provided. This project examined how worksheets, hint-sheets and solution-sheets can be used to provide guidance. In the worksheets a chain of questions and problems led to the new concept, relationship or technique. The hint-sheets contained leading questions, some suggestions on how to solve the problem, or the first step of the solution. Finally, the solution-sheets contained complete solutions to the problems.

The worksheets, hint-sheets and solution-sheets were designed to minimize the following difficulties with discovery learning. Students discover at different paces. Without worksheets, hint-sheets or some similar techniques only the fastest students would discover; others would be involved in reception learning, listening to the "discoverer". Students can discover only a tiny fraction of the accumulated knowledge of a culture (Skinner, 1968). With the use of the hint-sheets this fraction can be significantly increased. Research shows that discovery is very time-consuming (Orton, 1987). The hint-sheets can be of help here: if students are unsuccessful they can get help instantly, and they can move on to the next step or next problem in the chain of the discovery process. There may be some students who never come up with a discovery of their own. This can create jealousy, resentment or feelings of inferiority (Biehler and Snowman, 1982). With the use of hint-sheets all students may make at least *some* discovery of their own.

REFERENCES

Biehler, R. F. & Snowman, J. (1982). *Psychology applied to teaching*. Boston: Houghton Mifflin.

Orton, A. (1987). Learning Mathematics: Issues, theory and classroom practice. London: Cassell.

Skinner, B. F. (1968). *The technology of teaching*. New York: Appleton-Century-Crofts.

Algebraic Thinking

MATHEMATICAL CONTEXTS AND THE PERCEPTION OF MEANING IN ALGEBRAIC SYMBOLS

Anne R. Teppo, Montana State University
Warren W. Esty, Montana State University

This paper presents an analysis of the different types of meanings that an individual may assign to a collection of algebraic symbols depending on the mathematical context in which the symbols are presented and the mathematical knowledge possessed by that individual. Four contexts for the Quadratic Theorem are used to illustrate the ways in which generalization and abstraction develop the meaning of algebraic entities by changing focus from process to structure.

Research investigating students' construction of mathematical ideas can be enriched by including analyses of the mathematical structures under study. It is important for researchers to be aware of the "implicit, unspoken assumptions about the nature of the concepts being considered" (Tall, 1992, p. 508). Behr et al. (1994, p. 124) recommend a "deep, careful, and detailed analysis of mathematical constructs both to exhibit their mathematical structure and to hypothesize about the cognitive structures necessary for understanding them." This paper uses an analysis of the mathematical concepts embodied in the Quadratic Theorem to investigate mathematical structures and processes involved in the development of algebraic thinking.

The Quadratic Theorem: If $a \pm 0$, then $ax^2 + bx + c = 0$ is equivalent to

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Quadratic Theorem is used to solve equations. As with many other theorems, it expresses an abstract symbolic problem-pattern, "$ax^2 + bx + c = 0$" (if $a \pm 0$), and gives a corresponding solution-pattern,

$$\text{“}\, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \text{”}$$

This theorem aptly illustrates how the language of algebra can be used as a highly effective medium for expressing mathematical thoughts. However, the meaning that is assigned to such a symbolic sentence depends upon the knowledge of the reader and the mathematical context in which the sentence appears (Sfard and Linchevski, 1994). Four different contexts related to the Quadratic Theorem are presented to illustrate how the perception of meaning may vary according to the kind of mathematical constructs an individual is prepared to notice.

Context 1: Quadratic Formula

Evaluate $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for $a = 5$, $b = 2$ and $c = -3$.

The problem-pattern, "$ax^2 + bx + c = 0$," plays no role in this example. Only the second half of the theorem, known as the Quadratic Formula, is given. Here the collection of symbols performs the role of a formula, which represents a different type of conceptual entity than that of the whole Quadratic Theorem. In such a "plug in" problem the context is numerical. The reader only needs to interpret the symbol sentence as a set of directions for computing a number.

Context 2: Problems appear only in a limited, simplified quadratic form.

Use the Quadratic Theorem to solve the following:

a) $5x^2 - 7x - 12 = 0$.

b) $10 - 2x^2 + 7x = 0$.

c) $3x^2 = 6x + 14$.

d) $x(x + 1) = 7$.

e) $dx^2 + kx = c$.

This context requires the reader to focus attention on the pattern of coefficients in the problem-pattern of the theorem, a pattern that is ignored in Context 1. These five problems emphasize the concept of parameters and the role of symbols as placeholders (which are also called dummy variables). Although the computational process is the same as in Context 1, the numbers (or expressions), to which the process applies must first be identified. The reader must be able to perceive that the problem-pattern "$ax^2 + bx + c = 0$" represents a generalization that is common to the collection of symbols in each equation. Only equation (a) utilizes the theorem's given left-to-right alphabetical order. The other equations require interpretation of what the problem-pattern is intended to represent.

Equations (a) through (d) can also be solved numerically (instead of algebraically) simply by graphing the two component expressions of each equation and noting the x-values where they are equal, without any reorganization or identification of parameters. The numerical approach avoids using the theorem, and thus avoids the necessity to discriminate between the different symbolic roles of "a," "b," and "c."

The role of the problem-pattern in the Quadratic Theorem is to abstractly describe the type of problem to which the theorem applies. In Context 2 this type is distinguished by the appearance of "x^2" and "x" in each equation, using the particular symbol "x." As such, these equations are a very specific representation of the abstract problem-pattern. This application of the Quadratic Theorem does not require the reader to regard the symbolic "x^2" as representing the operation of squaring as opposed to the result of that operation applied to "x."

Context 3: Problems where the squaring does not apply to an unknown "x."

a) In the Law of Cosines, solve for b: $c^2 = a^2 + b^2 - 2ab \cos C$.

b) Solve for x: $\sin^2 x = \sin x - .2$ [given the ability to solve "$\sin x = c$"]

c) Using a graphics calculator, graph: $y^2 + 3xy + x^2 = 14$.
[When equations must be entered in the form "$y =$ ...."]

In the Quadratic Theorem "x" is just as much a dummy variable as "a," "b," or "c." The role of "x^2" in the problem pattern is to represent squaring (the operation) applied to *any* expression, not just to "x." The problems in Context 3 require a shift in understanding of the nature of the conceptual entities represented by the variables in the given Quadratic Theorem. In Context 2 the signifiers (x^2 and x) directly represented that which they signified. Even though the quadratic nature (the squaring) of the equations was apparent, it did not need to be the focus of attention since the theorem could be applied through a one-to-one matching of patterns of symbol strings.

In contrast, in Context 3 the algebraic symbols in the Quadratic Theorem must be perceived as representing sequences of operations, not just strings of similar symbols. Although "x^2 " may represent a number, the purpose of "x^2" in the theorem is now seen as representing squaring, even if it is not "x" that is squared. For example, in part (c) "y" plays the role of "x" in the theorem and "$x^2 - 14$" is represented by the symbol "c." To recognize that the Quadratic Theorem is relevant in Context 3 it is necessary to regard squaring as an object divorced from a particular symbolic representation.

The quadratic nature of the three equations can no longer be determined by a direct correspondence to specific symbols in the problem-pattern of the Quadratic Theorem. For example, in equations (a) and (c) squaring may appear more than once. In equations (a) and (b) it is not "x" that is squared and in equation (c) "x" does not represent the unknown. It may be particularly difficult to recognize the relevance of the Quadratic Theorem to graphing the equation in equation (c).

Context 4: A textbook's statement of a theorem.

a) The Quadratic Theorem.

b) The Theorem on Absolute Values: $|x| < c$ is equivalent to $-c < x < c$.

In Context 4 the theorem itself is the focus of attention. Meaning is assigned according to the symbolic *structure* of the theorem, which contains paired equations or inequalities, rather than through the interpretation of symbols within individual equations. As a conceptual entity, a theorem is perceived as describing *when* it may be used (with its problem-pattern) and *how* it may be used (with its solution-pattern). The collection of symbols in a theorem conveys information about the abstract family of problems to which the theorem applies and the problem-solving process the theorem describes, rather than about the end results of using such processes. This shift in perception represents a level of abstraction above that used in the preceding contexts which were focused at a parametric and operational, rather than a structural level.

Conclusions

The four contexts illustrate how different types of meaning can be assigned to the same collection of algebraic symbols according to the nature of the mathematical entities for which these symbols act as signifiers. Context 1 represents the use of algebraic symbols as a way to convey a generalization about a particular pattern of arithmetic computations. In Context 2 symbols are used to identify a single family of equations to which a single solution-process applies. The Quadratic Theorem is perceived as a description of the way in which this family can be represented and manipulated rather than as a process that is executed.

Context 3 requires an *expansive generalization* of the concept of a quadratic family of equations. This type of generalization extends existing cognitive structures rather than changes them (Tall, 1991). In this context the objects to which the operation of squaring applies need no longer be fixed unknown numbers represented by "x," but can also be variable quantities expressed by other algebraic expressions. This use of dummy variables where "x^2" can represent "y^2" and "c" can represent "x^2 - 14" does not have a parallel in English or other languages. The dummy variables in this context take on meaning for their ability to represent operations as objects.

In Context 4 the theorems describe certain types of problems and how to solve them by using an abstract problem-pattern/solution-pattern structure. This structure represents an abstraction of the operations used in previous contexts to solve specific families of equations.

What is different in each of the four contexts is the way that the collection of algebraic symbols representing the Quadratic Theorem is perceived. However, shifting perceptions is not a simple matter. Expansive generalizations, which create more complex contexts for conceptual entities, may perform an important role in preparing students to move to a new level of abstraction. If "we inadvertently present simplified regularities which become part of the individual concept image, [these] deeply ingrained cognitive structures can cause serious cognitive conflict and act as obstacles to learning." (Tall, 1989, p. 37)

Shifts in perception that involve conceptual reorganizations take place through the process of abstraction. According to Sfard (1991, p. 18), "First there must be a process performed on already familiar objects, then the idea of turning this process into an autonomous entity should emerge, and finally the ability to see this new entity as an integrated, object-like whole must be acquired." Students at a lower level of mathematical abstraction will not perceive the higher-level objects (Sfard and Linchevski, 1994). The new objects are apparent only when one has made an appropriate abstraction and shifted to a new perceptual focus.

The four contexts also exhibit another property of algebraic entities. Collections of symbols may be perceived operationally as processes or structurally as objects (Sfard, 1991). The specific abstractions that are required to shift perception from Context 1 to 2, and from Context 3 to 4 illustrate how meanings assigned to collections of symbols shift from one of using processes to one of studying the structure of these processes. According to Sfard, there are "differences between these two modes of thinking [that reflect different] beliefs about the nature of mathematical entities. There is a deep ontological gap between operational and structural conceptions" (p. 4).

The examples discussed in this paper illustrate the range of mathematical entities that may be perceived within the same collection of algebraic symbols and how specific contexts can elicit a procedural or a structural interpretation of these entities. Such an analysis has been used to formulate research tasks to study students' abilities to use particular algebraic constructs (Sfard and Linchevski, 1994; Teppo and Esty, 1994).

References

Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.) *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 123-180). Albany NY: State University of New York Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. & Linchevski, L (1994). The gains and pitfalls of reification: the case of algebra. *Educational Studies in Mathematics*, 26, 191-228.

Teppo, A. R., & Esty, W. W. (1994). Problem solving using arithmetic and algebraic thinking. In Kirshner, D. (Ed.) *Proceedings of the Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 2* (pp. 24-30). Baton Rouge: Louisiana State University.

Tall, D. (1989). Concept images, generic organizers, computers, and curriculum change. *For the Learning of Mathematics*, 9, 3 37-42.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 3-21). Dordrecht: Kluwer Academic Publishers.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (pp. 495-511). NewYork: MacMillan Publishing Company.

THE EMERGENCE OF THE SPLITTING METAPHOR IN A FOURTH GRADE CLASSROOM

Adalira Sáenz-Ludlow, The University of North Carolina at Charlotte

In this classroom a child introduced the word split when he was asked to describe his mental activity in performing the addition of two numbers; the teacher initiated a spatial representation using a broken line. Subsequently, the children used their own pictorial representations. The paper presents an analysis of the numerical diagrams used by these fourth graders to represent their splitting notion. Splitting evolved into a metaphor that fostered the emergence of these children's numerical meanings.

Introduction

Metaphors according to Lakoff (1987, 1994), Lakoff and Johnson (1980), and Johnson (1987) are not just a matter of language but of thought and reason. Metaphors, for them, are mapping "motivated by structures inhering in everyday experience" (Lakoff, 1987, p. 287). Metaphors, they contend, are mappings that put into correspondence two different domains of experience preserving their basic logic—an image schemata domain that structure our experience preconceptually and a conceptual abstract domain. In this fourth grade classroom, splitting became more than a peculiar way of speaking; it became a metaphor establishing a correspondence between the physical experiential domain of breaking and dividing into parts and the conceptual domain of unfolding or deunitizing a numerical unit into smaller subunits.

During the first months of the school year, the ***splitting metaphor*** and its spatial representations became, for the children, a thinking tool to conceptualize place value and solve word problems. Children used the ***splitting metaphorical expression*** to describe their mental actions of deunitizing a composite unit into simpler units to fit their particular goal when operating with natural numbers. These children's diagrams indicate the enactment of their mental actions to operate with natural numbers in different contextual situations. The ***splitting metaphor*** was represented through numerical diagrams that acquired different degrees of complexity and they were additive (decomposition of a unit into simpler units of different magnitude) or multiplicative (decomposition of a unit into simpler units of the same magnitude) in nature. Initially, the splitting metaphor emerged, as a linguistic device in a classroom discussion, from the numerical conceptual evolution of one child; the teacher legitimized it by initiating a spatial representation of it and incorporating it in the flow of the classroom conversation. The splitting notion seems to be a natural notion to children since they immediately started to

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The research reported in this paper is supported by the National Science Foundation under grant No. RED-9155734. The opinions expressed do not necessarily reflect the views of the Foundation. The author acknowledges the collaboration of Dr. Marcela Perlwitz during the first year of the teaching experiment, Mrs. Cathy Walgamuth, the collaborating teacher during the two years of the teaching experiment, and the work of the graduate students who participated with the project. The insights of Professor Merlyn Behr were of great help in the conceptualization of the paper.

use it to make diagrams or spatial representations of their own mental ways of operating with numbers.

The Study

The teaching experiment. During the school year, the fourth-grade arithmetic class was team taught by the researcher and the collaborating teacher. All instructional tasks for the arithmetic class were generated to make emphasis on natural numbers as composite units (Steffe, von Glasersfeld, Richards, and Cobb, 1983; Steffe and Cobb 1988) that could be themselves iterated to form a larger unit or decomposed into smaller subunits. Besides the instruction during the arithmetic class, these fourth graders were interviewed weekly in small groups of at most three students. These groups were constituted taking into account the inferred numerical understanding of each child and his or her willingness to work with the other members of the group. Every class and interview was videotaped and transcribed, field notes were taken, and the task pages and children's scrap paper were collected.

Organization of the teaching activity. Typically, children were given tasks on paper or verbally to solve individually while encouraged to consult and discuss with other classroom members. After the task appeared to be solved by the majority of the students, a whole class discussion took place. Children presented their solutions using the overhead projector or the board. A sequence of teacher's questions and children's explanations of their solutions characterized the interaction. Learning from Cobb (1989, 1990), and Cobb, Yackel, and Wood (1992), since the beginning of the school year there was an explicit mutual agreement between the teachers and the students about their responsibility to listen carefully to the solution of others and to express their agreement or disagreement by giving a reason for it. This agreement was consolidated throughout the school year. After the first three months of the school year, children's mental engagement in the classroom activity was manifested by collective applause when a child presented a solution that they considered to be novel or they perceived the presenter appeared to have difficulties but successfully completed the task.

The Emergence of Splitting as a Metaphor

These children's conceptualizations of units of ten and their understanding of the place-value structure of the Hindu-Arabic notation of numbers was minimal and their operations with numbers was strictly procedural and dependent on conventional algorithms. Our first concern was to help students to find their own meaningful ways of operating with numbers. To do this, we emphasized counting and mental computation. Most of the time, numbers were presented verbally to make relevant not only the units of ten but the relationship between number words and number symbols. To facilitate mental computation, at most three numbers were given to the children at once. At the beginning when numbers were presented in a written form, we used a rectangular 2-by-2 array of squares to locate the numbers in three of them and the fourth was empty for the children to write the sum.

By the middle of the third week of classes, the students were asked to add the numbers 80, 5, and 15. Numbers were located in the rectangular array. After children were given time to arrive at their result mentally, the teacher asked the children for their answers and explanations and she displayed them on the overhead. In this and the following dialogues, T stands for teacher and any two-letter set of an upper case letter followed by a lower case letter stands for the abbreviation of student's name and it will appear on italics in the body of the paper to avoid confusion.

1 Am: 80 plus 5 is 85 (counting on her fingers by one), 85 plus 15 is 100.

2 T: (writes 80 + 5 = 85, 85 + 15 = 100 as Am describes her solution). How did you add these two numbers (Pointing at 85+15)?.

3 Am: 85 plus 5, plus 5, plus 5. That's 15 (showing 3 fingers).

4 Ra: You split the 15 into 5, 5, and 5.

5 T: (simultaneously writes 85 + 5 + 5 + 5 = 100)

6 St: 5 and 15 is 20. 80 and 20 is 100.

7 T: (simultaneously writes 5+15=20, 80+20=100)

8 Pr: 80 and 15 is 95. 95 and 5 is 100.

9 T: (simultaneously writes 80+15=95, 95+5=100)

10 T: OK Ra, what did you do?

11 Ra: 80 plus 15 equals 95. I split the 15 into 5, 5, and 5. Then 95 plus 5 equals 100.

12 T: (simultaneously writes

$$80 + \underset{\overset{\swarrow\ \downarrow\ \searrow}{5\quad 5\quad 5}}{15} = 95\ ,\ 95 + 5 = 100)$$

The interaction between the teacher and the students illustrates a simultaneous event in which children's verbalizations are transformed or mapped into conventional numerical equalities through conventional symbols that children have used in their prior school years. The most significant part of the dialogue is the *split interpretation* that *Ra* made of *Am*'s way of acting on the numbers (line 4) and the description of his solution using the splitting notion coupled with the teacher's attempt to symbolize it (lines 11 and 12). It is worth noticing that, in the above solutions, children associated the numbers in the ways that were easy to operate for them. In general, the way children associated the numbers on the rectangular array were not prompted by their position given that there is not a particular direction in which the reading of numbers must be done in the matrix-type numerical arrangement, but instead children associated them according to their emerging or predetermined strategies to add the numbers. In the course of this and the following lessons, *Ra* was given full credit for the introduction of this splitting notion in the classroom mathematical discourse. By doing this, the teacher not only respected the intellectual contributions of one of the students but let the group know that their collaboration in the elaboration of common mathematical understanding was an ongoing process in which the students' partnership was needed and welcome.

Several questions emerge from the reflections on the above dialogue: Did the students pursue their own diagrammatic representations in their efforts to explain their answers? Did these representations coevolve with their numerical meaning-making process? Did the diagrams acquire some degree of sophistication? The analysis of children's solutions of mathematical tasks indicates that children autonomously generated more sophisticated diagrams and generated other ways of talking about the decomposition or deunitization of numbers into subunits.

The Splitting Metaphor and Operations with Natural Numbers

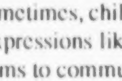
Once the teacher encouraged the splitting metaphorical expression as a way of speaking mathematically, the children considered it as a legitimate way of talking about numbers and they, on their own initiative, used it to communicate their mental actions on numbers. Sometimes, children used the term "splitting" explicitly; others, substituted it for expressions like *"take away"* or *"break into"*; still other children used their diagrams to communicate their solutions. In the following dialogue, we can observe how children, in the absence of paper and pencil, were able to conceptualize units of ten and use the splitting metaphor to find the sum of 26 and 25:

1 T: If I ask you to add, in your mind, 25 and 26, what do you get? (Several hands go up)

2 Mi: 6 and 4 is 10 ones, you can make a ten. 2 tens and 2 tens is 4 tens, so is 5 tens; that is 50. 50 and 1 is 51.

3 Pr: I split 20 into 2 tens and 20 into 2 tens; 6 and 5 is 11. 5 tens is 50 and 1 is 51.

4 Ra: 25 and 25 is 50 and 1 is 51. Let me show you ...(he goes to the board and makes the following diagram)



5 Li: 5 plus 6 is 11. 2 tens and 2 tens is 4 tens, another 10 is 50. One more is 51.

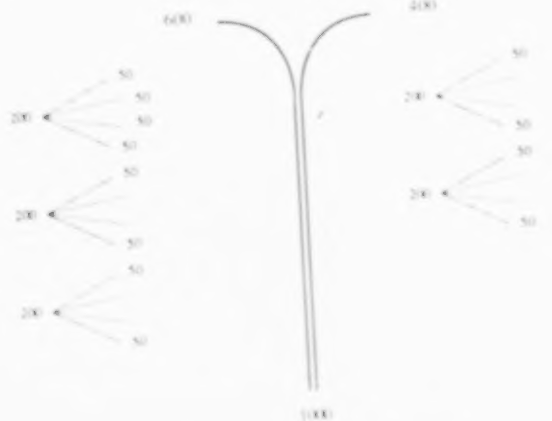
6 St: I took 6 away from 26 and 5 away from 25. 20 and 20 is 40.

6 and 4 is ten. One more is 51.

In line 2, *Mi*'s answer indicates that she conceptualized 26 as 2 tens and 6 ones, and 25 as 2 tens and 5 ones. It seems as if she had continued the splitting process for 5 as 4 and 1. *Mi* not only kept in mind the decomposition of 25 and 26 into units of ten and one but also was able to operate without confounding them. That is, 5 tens and 1 one were for her 51 and neither 6 tens nor 6 ones; *Mi*'s sophisticated reasoning seems to have been supported by some type of mental image. In line 3, splitting seems to be the mental purposive action that allowed *Pr* to deunitize 26 and 25 into units of ten and one. In line 4, *Ra* implicitly deunitized 26 into 25 and 1, added the two units of 25 and then added to it the unit of one. His verbalization was immediately followed by a diagram indicating that either he had a mental image of it to support his actions on the numbers or he generated it in the midst of verbalizing his explanations as a way of communicating with his peers. In line 6, *St*'s solution is essentially similar to that of *Mi* but she expressed the splitting action as *taking away from* as she took 5 and 6 *away from* 25 and 26 respectively.

The following task was posed by one of the students. He took a fake 1000-dollar bill from one of the banks (a bank was a plastic box with fake dollar bills of all the denominations which were kept classified) and asked his classmates if they could find the number of 50-dollar bills for which this bill could be exchanged. *Ri* offered this diagram as his solution.

Ri's diagram:



Ri's diagram is additive or a hybrid between additive and multiplicative. Additive because at level 1 (first split) the thousand unit was not deunitized into units of the same size. At level 2 (second split) the deunitization was done into units of 200, and at the level 3 (third split) the deunitization was done into units of 50. To make the diagram, *Ri* had to anticipate the consecutive unfolding of each of the units at each splitting level of the diagram. The diagram represents *Ri*'s ways of expressing his mental actions on numbers in terms of his physical experience of splitting. *Ri*'s cognitive behavior seems to be in accordance to Lakoff and Johnson's contention about "the essence of metaphor in understanding and experiencing one thing in terms of another" (p. 5, 1980).

Discussion

The splitting notion and its diagrammatic representation (whatever shape it took) became a collective way of speaking to express one's numerical reasoning and to understand the numerical reasoning of others. In the long run, splitting diagrams that were individually or collectively generated became a conceptualizing tool that fostered these children's numerical reasoning. That is, **splitting** became more than a way of speaking, it became a way of describing the mental action of decomposing numbers into subunits as conceptually perceived by the children according to their needs to operate with them. One question that would be of importance to consider is whether or not the physical notion of splitting projected into a numerical context, was used by children only in isolated instances at the beginning of the school year or if it occurred frequently and evolved throughout the school year to support children's conceptualization of fractions. An extended version of this paper will present evidence that supports this question in the positive. The role that metaphorical elaboration plays on children's numerical sense making, the conditions under which these metaphors emerge from children's efforts to operate with numbers, the influence of the social interaction on children's way of thinking, and the interdependent nature between mathematical thought and speech seem to be of interest for the teaching of arithmetic in the classroom.

References

Cobb, P. (1989). Experiential, cognitive, and anthropological perspectives in mathematics education. *For the Learning of Mathematics*, 9 (2), 32-42.

Cobb, P. (1990). Multiple perspectives. In L. P. Steffe, & T. Wood (Eds.), *Transforming children mathematics education: International perspectives* (pp. 200-215). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics*, *23*, 99-122.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination and reason*. Chicago, IL: The University of Chicago Press.

Lakoff, G. (1987). *Women, fire, and dangerous things*. Chicago, IL: The University of Chicago Press.

Lakoff, G. (1994). What is a conceptual system? In W. F. Overton & D. S. Palermo (Eds.), *The nature and ontogenesis of meaning* (pp. 41-90). Hillsdale, N J: Lawrence Erlbaum Associates.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago, IL: The University of Chicago Press.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb P. (1983). *Children's counting types Philosophy, theory and applications*. New York, NY: Praeger.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer-Verlag.

NEGATIVE NUMBERS IN THE TEACHING OF ARITHMETIC. REPERCUSSIONS IN ELEMENTARY ALGEBRA.

Aurora Gallardo, Centro de Investigación y de Estudios Avanzados del IPN, Mexico

This article reports the results of a questionnaire applied to 35 secondary school students in order to explore the efficiency in the resolution of equations in the domain of whole numbers and the spontaneous responses to problems leading to negative solutions. The most significant results obtained with the questionnaire are the lack of knowledge of the double use of brackets in arithmetic expressions, the partial comprehension of the operation of subtraction and the difficulty in the operativity of expression with double signs. The conclusions of this research suggest reccommendations for the teaching of whole numbers.

The Study

The present work describes the first stage of the project *"The Status of Negative Numbers in the Resolution of Equations"* (Gallardo, 1994a). This project deals with the study of negative numbers in their interaction with the languages and methods used to solve equations and problems. Other stages of the project are described in Gallardo & Rojano (1993, 1994) and Gallardo (1994b). This article reports the results of a questionnaire responded to by 35 secondary school students in order to explore proficiency in the resolution of equations in the domain of whole numbers and the spontaneous responses to problems leading to negative solutions. The interest in reporting the first stage of the project is the importance that the results obtained by the use of this questionnaire have for the teaching of whole numbers in the field of arithmetic and their later repercussions in elementary algebra. The questionnaire was responded to by students aged between 12-13 years, before they had received any formal algebra teaching, and covered the following topics:

1. *Operativity in the domain of whole numbers at the syntactic level and their representation in the number line.* The student is asked to solve additions and subtractions with whole numbers using the number line. The most difficult exercises were the following[1] : $a - (-b) =$ and $- a - (-b) =$ with a, b natural numbers. The students obtained a percentage of correct answers ranging from 37% to 6 % on the above items.

Regarding the operativity of whole numbers at the syntactic level, exercises of the following form were designed: $a + b=$, $a \bullet b=$, $-(a + b)=$, $(-a) + (-b) =$, and $a - (b-c-d)=$, with a, b whole numbers. Furthermore, the student is asked if the following expressions are true or false: $a + b = a + b$; $(a)(-b)=(-b)(a)$; $-a(b (c)=-(a (c)(b$; $a - (b-c)=(a-b)-c$; $a(b-c)=ab - ac$ with a, b natural numbers. Marks below 30%

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[1] All the examples in the questionnaire are numerical.

were obtained with this type of exercises. The greatest difficulty was due to the erroneous operativity of the minus sign together with the inadequate use of brackets.

2. *Location of the symmetric of a number in the model of the number line.* On the items corresponding to the symmetrics of numbers: $-(+a)$, $-(-(+a))$ with a as natural number, marks of 40% were obtained. The most difficult exercise of this theme corresponded to the symmetric of $(-(-a))$, that is, with a double minus sign. The percentage of correct answers in this case was 20%.

3. *Order in whole numbers.* The following questions were to be answered on this theme:

   Order these numbers from smallest to largest: -4, 3, 0, 14, -3, -8.

   Write three whole numbers greater than -3.

   Write three whole numbers smaller than -7.

   Write a whole number between -3 and -7.

   Write a whole number between -1 and 2.

   How many whole numbers are there between -5 and 0?

   How many whole numbers are there between -4 and 8?

The exercises 4 and 5 obtained the highest number of correct answers (from 83% to 77%). Item 6 achieved a higher percentage (66%) than item 7 (57%). The latter exercise is more difficult because *"you have to pass through zero"*. Exercise 1 corresponds to the order which *"appears natural to the student"*, that is, to order from smaller to larger (43% correct answers).

4. *Translation into symbolic language of situations expressed in words.* In these exercises different situations were presented and the student was asked to describe them using whole numbers. The following illustrates some of these situations:

   The temperature is 20 degrees below zero.

   Jose won 2 500 pesos.

   Archimedes was born in the year 267 before our era.

   The school is owed 25 000 pesos.

   Rosa neither won nor lost.

The lowest percentage of correct answers corresponded to item 5 (68%).

5. *Use of pre-algebraic languages in the context of equations.* In these excercises the student is asked to solve equations with the form: ☐

$\pm a = b$, $a \cdot \square = b$ and $a \cdot \square \pm b = c$, with a, b, c whole numbers. The most difficult items (40% correct answers of the total) were those where the number sought is negative.

6. *Resolution of word problems.* These problems revealed that the student has difficulty in formulating a subtraction when the statement contains the word difference. In the same way, problems with negative solutions are complicated for the students. In the latter case the percentage of correct answers is 7% (see results 3 and 4 of this article).

Results of the Study

Among the most significant results obtained with the questionnaire are the following:

1. *Lack of knowledge of the double use of brackets in arithmetic expressions.* Students are unaware that the bracket can be used as a symbol for grouping terms in an additive situation and as a multiplicative operator. To illustrate this we can take as an example one of the items on the questionnaire where students are asked to decide if the equality $20-(7-8)=(20-7)-8$ is true or false. Observe that in the first side of this equality the bracket indicates the grouping of 7-8. Moreover, the same bracket expresses a multiplication by -1, denoted by the minus sign which precedes it: -(7-8). The operativity is carried out with whole numbers. However, in the second side of the previous equality, the operations are effected in the domain of natural numbers and the brackets indicate only the grouping of 20-7.

These facts, which are not taken account of in the teaching of arithmetic, are inherited by algebra. The student does not understand expressions such as the following:

$$(x-y) + (w-z) = (x+w) - (y+z) \qquad (1)$$

$$(x-1)^2 = (x^2-2x) + 1 \qquad (2)$$

In the first side of (1) the brackets group terms and in the second side the bracket is used as a multiplicative operator (observe the minus sign in front of the second bracket). Again, in the first side of (2), the brackets indicate squaring an expression. In the second side, the brackets group the first two terms of a trinomial. In the terrain of algebra the situation becomes more complex because the literals do not reveal the numerical domain to which they pertain. It is very important to warn the student from the outset, that is, from the teaching of arithmetic, with which numbers they are working in the exercises they do.

2. *Wrong resolution of pre-algebraic expressions.* There is greater difficulty in equations of the form $\square \pm a = b$, $a \cdot \square = b$ and $a \cdot \square \pm b = c$ with a, b whole numbers when the value sought is negative. This situation permits the conjecture that a place-holder will contrib-

ute to the avoidance of the negative solution in algebraic equations. The place-holder has the inherent connotation of *"being filled"*. The student generally seeks to *"fill it"* with a positive number.

3. *Partial comprehension of the operation of subtraction.* The student does not solve word problems which indicate the difference between two whole numbers with a subtraction. Moreover, he/she erroneously conceives situations of *"complete to"* as in the following item: *"A person is going to copy part of a book, from page 29 to page 35. How many pages does he copy?"*. The majority of students effect the subtraction 35-29, that is, they understand the subtraction as "take away".

4. *Abandoning of arithmetic methods and use of literals in the formulation of word problems when the solution is negative.* This situation is found in those problems with an evident contradiction in the statement if the student supposes that the solution is positive. An example of this is the following problem: A says to B: *"If you give me all your money and I add it to mine, I can buy a horse which costs 1000 units"*. B answers A: *"If you had three times what you have and I had double what I have, altogether it would add up to the price of the horse"*. How much money did each friend have?

The student writes the response *"it can't be done"*, or else *"who do I pay attention to, A or B?"*. Other students try to formulate an equation even though they have not had any formal algebra teaching.

5. *Arithmetic methods make one of the conditions of some word problems unnecessary.* This happens, for example, in the following problem: *"A salesman has bought 15 pieces of cloth of two types and paid 160 coins. If one of the types cost 11 coins the piece and the other costs 13 coins the piece, how many pieces did he buy of each price?"* Fifteen students look for multiples of 11 and 13 that add up to 160 (this is equivalent to solving the equation $11x + 13y = 160$. The existence of $x + y = 15$ is ignored). The students do not see that one of the two solutions is negative. However, this problem can be solved arithmetically, changing the data in the statement in order to obtain contradictory facts and provoke a conflict. In this research, conflict is achieved by decreasing the numerical data 160, 15, 13 and 11 to 40, 3, 3, and 2 (see Gallardo & Rojano, 1993).

The considerations expressed above show the necessity of solving, via teaching, the difficulties presented with negative numbers in the field of arithmetic, before students begin formal algebra courses.

In the research literature, authors such as Glaeser (1981), Bell (1982), Freudenthal (1983), Fischbein (1987), Janvier (1985) and Peled (1989), among others, have indicated the conceptual and operative problems which arise during the process of teaching-learning of negative numbers. Specifically, Vergnaud (1989) points to the obstacles that these numbers represent when they are introduced into teaching, preceding the study of algebraic concepts. On the other hand, expressions such as (1) and (2) above are analyzed as propositional rules lacked visual salience (Awtry & Kirshner, 1994).

Recommendations

The conclusions of the research described in this article suggest the following recommendations:

1. To inform students of the numerical domain of the expressions to be dealt with, explaining the double use of brackets.
2. To use the method of teaching by diagnosis, which implies the identification of errors and false conceptions in a topic and later the formulation of a design for teaching in which the difficulties are exposed and solved through discussion-conflict.
3. To encourage the use of teaching models with whole numbers, different from the model of the number line, which permit other interpretations of the negative number, different from that of positions or displacements. We suggest the use of discrete models where the whole numbers represent objects of an opposing nature (protons, electrons; black balls, white balls, etc). In these latter models, the sign of operation is distinguished from the sign of number in the case of double signs *[-(-a), +(-a)* and *-(+a)]*.
4. In the resolution of pre-algebraic equations, we recommend the teacher not restrict his/her use of the place-holder as *"a place to be filled by a number"* but to encourage the methods of inversion of operations which propitiate *"operating the unknown"* and the extension of the numerical domain of solution in the terrain of true algebra (Filloy & Rojano, 1984).

References

Awtry, T. & Kirshner, D. (1994). *Visual salience in algebraic transformations.* Proceedings of the XVI Annual Conference for the Psychology of Mathematics Education, North American Chapter, Louisiana State University, USA, Vol. 1, pp. 131-137.

Bell, A. (1982) *Looking at children at directed numbers.* Mathematics Teaching. No. 100.

Fischbein, E. (1987). *Intuition in Science and Mathematics. An Educational Approach.* Reidel, Holland, Chapter XII: The practicality of intuitive meanings, analysis of an examen: The negative number. pp. 97-102.

Filloy, E. & Rojano, T. (1984). *From arithmetical to an algebraic thought: A clinical study with 12-13 year olds.* J. Moser (Editor). Proceedings of the VI Annual Conference for the Psychology of Mathematics Education, North American Chapter. pp. 51-56, Madison, Wisonsin, USA.

Freudenthal, H. (1983). *Mathematics as an educational task.* Chapter XII, Reidel Droodrecht, The Netherlands.

Gallardo, A. (1994a). Doctoral Dissertation. *El estatus de los números negativos en la resolución de ecuaciones algebraicas*. Centro de Investigación y de Estudios Avanzados del IPN, Departamento de Matemática Educativa, México, 2 Vols.

Gallardo, A. (1994b) *Negative numbers in algebra: The use of a teaching model*. João Pedro Da Ponte & João Filipe Matos (Editors). Proceedings of the XIX Annual Conference for the Psychology of Mathematics Education, Vol II, pp. 376-383, Lisbon, Portugal.

Gallardo, A. & Rojano, T. (1993). *Negative solutions in the context of algebraic word problems*. Proceedings of the XV International Group for the Psychology of Mathematics Education, University of Tsukuba, Japan; and North American Chapter, San José State University, USA, Vol. 1, pp. 121-127.

Gallardo, A. & Rojano, T. (1994). *School algebra. Syntactic difficulties in the operativity with negative numbers*. Proceedings of the XVI International Group for the Psychology of Mathematics Education, North American Chapter. Louisiana State Univesity, USA, Vol. I, pp. 159-165.

Glaeser, G. (1981). *Epistemologie des nombres relatifs*. Recherches en Didàctique des Mathématiques, Vol. 2, No. 3, pp. 303-346.

Janvier, C. (1985). *Comparison of models aimed at teaching signed integers*. Proceedings of the Nineth Annual Conference for the Psychology of Mathematics Education, State University of Utrecht, The Netherlands, pp. 135-140.

Peled, I., Mukhopadhyay, S. & Resnick, L. (1989). Formal and informal sources of mental models for negative numbres. Thirtheenth Annual Conference for the Psychology of Mathematics Education. Paris, France. pp. 106-110.

Vergnaud, G. (1989). *L'obstacle des nombres négatifs et l'introduction à l'algèbre. Construction des savoirs*. Colloque International Obstacle Epistémologique et Conflict Socio-Cognitif, CIRADE, Montreal.

CONSTRUCTIVIST LEARNING AND EXPLICIT TEACHING OUTCOMES: A COMPARISON OF VARIABLE REPRESENTATION

Bryan Moseley, University of California at Santa Barbara
Mary E. Brenner, University of California at Santa Barbara

Grasping the concept of a variable is a difficult but central portion of middle school students' first attempts to understand algebra. Instructional methods for presenting this material have gravitated toward two models: 1) an active teacher who details concepts and procedures explicitly 2) a constructivist approach in which the student is given autonomy to form new understandings by investigating new information and building upon their pre-existing knowledge. At issue are the differences in naive (i.e., inaccurate or incomplete) conceptions of variables and their notations, that students report after receiving different types of instruction, as well as students' ability to extend their mathematical knowledge to represent and solve word problems.

This research presents a qualitative comparison of seventh grade pre-algebra students' ideas of variables and their notations as a function of the type of instruction that they received. Three seventh grade pre-algebra classes, taught by the same teacher, were selected. Two treatment classes received a constructivist curriculum designed by the research team to emphasize group work, multiple representations, and student centered learning. The remaining class, designated as a comparison, received explicit teaching from a traditional text. Samples of five students, balanced for gender, ethnicity, and prior achievement were selected from each class, and interviewed immediately before and after the time of instruction for a total of fifteen respondents.

Qualitative analysis of the posttest interviews support quantitative findings (Mayer, Lewis, Hegarty, 1992) that students' conceptions of variables are important predictors of success, and that students who learn constructively display different types of naive conceptions about variables than their explicitly taught peers. Students in each treatment class made more attempts to link variables to real world constructs, but also displayed more naive conceptions overall than the comparison class. The data also suggest that students who learn in a constructivist paradigm develop greater facility in representing word problems with algebraic notation. Students in the two treatment classes were able to represent a difficult two step problem with an algebraic equation and find a correct solution with greater frequency than their explicitly taught peers.

Reference

Mayer, R.E., Lewis, A. B., & Hegarty, M. (1992). Mathematics misunderstandings: Qualitative reasoning about quantitative problems. In J.I. D. Campbell (Ed.), *The nature and origins of mathematical skill* (pp. 137-154). Amsterdam: North-Holland.

TEACHING SYSTEMATIC WORD PROBLEM SOLVING USING TWO EQUATIONS IN TWO VARIABLES WHEN THERE ARE TWO UNKNOWNS AS SOON AS POSSIBLE IN ALGEBRA I

Susann M. Mathews, Wright State University

This presentation reports an experiment in which Algebra I students learned to translate word problems with two unknowns from the prose representation to symbolic representation using a variable to represent each unknown when they first started solving word problems with two unknowns. Information processing theory states that the capacity for short-term memory is only about four chunks of information; experts have large chunks, and novices have small chunks. Because novice algebra students have only four small chunks of short-term memory to work with when solving a word problem, it is important for them to break a problem down into small pieces and write each of their pieces on paper to extend their short-term working memory. Thus representing each unknown with a variable and each relation with an equation extends students' working memory. Forcing students to represent all of the unknowns in terms of one variable, as is traditionally done, forces them to define and solve each relationship between variables in their heads, thus trying to use more short-term memory than they have. Experimental group students' performance on a test of word problems with two unknowns was compared to the results on the same test taken by students who had learned to solve word problems with two unknowns the traditional way, using only one variable to translate from prose to an algebraic equation. Four algebra teachers and 196 of their students participated in the study. A factorial block-randomized design was used. There was a statistically significant difference in the median overall problem-solving scores, and in the scores reflecting students' translation from the prose to a symbolic representation, of the experimental group and the control group on this word problem test with the experimental group scoring substantially higher. The statistical tests performed suggest that this difference is attributable to the experimental treatment.

USING THE BALANCE SCALE AS A CATALYST FOR THINKING: CHILDREN BUILD ON THEIR OWN LANGUAGE AND CONCEPTION OF EQUALITY TO CONSTRUCT A RICHER INTERPRETATION OF THE EQUALS SIGN

Barbara MacKay Gupta, Washington State University

Researchers have noted that students of all ages have difficulty moving beyond a barren, unidirectional interpretation of the equals sign (see Kieran, 1992). This study, unlike previous work, investigated second grade students' conceptions of equality, as distinct from their interpretations of the equals sign, and documented the students' construction of a richer interpretation of that sign, both in the classroom and interviews.

These students, as those in previous studies, declined to accept equations such as $5 = 5$ and $6 = 2 + 4$, stating that they were "not true" or that they "didn't make sense", and interpreted the equals sign only as a "do something signal" (Behr, Erlwanger, and Nichols, 1975), rejecting its use for comparison. However, students' conceptions of equality were spontaneously and intrinsically comparative. They used many synonyms to describe the equivalence of various sets of objects presented to them, such as "the same number as", "as many as", and "they both have two".

Using the balance scale as a device to compare the numerosity of sets of objects, and games which provided familiarity with the equals sign in new contexts, all children readily accepted the comparative process. Building on the synonyms they had produced earlier, and generating new ones tied to the use of the balance scale, such as "level" and "balanced", children gradually began to use the word "equals" in a comparative sense in arithmetic contexts. Soon they were using the sign as a comparative symbol as well.

Leslie said, "Ten equals ten because they are the same amount". Kate described her moment of change: "When I started looking at it [$10 = 4 + 6$] more with the equal sign there, I knew it must be a real sentence because 10 does equal $4 + 6 \ldots 6 + 4$." Joshua said, "It's just like if you put 10 cubes on the balance scale and another 10 cubes on the other side of the balance, it's going to be balanced".

Thus, children used a balance scale and related activities to change their patterns of thought regarding the equals sign: to construct a richer interpretation of the equals sign from their already rich conception and language of equality. The classroom teacher will base her next fall review on the balance scale as a result of our work together this spring.

References

Behr, M., Erlwanger, S., and Nichols, E. (1975) *How Children View Equality Sentences* (PMDC Technical Report #7). Project for the Mathematical Development of Children.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed), *Handbook of Research on Mathematics Teaching and Learning* (pp. 390-419). New York: Macmillan.

Assessment

A FRAMEWORK FOR ASSESSING YOUNG CHILDREN'S THINKING IN PROBABILITY

Graham A. Jones, Illinois State University
Cynthia W. Langrall, Illinois State University
Carol A. Thornton, Illinois State University

Based on a synthesis of the literature and observations of young children over two years, a framework for assessing probabilistic thinking was formulated, refined and validated. For each of four major constructs incorporated into this framework—sample space, probability of an event, probability comparisons, and conditional probability—four different levels of thinking were established which reflected a continuum from subjective to numerical reasoning. The framework was validated through data obtained from 24 children of grades 1 through 3 who served as case studies. Results suggest that while the framework produces a unified picture of children's thinking in probability, there is "static" in the system which generates inconsistencies among construct levels. The framework has implications for curriculum development and assessment.

Although there has been considerable research into young children's thinking and misconceptions in probability (Fischbein, 1975; Fischbein, Nello, & Marino, 1991; Garfield & Alhgren, 1988; Piaget & Inhelder, 1975, Tversky & Kahneman, 1982; Shaughnessy, 1992), none of this research has generated a framework for systematically assessing young children's thinking in probability. Given the call for including probability in the elementary school curriculum (National Council of Teachers of Mathematics, 1989) and the inclusion of probability in state and national assessments (e.g., Illinois Goal Assessment Program, 1993; Mullis, Dossey, Owens, & Phillips, 1993), there is a need to describe children's probabilistic thinking as a basis for generating appropriate curriculum and assessment programs.

Aims of the Research

Based on a synthesis of the research literature related to children's thinking about probability (e.g., Fischbein, Nello, & Marino, 1991; Piaget & Inhelder, 1975; Shaughnessy, 1992) and related neo-Piagetian research that postulates the existence of different levels of complexity in children's thinking (e.g., Biggs & Collis, 1991; Case, 1985), this study attempted to:

- develop and refine a framework for describing and predicting how young children think in probabilistic situations; and
- use the framework to generate assessment protocols to validate the framework.

Theoretical Considerations

The thesis of this study maintains that for children to exhibit probabilistic thinking, there is a need for them to understand probability concepts that are multifaceted and develop over time. In order to capture the manifold nature of probabilistic thinking, our Framework (Figure 1) incorporates four key constructs: sample space, probability of an event, probability comparisons, and conditional probability. In this study *sample space* refers to listing or identifying the complete set of outcomes of a one- or two-stage probability experiment. *Probability of an event* involves identifying and justifying which of two or three events are most likely to occur. *Probability comparisons* entail determining and justifying: (a) which probability situation is more likely to generate the target event in a random draw; or (b) whether the two probability situations offer the same chance for the target event. *Conditional probability* refers to recognizing and justifying why the probability of an event may or may not be changed by the occurrence of another event.

The first three of these constructs have been investigated by several researchers (Acredolo, O'Connor, Banks & Horobin, 1989; English, 1993; Fischbein, Nello & Marino, 1991; Piaget & Inhelder, 1975). Few studies on the fourth construct, conditional probability, have been directed at young children. However, interpretations have been made from data on tasks involving elements of conditional probability (Borovcnik & Bentz, 1991; Falk, 1988; Konold, 1989; Shaughnessy, 1992). Notwithstanding the extent of research into children's probabilistic thinking, it has seldom investigated the four constructs in combination, and has not produced universal agreement on the scope of children's thinking in probability (Shaughnessy, 1992).

In addressing this need, our framework enables young children's probabilistic thinking to be described and predicted across four levels for each of the four constructs. These levels have evolved from our observations of young children's probabilistic thinking over a two-year period. Moreover, the notion of levels of thinking within specific knowledge domains is also in concert with cognitive research that recognizes developmental stages (Piaget & Inhelder, 1975) and, more particularly, with neo-Piagetian theories that postulate the existence of sub stages or levels that recycle during stages (Biggs & Collis, 1991; Case, 1985).

As is highlighted in Figure 1, Level 1 is associated with *subjective* thinking, Level 2 is seen to be *transitional* between subjective and naive quantitative thinking, Level 3 involves the use of *informal quantitative* thinking and Level 4 incorporates *numerical reasoning*. Further it is claimed that a child's probabilistic thinking at a given time is stable across all four constructs.

Methodology

Subjects

The population for the study comprised children in grades one through three at a University laboratory school. Eight children, randomly sampled from each of these grades, served as case studies. None of these children had been exposed to prior probability instruction.

The Validation Process

To validate the framework we sought to: a) ascertain whether children's thinking at a particular level was stable across all four constructs; and b) confirm and refine the characteristics of each level within the framework. Cochran's Q test (Siegel & Castellan, 1998) was used to assess the stability of framework levels and qualitative analysis was used to address the rest of the validation.

Data Collection and Instrumentation

The framework and the validation process guided the design of the data collection instruments and procedures. A structured interview assessment based on the framework comprised 22 tasks—six tasks associated with *sample space*, four with *probability of an event*, seven with *probability comparisons*, and five with *conditional probability*. This interview, audiotaped for subsequent analysis, was administered by members of the research team to each of the case study students.

Each question in the interview assessment was scored according to a three-part rubric: 1) fully met, 2) partially met, and 3) didn't meet the framework criteria. Children's thinking on all questions was analyzed and coded by level for each construct of the framework using the double coding procedure described by Miles & Huberman (1984). As a result of this analysis, children's dominant level of thinking with respect to each construct of the framework was determined.

Validating the Framework: Results and Discussion

In validating the framework a major concern was to examine stability of children's thinking across the constructs of sample space, probability of an event, probability comparisons, and conditional probability. The results of Cochran's Q test indicated that there were no significant differences among the thinking levels generated by the four probability constructs. That is, each of the four constructs were generally in harmony in identifying a child's probabilistic thinking level.

Notwithstanding the results of these analyses, there were not more than five children for whom the thinking levels were in complete agreement across the four constructs. Our observations and interpretations suggest that while the framework produces a unified picture of children's thinking in probability, there is 'static' in the system which generates inconsistencies among the levels based on each of the constructs. Moreover, it is our contention that this static results from children's tendencies to unexpectedly regress back to subjective judgments, even when their probabilistic thinking is more indicative of "transitional" or "informal quantitative" reasoning.

A second area of interest in the validation process was the refinement of descriptors of children's probabilistic thinking at each level and across all four constructs. The analysis of children's thinking revealed that children exhibiting level 1 thinking were narrowly and consistently bound to subjective judgments. They did not provide a complete listing of the outcomes in a one-stage experiment and they almost always used subjective judgments rather than quantitative ones in situ-

CONSTRUCT	Level 1 (Subjective)	Level 2 (Transitional)	Level 3 (Informal Quantitative)	Level 4 (Numerical)
SAMPLE SPACE	• lists an incomplete set of outcomes for a one-stage experiment	• lists a complete set of outcomes for a one-stage sample space, and • lists the outcomes of a two-stage experiment in a limited and unsystematic way	• adopts and partially applies a generative strategy to make a complete listing of outcomes for a two-stage case	• adopts and applies a generative strategy which enables a complete listing of the outcomes for a two- and three- stage case
PROBABILITY OF AN EVENT	• predicts most/ least likely event based on subjective judgments • distinguishes "certain," "impossible," and "possible" events in a limited way	• predicts most/ least likely event based on quantitative judgments but may revert to subjective judgments • distinguishes "certain," "impossible," and "possible" events within reasonable parameters	• predicts most/ least likely events based on quantitative judgments including situations involving non-contiguous outcomes) • uses numbers informally to compare probabilities • distinguishes "certain," "impossible," and "possible" events, and justifies choice quantitatively	• predicts most/ least likely events for single stage experiments • assigns a numerical probability to an event (it may be a real probability or a form of odds.)
PROBABILITY COMPARISONS	• compares the probability of an event in two different sample spaces, usually based on various subjective or numeric judgments • cannot distinguish "fair" probability situations from "unfair" ones	• makes probability comparisons based on quantitative judgments (may not quantify correctly and may have limitations where non-contiguous events are involved) • begins to distinguish "fair" probability questions from "unfair" ones	• makes probability comparisons based on consistent quantitative judgments • justifies with valid quantitative reasoning, but may have limitations where non-contiguous events are involved • distinguishes "fair" and "unfair" probability generations based on valid numerical reasoning	• assigns a numerical probability measure and compares • incorporates non-contiguous and contiguous outcomes in determining probabilities • assigns equal numerical probabilities to equally likely events
CONDITIONAL PROBABILITY	• following a particular outcome, predicts consistently that it will occur next time, or alternatively that it will not occur again (over-generalizes)	• begins to recognize that the probability of an event changes in a non-replacement situation • can recognize when certain and impossible events will arise in non-replacement situations	• can determine changing probability measures in a non-replacement situation • recognizes that the probability of all events change in a non-replacement situation	• assigns numerical probabilities in replacement and non-replacement situations • distinguishes dependent and independent events

Figure 1. Probabilistic thinking framework

ations involving probability. Children reflecting level 2 thinking could list the complete set of outcomes in a one-stage experiment. However, they didn't always use these outcomes when responding to probabilities, especially in tasks involving conditional probability. Level 2 is a period of transition where probability constructs are not always coordinated.

While acknowledging the subjective 'static' discussed above, children exhibiting level 3 thinking characteristically used quantitative judgments when dealing with tasks based on probability constructs. They revealed a consistent predisposition to use numbers in describing and comparing probabilities, albeit not always expressed as correct probability measures or odds. This predisposition to use numbers carried across into conditional probability situations, where children were able to recognize that the probabilities of all events changed in a non-replacement experiment. Children typifying this level of thinking, also tended to use more generative strategies in listing outcomes of two-stage experiments. Moreover, our analysis of children's probabilistic thinking revealed that level 3 thinkers had begun to coordinate thinking in sample space and thinking in probability in a more systematic manner.

The move from level 3 thinking to level 4 thinking needs further investigation, as none of the children in our study exhibited level 4 thinking across all four constructs. There was, however, evidence in this study that some children had begun to use more precise measures of probability and listings of multi-stage sample spaces. Our observations suggest that lack of knowledge of fractions inhibited the thinking of children who were otherwise predisposed to more precise probability measures.

In validating the framework, we have described children's probabilistic thinking at each of the four levels in content-specific terms. That is, we have related the children's probabilistic thinking across the four constructs to a continuum of four levels of quantitative reasoning. Moreover, the notion of levels of probabilistic thinking appears to be in concert with the theoretical position of cognitive researchers such Biggs & Collis, 1991; Case, 1985. They claim the existence of more general cognitive structures which incorporate sub stages or levels of cognitive functioning that recycle across broader stages of development. Their theoretical position adds further support to the existence of distinct levels of probabilistic thinking among children found in our study.

The framework generated by this study enables children's probabilistic thinking to be described and predicted in a unified and systematic manner. It does have limitations in that the levels of children's thinking on the four constructs were not completely stable and appeared to be subject to "static" as children unexpectedly regressed to subjective reasoning. Future research may reveal more stable patterns if children whose thinking has generally progressed beyond level 1 probabilistic thinking, are assessed on the basis of their dominant level when they occasionally revert to subjective judgments. The framework has implications for curriculum development and assessment in relation to probability programs for children in the primary grades.

References

Acredolo, C., O'Connor, J., Banks, L., & Horobin, K. (1989). Children's Ability to make probability estimates: Skills revealed through application of Anderson's functional measurement methodology. *Child Development. 60*, 933-945.

Biggs, J. B., & Collis, K. F. (1991). Multimodal learning and the quality of intelligent behavior. In H. A. H. Rowe (Ed.), *Intelligence: Reconceptualization and Measurement* (pp. 57-76). Hillsdale, NJ.: Lawrence Erlbaum.

Borovcnik, M. G., & Bentz, H. J. (1991). Empirical research in understanding probability. In Kapadia, R. & Borovcnik, M. (Eds.), *Chance Encounters: Probability in Education* (pp. 73-105). Dordrecht, Netherlands: Kluwer.

Case, R. (1985). *Cognitive Development.* New York: Academic Press.

English, L. D. (1993). Children's strategies for solving two- and three-stage combinatorial problems. *Journal for Research in Mathematics Education. 24*(3), 255-273.

Falk, R. (1988). Conditional probabilities: Insights and difficulties. In Davidson, R. & Swift, J. (Eds.), *Proceedings: The Second International Conference on Teaching Statistics* (pp. 292-297). University of Victoria, Victoria, BC.

Fischbein, E. (1975). *The Intuitive Sources of Probabilistic Thinking in Children.* Dordrecht, Netherlands: Reidel.

Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgments in children in adolescence. *Educational Studies in Mathematics. 22*(6), 523-549.

Garfield, J. B., & Ahlgren, A. (1988). Difficulties in learning basic concepts in probability and statistics: Implications for research. *Journal for Research in Mathematics Education. 19*(1), 44-63.

Illinois State Board of Education. (1993). *Illinois goal achievement program: Mathematics.* Springfield, IL: Author.

Konold, C. (1989). An outbreak of belief in independence. In Maher, C., Goldin, G. & Davis B. (Eds.), *Proceedings of the Eleventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Volume 2, (pp. 203-209). Rutgers, NJ: Rutgers University Press.

Mullis, I. V. S., Dossey, J. A., Owens, E. H., & Phillips, G. W. (1993). *NAEP 1992: Mathematics report card for the nation and the states*, Report No. 32-FT02. Princeton NJ: Educational Testing Service.

Miles, M. B., & Huberman, A. M. (1984). *Qualitative Data Analysis: A Sourcebook of New Methods.* Beverly Hills, CA: Sage Publications.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: Author.

Piaget, J., & Inhelder, B. (1975). *The Origin of the Idea of Chance in Children.* (L. Leake Jr., P. Burrell & H. D. Fischbein, trans.). New York: W. W. Norton.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In Grouws, D. A., (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 465-494). New York: Macmillan.

Siegel, S., & Castellan, N. J., Jr. (1988). *Nonparametric statistics for the behavioral sciences* (2nd ed.). New York: McGraw-Hill.

Tversky, A. & Kahneman, D. (1982). Judgments of and by representativenesss. In Kahneman, D., & Tversky, A. (Eds.), *Judgment under Uncertainty: Heuristics and Biases* (pp. 84-100). Cambridge, UK: Cambridge University Press.

A FRAMEWORK FOR THE QUALITATIVE ANALYSIS OF STUDENT RESPONSES TO THE EXTENDED CONSTRUCTED-RESPONSE QUESTIONS FROM THE 1992 NAEP IN MATHEMATICS

Patricia Ann Kenney, University of Pittsburgh (PA)

The purpose of this investigation was to develop a general framework for analyzing the NAEP extended constructed-response questions qualitatively. The framework dimensions were based on information about the NAEP extended questions, and linked to important ideas in mathematics education and cognitive psychology. A set of student responses to an extended constructed-response question from the grade-4 assessment was analyzed qualitatively according to appropriate framework dimensions. The findings suggest that the student responses could be analyzed qualitatively, but further investigation is needed to verify the adequacy of the framework.

The National Assessment of Educational Progress (NAEP) is a Congressionally-mandated survey of the educational achievement of American students and changes in that achievement over time. Since 1978, NAEP has assessed student performance in mathematics, with the most recent assessment conducted in spring of 1992. Compared to earlier NAEP mathematics assessments, the 1992 assessment was different in some important ways including closer alignment to the vision for school mathematics as presented in the *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics (1989). An innovative feature of the 1992 NAEP mathematics assessment involved the introduction of a new item type called the extended constructed-response question. As opposed to multiple-choice questions (which require students to select the correct answer from a provided set of answers) and regular-constructed response questions (which require students to generate their own numerical answer or to provide a very short explanation), extended constructed-response questions not only require students to generate their own answers but also to express their mathematical ideas in writing and to demonstrate their depth of understanding.

The 1992 NAEP grade-4 mathematics test included five extended constructed-response questions, and the grade 8 and grade 12 tests each included six such questions. Students were instructed to allow themselves five minutes or more to work on the questions. Instead of being scored "right or wrong," as were the multiple-choice and regular constructed-response questions, the extended questions were evaluated according to a focused holistic scoring scheme with categories ranging from "minimal" (score level 2) to "extended" (score level 5). Quantitative information about student performance on the extended tasks used on the 1992 NAEP mathematics assessment formed the basis for a report by Dossey, Mullis and Jones (1993). A critical review (Silver & Kenney, 1993) of the Dossey, Mullis and Jones report noted that although the quantitative summary format was

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I wish to thank Dr. Jinfa Cai and Dr. Edward A. Silver for their helpful comments on previous drafts of this paper.

informative, the usefulness of the information was somewhat limited because no effort was made to analyze student responses with respect to the kinds of strategies and representations most frequently employed by students or with respect to the kinds of errors commonly made by students.

The purpose of the investigation described in this paper was to develop a general framework for analyzing the NAEP extended constructed-response questions qualitatively. The utility of the framework was then examined by conducting a preliminary qualitative analysis of student responses to a selected NAEP extended task administered to fourth-grade students.

Developing the Framework

Initially, it was deemed beneficial to inquire whether the NAEP extended constructed-response questions were developed with the idea that student responses would be analyzed qualitatively as well as quantitatively. Informal discussions with mathematics education professionals and test developers involved with the 1992 NAEP mathematics assessment revealed that the extended questions were not developed specifically to be analyzed qualitatively and that no qualitative analytic framework was ever developed. However, there was agreement among those most deeply involved with NAEP that a qualitative analysis of student responses to the extended constructed-response questions would be very beneficial, especially to classroom mathematics teachers, mathematics teacher educators, and curriculum developers.

Since no qualitative framework existed for the NAEP extended constructed-response questions, it was important to find sources of general information about these questions and the expected kinds of student responses to be evaluated according to a focused-holistic scheme, with the expectation being that this general information about the extended questions might suggest appropriate framework dimensions applicable at least to some (perhaps all) extended constructed-response questions. After existing information sources about the NAEP extended questions were consulted, it was determined that the general scoring guide for the NAEP extended constructed-response questions (called the "generic levels of performance" in Dossey et al., p. 89) provided the most useful information about possible dimensions for the qualitative framework. In particular, the general scoring guide recommends that student responses be evaluated according to important criteria such as conceptual understanding (e.g., "Response contains evidence of conceptual understanding" [quotations taken directly from the general scoring guide]), solution strategies, (e.g., "Methods of solution are appropriate and fully developed"), error patterns (e.g., "Response contains major mathematical errors"), evidence of reasoning (e.g., "Response is logically sound"), and justification of answer (e.g., [through the use of examples] - Examples provided are not fully developed").

Those criteria of conceptual understanding, solution strategies, error patterns, evidence of reasoning and justification of answer are also among the criteria recognized by cognitive psychologists (e.g., Glaser, Lesgold, & Lajoie, 1985; Royer, Ciscero, & Carlo, 1993) and mathematics education researchers (e.g., Charles &

Silver, 1989) as important dimensions for measuring students' high-level performance. The evaluation criteria in the NAEP general scoring guide are also reminiscent of the categories used in the QUASAR[1] project's qualitative analytic component. In QUASAR, a qualitative analytic framework has been used to report results of complex performance on open-ended, paper-and pencil tasks (which are similar to the NAEP extended constructed-response questions) with respect to dimensions such as solution strategies, mathematical misconceptions, mathematical justification, and modes of representation (Cai, Magone, Wang, & Lane, 1995; Magone, Cai, Silver, & Wang, 1993).

Based on information from the NAEP general scoring guide, important ideas from cognitive psychology, and the QUASAR qualitative analytic model, it was decided to select the following criteria as dimensions for the qualitative framework for the NAEP extended constructed-response questions: (a) conceptual understanding; (b) solution strategies or modes of representation; (c) mathematical errors or misconceptions; and (d) evidence of reasoning. It is worth noting here that it was not the expectation that every NAEP extended question be evaluated according to all four dimensions. Due to differences in problem situations and content, not all of the dimensions are equally appropriate for every extended question.

Using the Framework

The dimensions of the qualitative framework were used to analyze student responses to a selected NAEP extended constructed-response task. The qualitative analysis itself was structured according to the model developed for the QUASAR project (Magone, Wang, Cai, & Lane, 1993): 1) conduct a logical analysis of the question to identify its cognitive requirements and content; 2) select appropriate framework dimensions based on the results of the logical analysis; 3) apply the selected dimensions to a sample of student responses; 4) expand and modify the selected set of framework dimensions based on results from the sample of students responses; and 5) conduct the qualitative analysis using the final set of dimensions for analyzing student responses. The following sections of this paper focus on Steps 1-3 for qualitatively analyzing student responses to a 1992 NAEP grade-4 extended-constructed response question, hereafter referred to as "Pizza Comparison" and shown in Figure 1.

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[1] QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) is a Ford Foundation sponsored project designed to enhance mathematics instruction in middle schools with high percentages of students from economically disadvantaged communities. One aspect of the project has been the development of the QUASAR Cognitive Assessment Instrument [QCAI]), a performance assessment used to measure the impact of these enhanced instructional programs on students' mathematical reasoning, problem solving and communication. Information about the QCAI can be found in Lane (1993) Silver and Lane (1993).

José ate 1/2 of a pizza.

Ella ate 1/2 of another pizza.

José said that he ate more pizza than Ella, but Ella said that they both ate the same amount. Use words and pictures to show that José could be right.

Figure 1. NAEP grade 4 extended constructed-response question: Pizza Comparison

Step 1: Logical Analysis of the Pizza Comparison Question

The Pizza Comparison question was designed to "assess how well students are making the transition from whole number reasoning into using concepts associated with fractions" (Dossey et al., 1993, p. 91). Using a real-life setting of comparing quantities of pizza, the question measures students' understanding of the importance of the relative size of the object or unit in interpreting a fraction and taps into their knowledge of proportional reasoning. Concepts such as the importance of the size of the unit in fractions have been identified by mathematics education researchers (e.g., Behr, Harel, Post, & Lesh, 1992) as critical to the acquisition of a deep understanding of rational numbers.

Step 2: Selected Appropriate Framework Dimensions

Findings from the logical analysis of the Pizza Comparison question suggest the following as appropriate dimensions from the framework:

- *Conceptual understanding*: understanding of the effect of relative difference in the size of the unit ("whole pizza").
- *Modes of representation*: use of pictures only, words only, or a combination of pictures and words.
- *Mathematical misconception*: "1/2 is always 1/2."

Step 3: Preliminary Results Using a Sample of Student Responses

At the time this paper was written, the researcher had access to a small set of student response (n = 25) to the Pizza Comparison question. The preliminary findings from the qualitative analysis follow.

Conceptual understanding. Over half of the student responses (n = 13) showed evidence of conceptual understanding of the importance of the relative size of the unit in comparison of fractions. The most common method of demonstrating the importance of relative size involved drawing two pizzas, one smaller than the other, dividing each pizza approximately in half, and labeling the larger one "José" and the smaller one "Ella." In a few cases, students supplemented the labeled drawings of two different-sized pizzas with one-sentence explanations (e.g., "José could have had a bigger pizza than Ella."); some students even mentioned sizes commonly associated with commercially-made pizzas (e.g., "José ordered a large pizza and Ella ordered a medium pizza." "Her pizza was 8 inches, José could of had a 12 inch pizza").

Modes of representation. Most responses in the set involved a combination of words and pictures, a finding that is not surprising given these instructions to the student: "Use words and pictures to show that Jose could be right." Figure 2 below shows typical examples of word/picture combinations. A few responses were expressed in words only (e.g., "José's was large and Ella's was small [medium]."), while a few others consisted of unlabeled pictures.

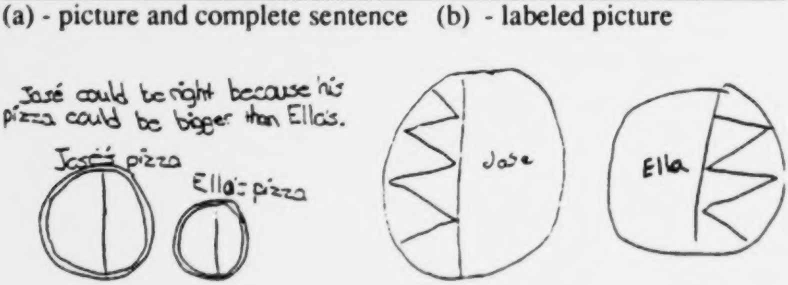

(a) - picture and complete sentence (b) - labeled picture
José could be right because his pizza could be bigger than Ella's.
José's pizza
Ella's pizza
Jose
Ella

Figure 2. Examples of word/picture combinations for the Pizza Comparison Question

Mathematical misconceptions. Twelve responses were based on the misconception that "1/2 is always 1/2." In some responses, students drew two equal-sized pizzas, divided them in half, and wrote a comment such as "José ate his half and Ella ate her half; they both had 1/2 and they both ate the same amount." However, other responses associated with the misconception that 1/2 is always equal to 1/2 were based on drawing one pizza, dividing it in half, and designating the halves as "José's" and "Ella's". This last example illustrates an error in understanding that was not anticipated in the logical analysis. The sentence, "Ella ate 1/2 of *another* pizza" (emphasis added) was a clue that the problem involved *two* different pizzas. However, some fourth-grade students most likely misunderstood or misread the problem.

Conclusion

This study focused on the development of a framework for analyzing student responses to the extended constructed-response questions from the 1992 NAEP mathematics assessment. Results from a preliminary analysis of a small set of responses to one extended question suggest that the student responses could be analyzed according to the framework dimension, but that further study of the adequacy of the framework is needed using more responses and using all NAEP extended questions.

References

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. G. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Cai, J., Magone, M. E., Wang, N., & Lane, S. (1995). *A cognitive analysis of QUASAR's mathematics performance assessment tasks and their sensitivity to measuring changes in students' thinking and reasoning.* Manuscript submitted for publication.

Charles, R., & Silver, E. A., (Eds.) (1988). *The teaching and assessing of mathematical problem solving.* Reston, VA: National Council of Teachers of Mathematics.

Dossey, J. A., Mullis, I. V. S, & Jones, C. O. (1993). *Can students do mathematical problem solving?: Results from constructed-response questions in NAEP's 1992 mathematics assessment.* Washington, D.C.: National Center for Education Statistics.

Glaser, R., Lesgold, A. M., & Lajoie, S. (1985). Toward a cognitive theory for the measurement of achievement. In R. R. Ronning, J. Glover, J. C. Conoley, & J. C. Witt (Eds.), *The influence of cognitive psychology on testing and measurement* (pp. 41-85). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lane, S. (1993). The conceptual framework for the development of a mathematics performance assessment. *Educational Measurement: Issues and Practice, 12,* 16-23.

Magone, M. E., Cai, J., Silver, E. A., & Wang, N. (1993). Validating the cognitive complexity and content quality of a mathematics performance assessment. *International Journal of Educational Research,* 21(3), 317-340.

Magone, M. E., Wang, N., Cai, J., & Lane, S. (1993, April). *An analysis of the cognitive complexity of Quasar's performance assessment tasks and their sensitivity to measuring changes in students' thinking.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Royer, J. M., Ciscero, C. A., & Carlo, M. S. (1993). Techniques for assessing cognitive skills. *Review of Educational Research, 63*(2), 201-243.

Silver, E. A., & Kenney, P. A. (1993, October). *Understanding students' mathematical problem solving: A commentary on the 1992 NAEP findings.* Pittsburgh, PA: Learning Research and Development Center, University of Pittsburgh.

Silver, E. A., & Lane, S. (1993). Assessment in the context of mathematics instruction reform: The design of assessment in the QUASAR project. In M. Niss (Ed.), *Cases of assessment in mathematics education* (pp. 59-69). London: Kluwer.

USING MULTIPLE SOURCES TO ASSESS COMPETENCY IN MATHEMATICS

Sitsofe Enyonam Anku, Nanyang Technological University

The use of multiple sources to assess competency in mathematics is one of the several recommendations for reforming mathematics education in North America (NCTM, 1989). Multiple sources provide avenues for monitoring students' evolving understanding and informal judgments of information from these sources can be useful for making important instructional decisions. In response to the NCTM challenge that mathematics educators implement the current reform in mathematics education, a major component of a 12-week introductory course in mathematics organized for prospective student teachers of a university was to assess the students' competence in the course using multiple sources. The 30 students who took the course were assessed through quizzes, journal entries, take-home assignments, group investigations, group presentations, student constructed questions, student assessment of the work of their peers, mid-term test, and end-of-course test.

Results indicate that about 40% of the students showed consistency of their competence whichever source was used. Others (30%) demonstrated their competence better using time-constrained sources like the quizzes, mid-term test, and end-of-course test, while the rest were better with the open-ended sources, like the journals and the group investigations. Also, the formative nature of the assessment package made assessment integral to instruction and made mathematics learning more meaningful to the students. The benefit to the instructor was that he could monitor students' progress from several sources and use information gathered to inform subsequent instruction.

However, there were several challenges to the class instructor and to the students. The instructor had to use a database to handle the large data set. Providing number grades and appropriate feedback for students' qualitative data were other major challenges. To the students, coping with the different assessment strategies and balancing their work with assignments from other subjects were major challenges.

An implication of the results for teaching is that expanding the scope of assessing students' competence in mathematics through multiple sources can provide each student with, at least *a medium*, for demonstrating mathematical competence. A major problem is how to handle the large amount of data generated through the use of multiple sources. However, technology can resolve the problem. What needs to be done is to train teachers to use technology confidently to handle such enormous data.

Reference

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

USING PORTFOLIOS TO ASSESS TEACHER DEVELOPMENT IN ELEMENTARY MATHEMATICS TEACHER PREPARATION

Anne M. Raymond, Indiana State University

Relying on only one method of assessment does not necessarily yield an accurate picture of student achievement. Thus there is a need to provide students alternative ways to demonstrate their understanding of course content. Portfolios have the potential to engage students in decision making, provide students a voice in their assessment, enhance their metacognitive awareness, and encourage students to take responsibility for their own learning (Gilman & Rafferty, 1995).

In an effort to learn more about portfolio assessment and to demonstrate alternative assessment techniques to future elementary mathematics teachers, I have begun to incorporate portfolio assessment in my mathematics methods course. My approach to portfolio assessment was informed by much of the current literature on portfolios (e.g. Gilman & Rafferty, 1995; Lambdin & Walker, 1994; Stenmark, 1991).

Each semester, my preservice elementary mathematics students participate in a midterm and final portfolio review conference in lieu of an exam. During that conference, students present chosen pieces for their portfolios which demonstrate their understanding of course objectives and document their mathematics field experiences. They provide verbal and written rationale for each piece selected. In addition, students discuss their strengths and areas for improvement in mathematics teaching as a means of goal setting. Students are primarily responsible for driving the conversation while my role is to ask questions and provide comments. In addition, students are given varying degrees of freedom in developing rubrics for grading the portfolio.

During the proposed presentation, I intend to (a) provide a general background about portfolio assessment, (b) discuss aspects of designing, implementing, and evaluating portfolios in the mathematics methods classroom, (c) share specific examples of preservice teacher portfolios and excerpts from portfolio conferences, (d) describe how my experiences in using portfolios have influenced my practice, and (e) present data gathered on preservice teachers' reactions to the portfolio process.

References

Gilman, D. A. & Rafferty, C. D. (1995). Portfolios: They're not just work folders anymore. Indiana State University, Terre Haute, Indiana: Curriculum, Research and Development Center.

Lambdin, D. V. & Walker, V. (1994). Planning for classroom portfolio assessment. Arithmetic Teacher, 41 (6), 318-324.

Stenmark, J. K. (1991). Mathematics assessment: Myths, models, good questions, and practical suggestions. Reston, VA: National Council of Teachers of Mathematics.

CRITERIA FOR PERFORMANCE-BASED TASKS IN MATHEMATICS

Kyoko Suzuki, University of Illinois at Urbana-Champaign
Delwyn L. Harnisch, University of Illinois at Urbana-Champaign

The objective of this study is to illustrate the criteria for a "good" item on a performance-based assessment. Earlier research (Suzuki & Harnisch, 1995) demonstrated seven criteria for performance-based (paper-pencil type "test") tasks: 1) modeling real-world phenomena, 2) having multiple strategies, 3) having ordered categories for measuring maturity levels, 4) connecting several concepts to solve, 5) depicting the achievement levels by verbal explanations, 6) detecting the discrepancy between an intuitive solution and a mathematical solution, and 7) matching complexity of task with a scaling system. Two tasks were evaluated by analyzing students' responses to illustrate the above criteria.

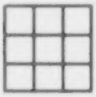
Telephone Area Code Problem

Telephone area codes in the U.S. and Canada consist of 3 digits, in which the first is a digit from 2 through 9, the second is either 0 or 1, and the third can be any digit except 0. Show all your work and explain how you found your answer.

(1) According to these rules, how many different area codes can begin with 6?
(2) How many different area codes can be an odd number?
(3) What is the probability that an area code is a multiple of 3?

Magic Square Problem

Arrange the integral numbers from 1 to 9 (1,2,3,4,5,6,7,8, and 9) into this square and make sure that each row, column and diagonal has the same sum: 15. Every number can be used only once. Show all your work and explain how you found your answer.



The students' responses collected in June 1993 were scored using a general scoring rubric from 0 to 4 with holistic perspective. The Telephone Area Code Problem revealed multiple strategies that were found to be ordered by achievement levels. Responses to the Magic Square Problem revealed many different solution approaches. For example, students drew many squares with numbers for finding their answers and explained "trial and error" or "I found the answer by chance" for their solution; however, these methods could not be ordered by achievement levels. Therefore, we can conclude that the Telephone Area Code Problem satisfies criteria 3 and 5, while the Magic Square Problem does not satisfy criteria 3 and 5. Performance-based assessment tasks must be distinguished from mathematical puzzles.

Reference

Suzuki, K., & Harnisch, D. L. (1995, April). *Measuring cognitive complexity: An analysis of performance-based assessment in mathematics.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.

HOW WELL IS CRITICAL THINKING EVALUATED? FINDINGS FROM STUDENTS' AND TEACHERS' ASSESSMENT OF EVALUATIVE ESSAYS

Howard Hansen, Western Illinois University
Judith Olson, Western Illinois University
Melfried Olson, Western Illinois University

What does it mean to be a 'better thinker'? In this study, we focus on critical thinking which is defined as the examination and evaluation—actual and potential—of beliefs and course of action. Critical thinking is often defined to include all good thinking including creative thinking. We should not conceive of critical thinking solely as a technique for settling the truth and justice of things, but rather as an enterprise of inquiry and understanding.

Our research involves the assessment of two distinct evaluative essays by junior high and senior high students as well as preservice and inservice teachers who were given a criteria and a rubric scoring scale from which to make their assessment. The essays were to be written using reasoned judgment and multiple criteria to support their conclusions. The data indicate that teachers, both preservice and inservice, do not appropriately use the rubric in the assessment process. The evaluation by the junior high and senior high school students do not differ dramatically from those of inservice and preservice teachers.

The data provide a note of concern for educators who are in the process of formulating open-ended alternative assessment instruments. If one component is to evaluate 'reasoning', then a significant amount of practical experience and philosophical discussion are needed by inservice teachers. Preservice teachers must also be provided with both instruction that incorporates alternative assessments as well as opportunities to evaluate such assessments.

In addition to a wide range of values for a given essay, there appears to be differences in the ratings given by elementary and secondary preservice teachers. Questions are raised as to whether this indicates differences in critical thinking resulting from educational background or a gender difference since the preservice and inservice elementary teachers were all females and all but one of the secondary students were male.

Cognitive Modalities

A CONSTRUCTIVIST USE OF TECHNOLOGY IN PRE-ALGEBRA

Michael L. Connell, Ph.D., University of Houston

This paper will present two examples where technology, in this case a fairly sophisticated authoring system - ToolBook, was used as a tool to construct student understandings in mathematics. In doing so, students were able: (a) to successfully identify the variables (unknowns) and the information given (data) in the problem; and (b) to create meaningful links between the data and givens which enable successful problem solution.

These examples were from work in a seventh grade pre-algebra classroom of below average ability students in a middle class urban setting and are from a single classroom. The curriculum used was conceptually based and utilized a five phase approach which allowed students to construct mathematical intuition via physical materials and computer use (Connell, 1994, Connell and Peck, 1993).

In this method, the initial two phases require use of physical materials to present problems and actively engage students with the materials to model mathematical situations, define symbols, and develop solution strategies. The third phase uses sketches of physical materials and situations experienced by the students to encourage a move toward abstraction. These student sketches, many of which were constructed using the object based graphics of ToolBook on the computer[1], then serve as the basis for additional problems and as referents for thinking. In the fourth phase, the children construct mental images through imagining actions on physical materials and manipulating the computer sketch. Following these experiences students construct arithmetic generalizations and problem solving skills through scripting their understandings using ToolBook[2].

This sequence might be visualized somewhat like Figure 1 which, although not complete, does capture the *look and feel* of the approach fairly well (Wirtz, 1979; Connell, 1986).

	Memory/Recall	Teacher Posed Problems	Self Posed Problems
Manipulatives			
Sketch			
Mental Picture			
Abstraction			

Figure 1. Simplified model.

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[1] It is interesting to note the parallels between the mathematical objects these students created and the objects of analysis mentioned by Sfard and Thompson (1994).

[2] I have found HyperCard on the Macintosh works equally well. The key is not in the type of computer, but rather in how it is used.

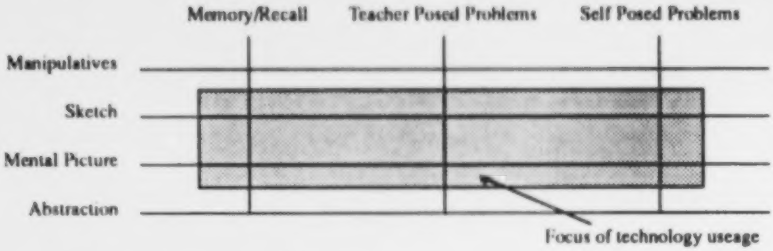


Figure 2. Focus of technology use.

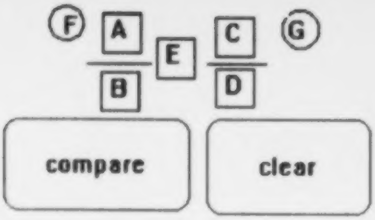
This approach was implemented using an object oriented computer authoring language, ToolBook in this case. The nature of this language allowed for a wide range of powerful tools, such as drawing and painting, to be available for student use, and yet still had relatively simple syntactic requirements conducive to expressions in algebraic terms.

Students did work on computers until immersed in their problems via physical materials. Students commonly developed initial working representations on the computer and identified what the relevant information should be through creating appropriate input and output fields. The developing representations at this time had features common to both sketches and mental pictures. Abstraction began as they constructed their method of procedure and expressed it in algebraic terms by scripting buttons. I think of this usage of technology as providing for student construction of a bridge between sketch and abstraction as shown in Figure 2.

The computer acted as a tool and an active listener doing what it was told, not as an instructor requiring a specific answer. This "tool" helped students identify variable(s) and information (data) necessary for problem solution and to construct appropriate linkages. The student representations on the computer reflected their own ongoing construction of meaning. Family resemblances were observed in observing student work. First, students began by using the sketch tools to create a working sketch. This seems to indicate a tight linkage between the curriculum and the technology. Second, with sketch in place, students created and named fields corresponding to variables. This appears to have been highly helpful in their thinking. Third, buttons were scripted linking fields and solving the problem. The drawing tools of ToolBook and the ability to create almost any representation appeared to liberate student thinking and contributed to a natural integration of computers in the classroom.

Two student examples

These two examples illustrate how the scripting of computer objects created an entry point into the algebra. In presenting these, several modifications were necessary. First, colors were changed to black and white (originally they were highly colorful) Students learned about paint options quickly. Second, field names are indicated to aid discussion. Lastly, spacing was added to scripts to discuss the function of each section. The examples were originally for students' use so formatting and annotating were not high priority. All else is as it was.



Example 1. *Comparison of fractions.* This tool was created after a review of fractions during which the cross-multiply method emerged. It is highly unlikely that this was a spontaneous creation - most likely it was a "rediscovery" or a "remembrance" of old learning.[1] The example shows how the students used ToolBook, however, and provides examples of scripting.

The first thing was to lay out the problem space using fields and graphic objects. As shown, the fields used by the student have been labeled **A - G**. The buttons, **compare** and **clear**, were then added and scripted to solve the problem.

The scripting for the button **compare** is shown. The script breaks down into some well defined sets of instructions bracketed between the *to handle buttonup* and *end buttonup* statements. These tell the button to execute these instructions when clicked.

```
to handle buttonup
     put the text of field  A  into a
     put the text of field  B  into b
     put the text of field  C  into c
     put the text of field  D  into d

     put d*a into the text of field  F
     put b*c into the text of field  G
     put the text of field  F  into f
     put the text of field  G  into g

     if f>g then
          put  >  into the text of field  E
     end if

     if f<g then
          put  <  into the text of field  E
     end if

     if f=g then
          put  =  into the text of field  E
     end if
end buttonup
```

1) The student first assigns variables (a, b, c, and d) which correspond to the fields **A - D** used for input by the comparison tool.
2) These variables—containing values input when using the program—are used in calculation of values which are placed in fields **F & G**.
3) Logical conditions are then checked to see which of the comparison symbols (<, > or =) is to be placed in field **E**.

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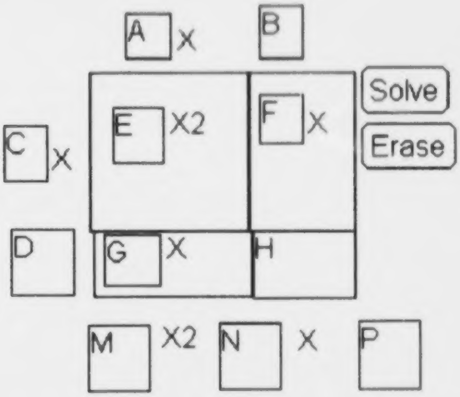
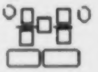
[1] This is not to say that students are incapable of constructing this method. For a discussion of one class in which students *did* construct this method see Peck and Connell, 1991.

The button **clear** was a much easier task and merely required a blank, or " ", to be placed into each field where either a character or numeral might be. This proved to be of such great utility that a version of **clear** soon became common in each of the student created tools. This illustrates creation of new objects by combining features of previously created objects. This not only enabled the **clear** button to migrate, but also allowed for the tools themselves to be shared and used by the entire class.

```
to handle buttonup
        put " " into the text of field "A"
        put " " into the text of field "B"
        put " " into the text of field "C"
        put " " into the text of field "D"
        put " " into the text of field "E"
        put " " into the text of field "F"
        put " " into the text of field "G"
end buttonup
```



For example, the student who created the tool shown here made a copy of it and shrank it down very small - like this shown here. Then when needed, the student would select it, expand it to useable size, and then shrink it back when it was no longer needed. It was not uncommon to see created tools of various types throughout any given "page".



Example 2. *Multiplication of binomials.* The similarity of approach students brought to bear between these two examples is easily seen. As in the fraction tool, the first thing done was to lay out the problem space using fields and graphic objects.

In the sketch shown the fields used by the student have been labeled **A - H, M, N, & P**. Original field names were not nearly so terse. Snoopy, Wimpo, and REM all ap-

```
To handle buttonup
        Put the text of field "A" into a
        Put the text of field "B" into b
        Put the text of field "C" into c
        Put the text of field "D" into d

        Put a*c into the text of field "E"
        Put c*b into the text of field "F"
        Put a*d into the text of field "G"
        Put b*d into the text of field "H"

        Put the text of field "E"into e
        Put the text of field "F"into f
        Put the text of field "G"into g
        Put the text of field "H"into h

        Put f+g into the text of field "N"
        Put e into the text of field "M"
        Put h into the text of field "P"
end buttonup
```

peared during early experiences, but proved awkward for students to remember and took longer to type. Soon single letters were adopted.

The buttons, **Solve** and **Erase**, were then added and scripted to solve the problem. As **Erase** is a modified copy of the **clear** button it will not be described.

1) Once more, the student first assigns variables (a, b, c, and d) which correspond to the fields **A - D** used for input by the multiplication tool.
2) The student then uses these variables in calculation of values which are then placed in fields **E, F, G & H**.
3) Then, in a rather interesting piece of scripting, the student then reads the numbers which the computer has put into fields **E, F, G & H.**
4) Finally, these values are used to perform the final calculations and output necessary for the answer to be in a more useable form for the student.

Implications for mathematics education

Technology in mathematical exploration typically takes the form of a black box with only outcomes visible. Methods of solution leading to the answer and rationale for them is invisible. We must provide more than a black box giving right answers, the box must be subject to student control and exploration. The work reported in this paper illustrates an alternative to black box approaches which places the student in control of the computer. As the results clearly show, this in turn results in the student being in control of the content.

References

Connell, M. L. (September, 1994). *Making Mathematics:Teaching and learning mathematics in a technologically rich environment.* Paper presented to the 75th anniversary research symposium of the University of Illinois at Champaign Urbana. Champaign:IL.

Connell, M. L. (1986). *Conceptual based evaluation and student tracking in elementary mathematics.* Unpublished masters dissertation. p. 75-80.

Connell, M. L., & Peck, D. M. (1993). Report of a conceptual change intervention in elementary mathematics. *Journal of Mathematical Behavior. 12(4)*, 329-350.

Peck, D. M. & Connell, M. L. (1991). Using physical materials to develop mathematical intuition. *Focus on learning issues in mathematics. 13* (4), 3-12.

Sfard, A., & Thompson, P. (November, 1994). *Reification in Post Secondary and K-12 Mathematics.* Paper presented at the 16th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Baton Rouge:LA.

Wirtz, R. (1979). *Mathematical problem solving.* In D. M. Peck (Chair), Conceptual issues in mathematical problem solving. Symposium conducted at Department of Educational Studies. University of Utah.

ENUMERATING CUBES IN 3-D ARRAYS: STUDENTS' STRATEGIES AND INSTRUCTIONAL PROGRESS

Michael T. Battista, Kent State University
Douglas H. Clements, State University of New York at Buffalo

This investigation is part of a combined research/curriculum development project in which children's learning is being examined in the context of developing and testing instructional units on 3-D geometry at grades 3, 4, and 5. There are two components to the article. First, we describe the strategies and cognitive constructions students utilize to conceptualize and enumerate the cubes in 3-D arrays. Second, we examine the change in thinking of students as they are involved in instructional tasks that have been utilized to help students develop more sophisticated thinking about enumerating cubes in 3-D arrays.

Enumeration Strategies and Cognitive Constructions: An Overview

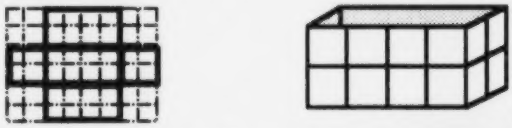
In previous research, we have described in detail students' strategies and difficulties in enumerating 3-D rectangular cube arrays (Battista & Clements, in press). Our theory suggests that students' initial conception of a 3-D rectangular array of cubes is as an uncoordinated set of faces of the prism formed. These are the students who count all or a subset of exterior cube faces. Eventually, as students become capable of coordinating orthogonal views of the array, and as they reflect on experiences with counting or building cube configurations, this conception is perturbed. They see the array as space filling and strive to restructure it as such. Those who complete a global restructuring of the array conceptualize the set of cubes organized into layers. Those in transition, whose restructuring is local rather than global, conceptualize the set of cubes as space-filling, attempting to count all cubes in the interior and exterior, but do not consistently organize the cubes into layers. They have not yet formed an integrated conception of the whole array that globally coordinates its dimensions. Indeed, our data supports this hypothesized sequence of conceptions. From 3rd to 5th grade, we saw that students made a definite move from seeing a 3-D cube array as an uncoordinated medley of faces toward seeing it in terms of layers. We also saw a significant number of students in transition, with these students exhibiting a wide range of sophistication in their structuring of such arrays.

Our research suggests that *spatial structuring* is a fundamental notion in understanding students' strategies for enumerating 3-D cube arrays. We define spatial structuring as the mental act of constructing an organization or form for an object or set of objects. We found that in the process of determining the number of cubes in an array, students' spatial structuring of the array determined their enumeration of it; sometimes their spatial structuring supported a correct enumeration, sometimes it inhibited it.

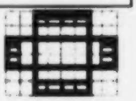
The Evolution of Students' Thinking during Instruction

A fifth grade class was divided into pairs of students, each working on an activity sheet consisting of problems in which students were to predict how many cubes would fit in a box, then check their answer by making the box out of grid paper and filling it with cubes. The teacher circulated about the room, listening to students' conversations and asking questions. The first researcher observed and recorded the work of one pair of students, N and P, throughout the instructional unit. We will trace the course of these students' construction of a viable structuring and enumeration scheme for 3-D cube arrays.

Box A



Pattern Picture *Box Picture*



For Box A, N counted the 12 outer squares on the 4 side flaps, then multiplied by 2: "There's 2 little squares going up on each side, so you times them."

P counted the 12 visible cube faces showing on the box picture, then doubled that for the hidden lateral prism faces. So both students agreed on 24 as the prediction. After putting 4 rows of 4 cubes into the paper box, the boys exclaimed:

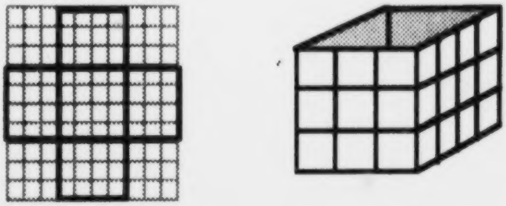
N&P: We're wrong. It's 4 sets of 4 = 16.

N: What are we doing wrong? [question directed at himself and his partner]

P: I know; we counted these twice [pointing to the column of 2 cubes on the right front corner of the box picture].

The boys then examined the box they constructed and concluded that they should have subtracted 8 for the 2 double-counted cubes at each of the 4 vertical edges (which would have given them a correct answer). So their reflection on the discrepancy between the actual and predicted answers caused them to discover their double counting.

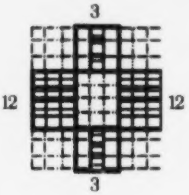
Box B



Pattern Picture *Box Picture*

For Box B, P counted 21 visible cube faces on the box picture, then doubled it for the hidden lateral prism faces. He then subtracted 8 for the double counting (not taking into account that this box is 3 high, not 2, like Box A). He predicted 42 - 8 = 34.

N added 12 and 12 for the right and left side flaps on the pattern, then 3 and 3 for the front and back flaps, explaining that the outer columns of 3 on the front and back flaps were counted when he counted the right and left flaps. His prediction was 30.

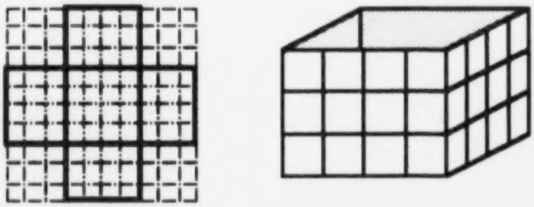


Both N and P accommodated their structuring and enumeration schemes in attempting to deal with the double-counting error. P compensated for the error by subtracting double-counted cubes. N tried not to double count.

After the boys constructed the box, filled it with cubes, and discovered that their answer was incorrect, P tried to figure out why their predictions were wrong: "If there's 21 here and 21 there, there's still some left in the middle. We missed 2 in the middle."

In this episode, the boys discovered yet another shortcoming of their original counting strategy—it ignored cubes in the middle. But as they attempted to compensate for this error, they focused on numerical differences, rather than carefully analyzing the spatial structure of the cube arrays. P concluded that they missed 2 cubes in the middle because 2 was the difference between his prediction of 34 and the actual answer of 36. The boys used a similar line of reasoning in making their prediction for Box B. They subtracted 8 for double counting because they needed to subtract 8 to make the prediction for Box A correct. However, although neither P nor N had yet developed a structuring of 3-D arrays that lead to correct enumeration of cubes, they were abstracting important aspects of the spatial organization of the cube arrays that would help them make the needed restructuring.

Box C


Pattern Picture

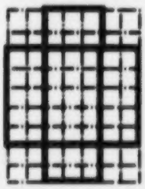
Box Picture

For Box C, N and P counted 24 visible cubes on the box picture then multiplied by 2 to get 48. They subtracted 12 for double counting the vertical edge cubes, getting a total of 36. But they decided that Box C was bigger than Box B, so they tried another analysis. This time they counted 21 outside faces (not double counting cubes on the right front vertical edge), times 2 for the hidden lateral faces. They then added 2 for the middle cubes (which is how many cubes they concluded they missed in the middle of Box B) to get a total of 44. They filled the box and found it contained 48 cubes.

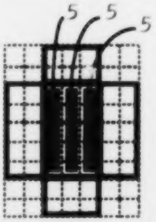
The next day, N and P began class by trying to figure out what they did wrong with their prediction for Box C. They reviewed their method and concluded that they didn't add enough cubes for the middle—they needed 6 instead of 2. But they derived this conclusion by comparing their predicted amount, 44, to the actual number, 48, not by analyzing the spatial structure of the cube arrays. Analyzing only numbers can easily lead one astray in spatial situations. At this point, N and P's numerical reasoning was not properly linked to the spatial structure of the arrays.

Box D

Pattern Picture

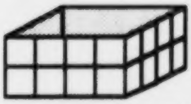


For Box D, N said there would be 30 cubes: "5 + 5 + 5 for the columns in the bottom, times 2 because there are 2 up." The boys cut out the pattern, filled it with cubes, and determined that it had 30 cubes. This excited them because it was the first time their prediction was correct. N explained his procedure to P, and P said he understood it: "You find how many are on the bottom, then you count how many you go up; 5 by 3 by 2 up. Add 15 and 15 and get an answer of 30."



When the observer asked N how he developed this strategy, N said that he generated the idea while looking at Box D, then tested it (silently) on Box C and found that he got the correct answer. N had been staring off into space for a while, clearly thinking about the problem. It seemed that he knew there was a better way to solve the problem, that he was reflecting on and analyzing the situation. On this problem, the boys seemed to abandon the counting of exterior cubes to find another structuring, possibly because of the shortcomings they were finding with their previous methods.

Box E



At this point, the boys' method for enumerating cubes was confined to examining box patterns, so problem 5 presented some difficulty for them. P counted 16 around the bottom and 16 around the top. But N replied, "Wait, that's not right because you counted these 2 twice [at the right front vertical edge]." P agreed, so they decided the count for the bottom layer should be 14. P said there were two horizontal layers, and predicted 32 - 8 = 24; taking 8 away because of the double counting on the 4 edges. But N said, "We don't know there's only two rows in this [meaning horizontal layers]. I think there might be 3." N predicted 28, which he arrived at by counting 14 on the front and right sides (not double counting the corner cubes), then multiplying 14 times 2, saying to P: "Maybe you should only take 4 away [so their predictions would be equal]."

After the boys correctly made the pattern, the observer asked them if they wanted to stick with their predictions, now that they could see the pattern. P said it was 16×2 = 32, + 16 = 48. N said it was just 32. But they decided to stick with their original predictions.

The boys seemed to be confusing parts of their old and new strategies. For instance, when only the box picture was available, it's possible that N used his count of the front and right sides in the same way he used his count of the bottom of a pattern. He didn't seem to be able to visualize what the bottom would look like. Even though looking at their pattern seemed to enable the boys to conceptualize the array in terms of layers, they didn't change their original predictions, seemingly unable to decide which of the two strategies was appropriate. However, when the boys put the cubes in the box and found that 32 fit, N said, "It is 32," as if coming to some realization.

Box F The bottom of the box is 4 units by 5 units. The box is three units high.

For Box F, the boys were unwilling to make a prediction until after they had made the pattern. N counted all the bottom squares in the pattern one by one—once for each of the 3 layers—counting 1-20 the first time, 21-40 the second, and 41-60 the third.

P: You counted 3 times, no 4.

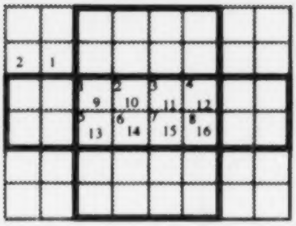
N: Why 4, it's 3 up? [with assurance]

The boys predicted 60 cubes, seeming quite confident in their prediction. They built the box and filled it with rows of 5 cubes, then counted the cubes by fives to 60. However, they didn't seem relieved that they were correct. Instead, they expected that their answer would be correct. Later, the boys read aloud the procedures they had written for determining the number of cubes in a box:

N: You count how many are on the bottom. Then you add how many go up.

P: You multiply to find the bottom. Then you multiply by how many high.

To test P's understanding of his procedure, the observer asked him how many cubes would be in a box that was 3×2 on the bottom and 5 high. He drew a 3×2 rectangle on graph paper, then correctly drew the four sides: "3×2 on the bottom, 6. 6 × 5 = 30."



Both boys seemed to have come to an understanding of a layering approach. They struggled, but they found viable methods to solve the "How many cubes?" problem and seemed pleased with themselves for doing so.

On the third day of the unit, N showed the observer an alternate way of finding the number of cubes in a box. He had described and illustrated his method in his journal. "There are two up, so you have to count two for each on the bottom." N demonstrated by counting by ones from 1 to 8 as he pointed to the 8 squares on the bottom, then counting from 9 to 16 as he pointed to each of these squares again.

Finally, the observer asked the boys how many cubes would be in a box that had the same bottom as Box A but was 3 cubes high.

P: 8 times 3 = 24.

N: Yeah, 8, 16, 24. I'm not too good at my multiplication facts.

Analysis

Throughout this account, N and P were trying to develop a theory of how to make correct predictions. The discrepancies between what they predicted and what they actually found caused them to reflect on their prediction strategies and their structuring of the cube arrays. At first, their enumeration strategies were based on more primitive, spatial structurings of 3-D arrays—seeing them in terms of the faces of the prism formed. The boys seemed to focus more on numerical strategies than a deep analysis of the spatial organization of the cubes. However, because their initial spatial structuring led to incorrect predictions, the boys refocused their attention on the structure of the cube arrays, which led to a restructuring of their mental models of the arrays. In fact, during their work on Box D, N and P seemed to develop a layer structuring of the array, a structuring that they verified and refined on subsequent problems.

The gains for N and P were typical of those achieved by students in this instructional unit. For instance, in one class of 47 fifth-graders, of the 31 students who did not use layering strategies on all the pretest items, 16 were doing so on the posttest, 9 increased their use of layering strategies, 4 did not increase, and 2 decreased. So, 81% increased their use of layering strategies. And 5 out of 6 of the students who did not increase used layering strategies on a box item similar to those on the student sheet discussed above. Forty-three of the students got this item correct; 2 of the 4 students who missed the item made computational mistakes.

Conclusions

Consistent with constructivist accounts of the learning process, two of the essential components of learning for N and P were reflection and cognitive conflict. Reflection and cognitive conflict were promoted by focusing students on *predicting* the number of cubes of 3-D arrays. Errors in predictions—which the boys themselves discovered—caused cognitive conflicts, or perturbations in the boys' current mental models for arrays. The boys attempted to resolve these conflicts by reflecting on the strategies they were using, all the while examining and restructuring their mental models of the arrays. In fact, the boys moved from an incorrect conception of the arrays, to a period confusion in which they vacillated between different conceptions, to a viable conception that resolved their confusion.

The account of N and P's work illustrates the constructivist claim that, like scientists, students are theory builders. They build conceptual structures to interpret the world around them. Cognitive restructuring is engendered when students' current knowledge fails to account for certain happenings, or results in "obstacles, contradictions, or surprises. The difference between the scientist and the student is that the student interacts with a teacher, who can guide his or her construction of knowledge as the student attempts to complete instructional activities" (Cobb, 1988). This guidance is often covert; in the present situation, the guidance came through the sequence of tasks, not by telling N and P problem solutions.

References

Battista, M. T., & Clements, D. H. (in press). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education.*

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist, 23,* 87-103.

INFERRING INTERNAL STRATEGIC PROBLEM REPRESENTATION AND ITS DEVELOPMENT: A TWO-YEAR CASE STUDY WITH MARCIA

Richard A. Zang, University of New Hampshire at Manchester

This study is part of a longitudinal study which entails observing 22 elementary school children over a 3-year time span in the classroom and in individual task-based interviews. Through the examination of videotaped structured clinical interviews of Marcia, the first in 1992 when she was 9 years old (4th-grade), this study seeks to elucidate inferences of her internal strategic problem representations while engaged in problem-solving activity, and to analyze how these representations developed over a two-year time span. Marcia's analysis shows that strategies are accessible to her, and even more importantly, that they do not necessarily need to be "taught". The research reported here shows that Marcia, when left to her own devices, not only invented strategies and representations to aid her in finding a solution to the problem, but often she did so in noncanonical ways.

If one views learning as the acquisition of competencies, and further, internal representations as descriptors for such competencies, it would stand to reason that children with rich systems of internal representation developed from exposure to appropriately chosen task-based situations, would be in a better position to assimilate, and thus learn, new competencies when exposed to new problem solving situations. It follows then, that we need to better understand these internal systems of representation to foster learning. Moreover, we need to develop ways to assess internal representation acquisition of individual children.

By designing task-based interviews with a homomorphic nature to them (sequences of geometrically presented figurate numbers), and by conducting these interviews two years apart, cognitive representations were able to be compared over that time span. Moreover, as cognitive representations describe competencies, various types of competencies (e.g., reversibility of strategy) could be compared. The richness of possible behaviors allowed for by the protocols, and subsequently observed, enabled Marcia's cognitive development to be charted.

The focus of this study is Marcia's *choice of representation* of the task, and her *strategic decisions* and methods of solution. In each structured clinical interview, an analysis of Marcia's external representations (what the child constructed and verbalized) has been conducted, and inferences of internal strategic representations have been made consistent with Marcia's observed behavior (Zang, 1995). Among the features of strategic representations looked for were: 1) spontaneous use of formal symbolic representation in place of concrete manipulatives; 2) heuristic processes; 3) ability to generalize; and 4) reversibility of reasoning.

This is a qualitative case study, the purpose of which is purely exploratory and descriptive. There is necessarily a certain subjective nature to the reporting of the

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The research reported in this study was partially funded by a grant from the National Science Foundation (NSF). "A Three-Year Longitudinal Study of Children's Development of Mathematical Knowledge", directed by Robert B. Davis and Carolyn A. Maher of Rutgers University. The opinions expressed and conclusions reached do not necessarily represent those of the NSF.

results of such a study. No attempt was made to conduct independent analyses by more than one researcher to achieve intercoder reliability; i.e., this technique was not used. *Ergo*, no assertion is made about the reliability of the inferences drawn.

However, the research team involved has allowed for an essential feature of scientific progress, by carefully preparing scripted protocols for the structured task-based interviews (Goldin, DeBellis, DeWindt-King, Passantino, and Zang, 1993; Zang, 1995). This had the dual effect of not only preparing the clinicians involved to mitigate their individual interview styles and adhere to a planned sequence of questions and contingencies, but also to permit a degree of comparability and reproducibility so that these results can be further extended and compared with those of other researchers.

The Tasks

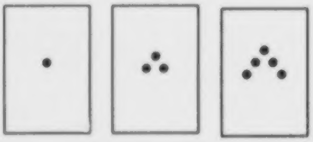


Figure 1. Task #1

The domain of the problem solving activities revolve around 5 problem contexts involving number sequences with attending figure/geometric representations. Materials (index cards, red and black chips, markers of different colors, paper, and pencil) were placed ahead of time on the table in front of the child for each of the interviews. The first task-based interview consisted of laying 3 cards, one at a time, in front of the child as illustrated in figure 1. The following series of questions was then asked: 1) "What card do you think would follow that one?" [then asked again with reference to the 4th card]; 2) "Do you think this pattern keeps going?"; 3) "How would you figure out what the 10th card would look like?"; 4) "Here's a card (showing one with 17 dots in the shape of a chevron) ... can you make the card that comes *before* it?"; and 5) "How many dots would be on the 50th card?"

For each question discussed above, the following stages exist: 1) Posing of the question (free problem solving); 2) Heuristic suggestion (if not spontaneously evident; e.g., "Can you show me using some of these materials?"); 3) Guided use of heuristic suggestion (e.g., "Do you see a pattern in the cards?"); and 4) Exploratory (metacognitive) questions (e.g., "Do you think you could explain how you thought about the problem?"). The clinician always sought to elicit a complete, coherent verbal reason and a coherent external representation before proceeding to the next question. It is important to note that the canonical card did not have to be drawn or the canonical pattern described, in order for a response to be considered a complete and coherent reason and a complete coherent external representation.

The next task-based interview, as it relates to this present study, was conducted two years later (1994). Task #2 involved 4 problem contexts as illustrated in figure 2. Each of the 4 problems utilized the same basic paradigm as in the first task. In formulating/designing this task-based interview, and what task(s) it would encompass as its domain, it was decided that the richness of variety embodied in figurate number sequences and their interrelatedness (i.e., homomorphic nature) to task-based interview #1 would be the most appropriate vehicle to look at developing internal strategic problem representations. In light of and in recognition of the fact that we as researchers cannot see the cognitions being employed as problem solving occurs, and instead we must infer these cognitions (with the aid of self-reporting by the child and their accompanying external representations), it is prudent to provide a cornucopia of figurate number tasks (each one building on the other) such that externalizing the internal representations provides as much of a cognitive window as possible to viewing their internal strategic representations.

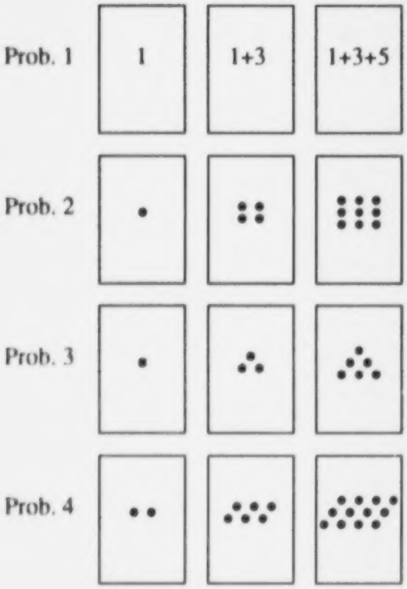


Figure 2. Task #2

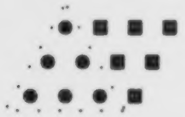
Some Highlights

During Task #1, Marcia developed an add-two strategy, but not in the canonical fashion (i.e., adding two dots to the bottom of the previous figure). Instead, she would draw each dot in contiguous order starting from the lower left portion of the chevron. The two dots she said she was adding, were always the last two dots on the bottom right side of the chevron (she gave quite a lengthy explanation of this). When asked what the 10th card would look like, she appeared to discover that the number of dots on the left side of the chevron represents the number of that card (e.g., 3 for 3rd card, 4 for 4th card, *et cetera*), and that the number of dots on the right side was one less. She stopped constructing the cards *en route* to the 10th card, after the 7th card (having constructed the 4th, 5th, 6th, and 7th cards), and proceeded to give a verbal accounting of what the 10th card would look like. The inference made is that this is when the discovery took place, or at least the point that it manifested itself externally, for it was here that she was able to apparently generalize, and extend mentally, the operations necessary to form an internal representation of the 10th card. The inference made is that she then held a more imagistic internal representation, in that, for the first time, there was a direct correlation between the number of dots she was focussing on (left side of the chevron) and the geometry of the shape; all done simultaneously (tieing the numeric concept into a portion of the shape).

When asked the "reversibility" question--Here's a card (showing one with 17 dots in the shape of a chevron)...can you make the card that comes *before* it?" —

Marcia said and gestured that there were 9 dots going up on the presented card, and so there would have to be 8 dots going up (referring to the left side of the chevron) on the card before it.

Two years later when the "reversibility" question was posed during Task #2, Problem 3, she apparently realized it was unnecessary to know that number (the number of dots on the left side of the triangle); one merely had to remove the bottom row (no matter how many dots may be in it, or any other part of the figure).



Marcia was asked if she noticed any relationship between the 3rd and 4th set of cards during Task #2. She noticed that two of every card from the 3rd set is embodied in the corresponding card of the 4th set (albeit one of the two from the 3rd set is inverted). Along with a good verbal accounting, she explained as follows:

Here, it was inferred, she utilized an internal imagistic representation(s); to "see" instances of certain problems embodied in still other instances of other problems.

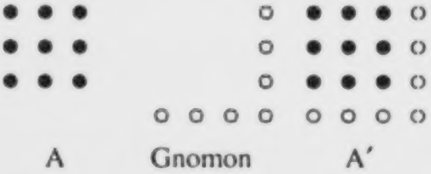


Figure 3. Cognitive Dots

Another interesting highlight during Task #2 was when Marcia employed a gnomon-like strategy while engaged in Problem 2. A gnomon to a geometric figure (A), is another geometric figure, such that when the gnomon is suitably attached to A, the resulting figure (A') is similar (in a geometric sense) to A. Because of her verbal accounting, gesturing, and drawing (a representative example of the type of behavior she exhibited is seen in figure 3), the inference made is that the dots that comprise the gnomon are the dots she is focussing attention on (I refer to these as cognitive dots), and thus are the dots she is using to draw a pattern from. The inference made is that the focus on these cognitive dots is her strategy, whereas the overall process of attaching (mentally) part A with a gnomon (resulting in A') is the heuristic process. This discussion serves to illustrate McClintock's (1984) suggestion that heuristic processes may be viewed from another standpoint (i.e., as inherent in and thus residing in the mathematical problems themselves).

Conclusion

The structured individual task-based interviews proved to be an appropriate research tool as they were able to draw out the *processes* Marcia used (e.g., strategy use), as opposed to the more traditional *product* so often emphasized in the classroom (Goldin et al., 1993; Zang, 1995). This is an important distinction in that teachers have always manifested an overriding concern to *measure* learning. In their quest to measure, they inevitably turn to the product of the problem solving (test scores, time to solution, etc.), and have traditionally not focused on the process.

One possible implication of this research (to the extent that results are generalizable) is that we as mathematics educators should be mindful that a teacher's expectations of the canonical response can have the cognitive effect of being so overriding, that they can interfere with the teacher being receptive to really insightful ways of thinking by a child. As such, this research suggests that one possible way of teaching these strategic representations, is not to offer instruction in them *per se* (i.e., in some procedural way), but rather to provide a rich environment wherein the children will construct them on their own, in much the same way as occurred during these task-based interviews with Marcia. Another possible implication of this research is that we can introduce geometric concepts much earlier than traditionally thought — teaching children to exploit the visual aspects of a problem in their early years of problem solving, long before the more formal operational approach is encountered

References

Goldin, G. A., DeBellis, V. A., DeWindt-King, A. M., Passantino, C. B., & Zang, R. A. (1993). Task-Based interviews for a longitudinal study of children's mathematical development. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Eds.), *Proceedings of the 17th International Conference of the Psychology of Mathematics Education (PME)* Volume I (pp. 197-203). Tsukuba, Ibaraki, Japan.

McClintock, C. E. (1984). Heuristic processes as task variables. In G. A. Goldin & C. E. McClintock (Eds.), *Task variables in mathematical problem solving* (pp. 171-234). The Franklin Institute Press.

Zang, R. A. (1995). Inferring Internal Strategic Problem Representation and its Development: A Two-Year Case Study with Four Elementary School Children. *Dissertation Abstracts International*, *55*(12A), 3775. (University Microfilms International No. 9514127)

Acknowledgments. Special thanks are due to many people at Rutgers Univrsity where this research was conducted: the research team; the AaPS project video team; and most importantly, Gerald A. Goldin, who was my thesis advisor.

PROCEDURAL AND CONCEPTUAL UNDERSTANDINGS OF THE ARITHMETIC MEAN: A COMPARISON OF VISUAL AND NUMERICAL APPROACHES

Elizabeth Ann George, University of Pittsburgh

Although the average, or arithmetic mean, has a rich conceptual meaning, it is often simply defined as the outcome of a procedure. The purpose of this study was to compare the nature and extent of the procedural and conceptual understandings developed by two groups of students who had received different forms of instruction, one based on the traditional numerical algorithm and the other on a visual algorithm. When confronted with tasks varying along several dimensions, students adjusted or extended their basic approach for finding the arithmetic mean in ways that give insight into their understanding of this mathematical concept. While both groups of students showed a degree of understanding and flexibility with the procedure they had been taught, students who had learned the visual procedure showed a deeper conceptual understanding of the arithmetic mean.

A growing amount of information in today's world is presented and must be processed numerically. Therefore it is crucial to understand the relationship between a set of numbers and the representative numbers, or statistics, used to describe the set. One commonly used descriptive statistic, the arithmetic mean, is usually introduced in elementary and middle school mathematics classrooms. Traditional instruction on this topic primarily focuses on a numerical algorithm which is executed when a set of numbers is given and determining the average value is the intended goal. The arithmetic mean is rarely taught as a concept, but rather as the outcome of a computational procedure—the result of dividing the sum of the numbers in the given set by the number of numbers in the set.

If a student's sense of the arithmetic mean is too closely tied and limited to the outcome of a procedure, an impoverished understanding of the arithmetic mean is often the result. A series of studies have probed students' understanding of the arithmetic mean. Strauss and Bichler (1988) identified the concept of the mean as having seven different properties and found that it was particularly difficult for children to view the arithmetic mean as representative of the values that had been averaged. Mokros and Russell (1995) further examined the relationship between students' ideas of representativeness of a set of numbers and their understanding of the arithmetic mean and found that students who approached the mean as an algorithm rarely understood the average as a number which represents a data set. These students were limited in the strategies they had available and confused about the meaning of the total sum, the arithmetic mean, and the numbers in the data set. Earlier investigations of students' understanding of the arithmetic mean (Pollatsek, Lima, and Well, 1981; Mevarech, 1983), showed that even college students who relied on the numerical algorithm to find the average of a set of numbers displayed

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The author wishes to acknowledge the helpful comments of Dr. Edward A. Silver on an early draft of this paper.

misconceptions when confronted with more complex tasks involving the arithmetic mean.

A visual alternative to the traditional numerical approach for finding the arithmetic mean is offered in the middle school curriculum, *Visual Mathematics* (Bennett and Foreman, 1991). Students build a column of wooden cubes to represent each number in a given set, then level-off the columns of cubes; the height of the leveled-off columns is defined as the arithmetic mean. Students are encouraged to move from physically constructing columns of cubes to using diagrams. The authors claim that discussion and practice with this visual "leveling-off" method reinforces the concept of the average as students are forced to consider the relationship between the numbers in the set and the average itself.

The two instructional methods described above both involve finding the arithmetic mean through the use of a procedure, whether numerical or visual. Both procedures move in a linear fashion from a given set of numbers through an algorithmic series of actions taken on those numbers, to produce a numerical outcome which is called the arithmetic mean or average. With either procedural approach students can easily come to interpret average as a "do-something signal", in much the same way that Kieran (1981) described students' view of the equal sign as an operator, not a relational symbol. Just as students must come to understand the equal sign as expressing a relationship of equivalence, students must come to see the relationship between the numbers in a set and the arithmetic mean. Understanding this relationship should allow students more flexibility in solving problems involving the arithmetic mean. With this deeper conceptual understanding, they should be able to move back and forth between the numbers in the set and the average, not simply proceed in one direction from the given numbers to the average.

Both numerical and visual procedures for finding the arithmetic mean have strengths and limitations dependent on the size of the numbers in the set, the size of the set, and the adaptability of the procedure for less straightforward problems. Both instructional methods could potentially produce a limited understanding of the arithmetic mean. Of interest is whether either or both of the procedural based instructional methods can help students construct rich conceptual understandings. Examining the nature and extent of students' procedural and conceptual understanding of the arithmetic mean involves accessing students' understanding of the procedure they were taught, their flexibility with and willingness to extend that procedure, and their ability to move between visual and numerical procedures. The purpose of this study is to compare the nature and extent of the procedural and conceptual understandings developed by two groups of students who have received different forms of instruction, one based on the traditional numerical algorithm and the other on a visual algorithm.

Method

Six students participated in this study during the fall of their seventh-grade year. None of the students had yet covered the topic of the arithmetic mean in their seventh-grade math classrooms, though all had received instruction during their sixth-grade year. Two of the subjects were enrolled in a school whose mathematics instruction was primarily drawn from the *Visual Mathematics* curriculum and had learned the visual "leveling-off" method for finding the average of a given set of numbers. The other four subjects had received instruction based on the numerical "add and divide" algorithm.

Five tasks involving the arithmetic mean were administered individually to each student. The goal of the first task was simply to find the average of a set of four relatively small numbers; the results of this initial task served as a baseline from which responses to the subsequent tasks were examined. The other four tasks varied along several dimensions, such as the size of the numbers, the size of the set of numbers, the initial representation of the task in either visual or numerical form, and the goal of the problem. Tasks 2 and 3 were presented in the context of a story, with the term "average" embedded in the text. Both tasks stated three of four numbers in the set and the average of the set; the goal was to find the fourth number in the set. While both tasks presented information visually, task 2 used discrete objects and task 3 used a bar graph to model the given situation. These tasks were adapted from the QUASAR Cognitive Assessment Instrument (Lane, 1993). The fourth task shared the same goal as the initial task, to find the average of a given set of numbers, but used larger and more numbers. Task 5 asked students to construct sets of numbers having an average of 12. (See Appendix for selected tasks.) To fully capture student thinking, students were asked to think aloud as they completed the tasks. These verbal protocols ranged from 20 - 35 minutes and were transcribed from audiotapes. Coding and analyses of the transcriptions and of students' written work form the basis for this study.

Analysis focused on the approaches each subject used to complete the tasks. The strategy used on each task was first coded as primarily visual versus primarily numerical. Visual strategies were further coded as involving wooden cubes or diagrams. Responses were also coded as successful or unsuccessful in arriving at a correct solution. Both successful and unsuccessful attempts were analyzed for evidence of student understanding of the arithmetic mean concept and sources of errors were identified as computational, counting, or conceptual.

Results

Similarities and differences in students' procedural approaches and conceptual understandings become evident as patterns of behavior appeared for each subject across tasks. When presented with the initial task, most students used the method that had been the basis of their classroom instruction. Examining each individual student's responses across the subsequent tasks revealed that most found ways to adjust or extend their basic approach to finding the arithmetic mean as the format and demands of the tasks changed. Table 1 displays the results of this analysis.

The two students whose instruction had been from the *Visual Mathematics* curriculum materials (S1 and S2) continued to use visual strategies in approaching each task. Understanding the relationship between the heights of the columns of cubes, representing the given numbers in the set, and the height of the leveled-off

Table 1.
Strategies used and success in solving averaging tasks

	Task 1	Task 2	Task 3	Task 4	Task 5
S1	Visual - cubes Successful	Visual - cubes Successful	Visual - cubes Successful	Visual - numbers representing cubes Successful	Visual - cubes Successful
S2	Visual - cubes Unsuccessful counting error	Visual - diagram Successful	Visual - diagram Unsuccessful - counting error	Visual Unsuccessful - good approx.	Visual - diagram Unsuccessful - good approx.
S3	Numerical Successful	Visual - diagram Successful	Numerical Unsuccessful - computation error	Numerical Successful	Numerical Successful
S4	Numerical Successful	Numerical Successful	Numerical Unsuccessful - conceptual error	Numerical Successful	Numerical Successful
S5	Numerical Successful	Numerical Unsuccessful - no solution found	Numerical Unsuccessful - conceptual error	Numerical Successful	Numerical Successful
S6	? Unsuccessful	Visual - diagram Successful	Visual - diagram Successful	? Unsuccessful	? Unsuccessful

columns, the average, provided these students with flexibility in using the method of transferring of cubes in tasks posed in various ways. S1 consistently chose to actually build the columns of cubes, even when problems were presented in diagrammatic form, and arrived at correct solutions. S1 described how she was modeling the fourth task with imaginary columns and, though she proceeded to work solely with the numerical values, her explanation focused on her actions in moving imaginary cubes. S2 quickly moved into sketching diagrams. Her errors related to difficulties which arose in depicting the movement of cubes in her sketches and in keeping track both visually and numerically of the current state of the problem. The greatest challenge that these students faced was in applying and extending the visual solution process to the fourth task, where the numbers given were too large to be modeled directly with the cubes. The actions of the students demonstrated their ability to adapt the visual procedure in finding the average and to work back and forth between the numbers in the set and the average.

Students whose instruction was based on the traditional numerical algorithm depended on the one-way application of that algorithm to solve a majority of the tasks. S4 and S5 approached each of the five tasks with the algorithm and were successful in solving all but the third task. No mention was made that the fourth task used larger or more numbers than the first task. The second and third tasks did prove more challenging as they utilized a trial-and-error approach to find the missing number in the set. Their preliminary attempts resulted in errors primarily involving the divisor in the algorithm. Neither subject chose to work visually, even though the numbers in the data set for tasks 2 and 3 were given in a diagram.

S3 and S6 did choose to move at least once from the use of the numerical algorithm into a visual solution strategy. S3 employed a visual strategy successfully to solve the second task; he used the same visual method described by S2. He chose to return, unsuccessfully, to the numerical algorithm in the third task. S6 was unable to solve any task which depended on a decontextualized understanding of average. Although she commented that she recognized this type of problem, she did not know the numerical algorithm. But the meaning of the average was implicit for her in the contexts of the second and third tasks and she was able to find the solution to the third task more quickly and directly than any of the other subjects. In fact, neither S3, S4, nor S5 was able to solve this task.

Discussion

The results of this study show that the same method which formed the basis for classroom instruction on averaging was used by students when presented with the initial task of finding the average of a set of numbers. Students overcame the obstacles found in variations on the initial task by adjusting their use of the method learned or by finding a new problem space in which to work. No student whose experience was in Visual Mathematics used any form of the numerical algorithm, while two of the four students whose instruction involved the numerical algorithm did work with the diagrams when tasks were represented in visual form.

Analysis of student responses showed how task demands presented different challenges for students who had learned the numerical versus visual procedures. Students who had learned the numerical algorithm were confident and successful in finding the arithmetic mean when a complete set of numbers was given, regardless of the size of the numbers or the size of the set. When given an average and asked to find a number(s) in the set, they were often successful in identifying a solution, but consistently worked from the numbers in the set to the average, moving unidirectionally and using a trial-and-error approach. Regardless of context, students who had learned the numerical algorithm referred to the average as the outcome of a procedure, "what you get".

Students who had learned the visual approach revealed greater flexibility in moving back and forth between the numbers in the set and the average. Recognition that the sum of the deviations from the average is zero, one component of the average concept identified by Strauss and Bichler (1988), appeared to be attained as students focused on the relationship between the heights of the original columns (the numbers in the set) and the leveled-off height (the arithmetic mean). While both groups of students showed a degree of understanding and flexibility with the procedure they had been taught, those who learned the visual procedure showed a deeper conceptual understanding of the arithmetic mean.

References

Bennett, A. & Foreman, L. (1991). *Visual Mathematics Course Guide, Volume 1*. Portland, OR: Math Learning Center.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

Lane, S. (1993). The conceptual framework for the development of a mathematics performance assessment. *Educational Measurement: Issues and Practice, 12*, 16-23.

Mevarech, Z. R. (1983). A deep structure model of students' statistical misconceptions. *Educational Studies in Mathematics, 14*, 415-429.

Mokros, J.R. & Russell, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education, 26*, 20 - 39.

Pollatsek, A., Lima, S., & Well, A.D. (1981). Concept or computation: Students' understanding of the mean. *Educational Studies in Mathematics, 12*, 191-204.

Strauss, S. & Bichler, E. (1988). The development of children's concepts of the arithmetic average. *Journal for Research in Mathematics Education, 19*, 64-80.

Appendix: Selected Tasks as Presented to Subjects

TASK 2: In order to raise money for a trip, the seventh grade class is selling candy bars. The class is divided into teams of four students. Joe, Carol, Michael, and Keisha make up one of the teams. If the team sells an average of 8 candy bars each day, they win a prize. The picture below shows the number of candy bars sold by Joe, Carol, and Michael.

How many candy bars does Keisha have to sell in order for the team to win a prize?

TASK 4: Jim recorded the amount of time he spent watching television for five days.

Joe	▨▨▨▨▨▨▨
Carol	▨▨▨▨▨▨
Michael	▨▨▨▨▨▨▨▨▨▨
Keisha	

Monday - 120 minutes
Tuesday - 100 minutes
Wednesday - 60 minutes
Thursday - 90 minutes
Friday - 180 minutes

What is the average number of minutes Jim spent watching television?

RELATIONSHIPS BETWEEN UNDERSTANDINGS OF OPERATIONS AND SUCCESS IN BEGINNING CALCULUS

Barbara J. Pence, San José State University

In an effort to examine the impact of the changes being made at San José State University in the calculus curriculum, multiple measures were collected and analyzed. This study focuses on the relationship between performance on a pretest and the class grade. Through written responses on the pretest, a belief and knowledge profile for each student was constructed. Students were grouped according to their answers on an item which asked them to graph 2, x, x^2, and 2^x. Profiles of student perceptions and knowledge were consistent within groups and varied across groups. Results showed that the concept of multiplication was not well understood, and closely related to success in first semester calculus. Multiplication was itself still a process and in some cases, this process produced multiple concept images within cognitive neighborhoods.

San José State University (SJSU) is in the process of making changes in the calculus curriculum. In an attempt to trace the impact of these changes, several assessment efforts are in progress. This paper examines data from one of these studies for the purpose of investigating the relationships between understandings of operations and understandings of concepts studied during first semester calculus.

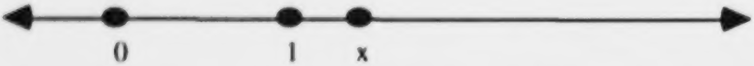
Background

Key concepts in beginning calculus involve the study of processes on functions. The road from seeing functions as processes to thinking about them as an object and finally using functions in other processes is difficult. In order to work with functions found in first semester calculus there exists a need for the encapsulation (Dubinsky, 1992) of many operations. The idea of *cognitive root* described by Tall (1992, p. 497) as "*concepts that have the dual role of being familiar to students and providing the basis for later mathematical development*" seems to apply to the role of understandings of operations relative to work with functions. At the stage when each function is still a process [take a point on the x-axis, trace a vertical line and then a horizontal line to find the value of $y = f(x)$], one basis for mathematical development includes operations such as multiplication, powers and exponents. Operations are familiar to the students; they have been using multiplication and powers in variable expressions for years. If, however, operations are not yet at the object level, then students must overcome additional obstacles in order to encapsulate the process into a single concept. This paper will investigate the linkages between the concept images of operations and understandings of processes on functions.

Methodology

Multiple measures were collected during the fall semester of 1994 for nine sections of beginning calculus including a pretest, a mid-semester test, a final, the course grade and the course grade from the second semester calculus class. The pretest and the mid-semester survey items elicited information about student cognitive knowledge, perceptions and beliefs while the other measures focused on student achievement. Complete pretest data existed for four classes, one class using the Harvard Consortium materials and three classes using Stewart. This study will concentrate on understandings as seen through the lens of the pretest and the course grades. The pretest was developed to gain insight into the students' entry level perceptions, attitudes and understandings of operations and functions. Of the eight written pretest items, the first six items elicited student comments regarding the anticipated difficulty level of the course, the grades expected, the key concepts of calculus, the perceived difficulty of representational forms, the expected applications of the content and the role of technology in the course. Content knowledge was examined through two questions. One of the content questions asked students to examine three graphs and in each case tell whether each graph was or was not a function and why. Graphs used in this question came from the research by Dreyfus and Vinner (1989). The second content question was motivated by faculty concerns that student understandings of powers and exponents are weak. This item is shown below:

> On the following number line, you will see the points representing 0, 1, and x indicated. Approximate the location of the points corresponding to 2, 2x, x^2 and 2^x.



For this item students are asked to connect symbolic and visual representations and to link units, variables, and operations on the number line. Although the original problem was conceptualized for use on a computer using a dynamic geometry system such as Cabri Geometry, this became impossible due to the lack of availability of computers. Thus, the problem became static with the variable x located so that it was less than 1.5. Actually it was placed at the point corresponding to the square root of two. Each of the four functions corresponds to an operation. Locating the point corresponding to 2 required repeated addition with the input being that of the location of the unit. As with 2, the function of 2x could be processed by repeated addition. On the other hand, the image of x^2 and 2^x required both the location of the unit and x. For each of the four functions students were required to process an operation.

Analysis

Data collected was examined both quantitatively and qualitatively. Results from 76 completed pretests and class grades for each of these students was examined collectively. The first step of the analysis was to sort the pretests into levels of understanding on the operation item. The sort produced six categories which were:

(1) students who were not able to locate any of the four functions correctly;

(2) students who located 2 but were not able to locate 2x;

(3) students who located 2 and 2x but were unable to locate x^2 or 2^x;

(4) students who located 2, 2x and x^2 but had trouble locating 2^x;

(5) students who located 2, 2x and 2^x but had trouble locating x^2; and

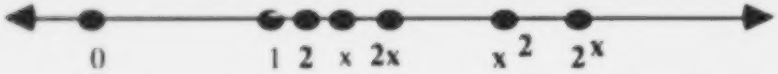
(6) students who located all four functions 2, 2x, x^2 and 2^x.

The six categories are hierarchical with categories 4 and 5 conceptualized as parallel. In the table, the categories form the structure for examining relationships between understandings and average anticipated grade, average actual course grade, failure rate, average score on the function item, and the representation reported to cause the most difficulty when solving a problem. Patterns exist both between and within categories. First, students in categories 1 - 3 were more likely to be repeating the course of calculus and the difference between the anticipated and actual grade was larger. Second, the score on the function item did not seem to be related to understandings of operations. Since the vertical line test was the major justification for answers to this question, the lack of connection with the understandings of operations is not surprising. Third and one of the most interesting pattern was that of self monitoring. Students in categories 1 - 4 (79%) expected grades of at least a B^+ but earned grades of D or lower or dropped out. Students in category 6 were better able to monitor their progress or at least their progress against standards set by the instructor. Could it be that the category 1 - 4 students are progressing in the development of their own understandings but are not to the stage of contrasting the result of these understandings across conflicting concepts in their own cognitive structures much less in comparing their structures with those being set forth by their instructors?

Category #	# of Students	# Repeatg	Ave Anticip grade A=4	Ave Actual grade A=4	% of students who failed	Ave score on funct item (n=3)	Most difficult representation (graph, table, equation)
1	5	4	3.0	0.5	60%	2.8	(3, 1, 1)
2	34	17	3.6	1.4	53%	2.5	(16, 5, 5)
3	19	11	3.5	0.8	63%	2.7	(3, 2, 11)
4	2	1	3.5	0.0	100%	2.0	(0, 0, 2)
5	2	1	3.5	3.5	0%	2.0	(0, 0, 2)
6	14	8	3.4	2.5	36%	2.4	(5, 2, 3)

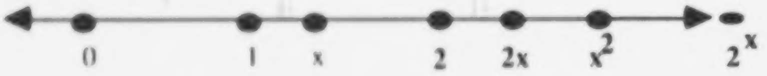
Although the tables describe both within and across category patterns, examination of sample student work helped in the exploration of student understandings of operations. Due to space restrictions, student work will be shared for categories 1, 2, 3 and 6 only.

Category 1. The five students in this category had trouble locating 2.



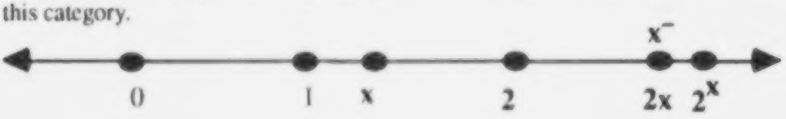
This student is repeating calculus but expects to get a B this time. She finds graphs to be the most difficult representational form to work with. Her graph is similar to the other four students in this category. There is an attempt to process most of the functions and there seems to be a belief that the constant 2 must come before the variables. The conflict between the location of 1 and 2 is not resolved. In fact, this pattern between 1 and 2 seems to be carried over to the relationship between x and 2x. To carry this analysis any further when the role of the unit is in doubt makes little sense. This woman continued in the class through the mid-semester exam. On the mid-semester exam, she was able to produce only a little work on one out of four problems, the one symbolic problem, and eventually withdrew from the class.

Category 2. In the second category, the students were able to locate 2 but were unable to find the point corresponding to 2x. Although students rarely provided any more than the diagram, this student actually gave sufficient work to help explain his thinking.



To construct the location of 2 he replicated the interval between 0 and 1. This logic of repeated addition was continued through his work with both 2x and x^2. That is, 2x was 2 plus x and x^2 was found by taking the interval from 0 to x and marking it off from x ($x^2 = x + x$). The location of 2^x seemed to be something beyond the others. Even though this student entered calculus class with high expectations, he was forced to drop the course before the final.

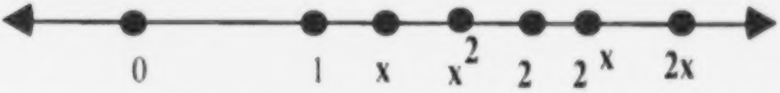
Category 3. Students in the third category correctly identified 2 and 2x. They either stopped at this point or went on to mappings which incorrectly represented both x^2 and 2^x. Many interesting linkages can be found in the work of students in this category.



The actual relationship that this student, who is repeating calculus after taking it in high school, wanted to communicate between x^2, 2x and 2^x is unclear. But, it seems as though each concept is closely related to 2x while being unrelated to 2. That is as a neighborhood is drawn closer and closer to 2x, it would always include x^2 but not 2. At what point in the shrinking of the neighborhood the location of 2^x

would be separated from 2x and x^2 it is difficult to say. Since this clustering of the concepts of 2x and x^2 appeared in more than 10 papers it is an example of a cognitive neighborhood, a construct introduced by Ervynck (1994).

Category 6. On the opposite end of the spectrum, the group of 11 students who were able to locate all four functions passed calculus with a grade of C or better. In fact, there were three A^+ grades given in the fall semester, with all three of them appearing in this category. This group included 7 students who were repeating the course, 4 of whom took the course in high school. The graph of an A^+ student is found below. This student was repeating the class and report that he



found the three representational forms equally easy to work. The location of x^2 and 2^x are not exact but the relative positions are close thus it was counted as correct. He also did well on the mid-semester exam, getting 3 out of 4 of the problems correct but did not feel confident with his results.

Discussion

Although the pretest was a written task, the results identified some interesting relationships which need further exploration. Work from students who either dropped out or failed first semester calculus showed patterns of incomplete understandings of the operation of multiplication. Their image of multiplication reflected difficulty in extending the models of multiplication beyond repeated addition with constants. Multiplication was itself still a process and in some cases, this process produced multiple concept images within cognitive neighborhoods.

This study supports the cognitive root conjecture. The idea of classification of functions by operations may be a step in the development from functions as process to function as object. Operations are familiar, that is, students have used the symbolic representations and they form the basis for later mathematical development. Thus, they may be a candidate for a cognitive root for advanced mathematics.

Trends of repeated failure among these students who have passed all of the necessary prerequisites to enter college calculus is perplexing. Why are 76% of these students unable to move beyond processing the functions of 2 and 2x? Why were these advanced students not monitoring and resolving conflict between function processes? What role does the belief system play in the cognitive image and conflict resolution? Does the multiple representation in this static task mask the potential for identification of conflict? Would a dynamic task encourage students for whom multiplication was still a process to reduce the multiple concepts contained in cognitive neighborhoods and support their movement from seeing functions as processes to thinking about them as objects and even using functions in other processes as required in their study of calculus?

References

Dreyfus, T., & Vinner, S. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20, 356-366.

Dubinsky, E. (1992). A learning theory approach to calculus. In Z. Z.. Karian (Ed.) *Symbolic computation in undergraduate mathematics education*, MAA Notes and Reports, No. 24. Washington, DC: Mathematical Association of America.

Ervynck, G. (1994). Cognitive neighborhood. Unpublished presentation given to the AMS/MAA Joint Meeting. San Francisco, CA.

Tall, D.O. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.495-511). New York, NY: MacMillian Publishing Company.

TEACHER GUIDANCE IN AN EXPLORATORY MATHEMATICS CLASS

Bey-bey Li, Washington State University
Verna M. Adams, Washington State University

This paper is concerned with understanding how a scaffolding process is utilized in a natural setting of a middle-school mathematics class. Wood, Bruner, & Ross (1976) characterize scaffolding as a learning process of a novice which is assisted and dominated by the adult. Rogoff and Gardner (1984) also point out that "to make messages sufficiently redundant" (p.109) is one way to provide scaffolding. In this study, we examine the classroom discourse when a new topic is introduced to the class. The teacher connects a new topic (combinations) to old content (permutations). He uses abundant, similar, but condition changed slightly, examples as referents to help the students attach meaning to symbols for permutations and then begins to turn over the discourse to the students to support students' development of that new content.

To identify scaffolding, we draw on Wood, Bruner, & Ross's (1976) description of scaffolding as "a process that enables a child or novice to solve a problem, carry out a task or to achieve a goal which would be beyond her/his unassisted efforts. This scaffolding consists essentially of the adult 'controlling' those elements of the task that are initially beyond the learner's capacity" (p.90). Rogoff and Gardner's (1984) use of data from a study of mothers preparing their children for a memory test illustrate how an adult's instruction serves as a scaffold for the learner. They emphasize that "to make messages sufficiently redundant" (p.109) is one way to provide scaffolding. They also conclude that the adult assists children with new challenging problems by guiding children to make connections to more familiar contexts. We view scaffolding as a process that transfers responsibility back and forth between the teacher and students until it is completely turned over to the students in problem-solving situations. Errors and uncertain responses of students in the classroom dialogue function for the teacher as signals that students are in or beyond their zones of proximal development (Vygotsky, 1978, 1986; Wertsch, 1985). If students are functioning in their zones of proximal development, then teacher intervention may help them function successfully. Greenfield (1984) pointed out that "Errors, either anticipated or actual, are used as a signal to upgrade the scaffold, transferring responsibility from the learner to the teacher" (p.136). The challenge for the teacher is to communicate in a way that helps the teacher identify the students' thinking and that allows the students to participate and redefine the task.

We also draw on Hiebert's (1988) theory of cognitive processes involved in increasing students' competence with written mathematics symbols. Five sequential types of processes are distinguished: "(1) *connecting* individual symbols with

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Preparation of this paper was supported in part by National Science Foundation, Grant No. RED-9254922. Any opinions, conclusions, or recommendations are those of the authors and do not necessarily reflect the views of the National Science Foundation.

referents; (2) *developing* symbol manipulation procedures; (3a) *elaborating* procedures for symbols; (3b) *routinizing* the procedures for manipulating symbols; and (4) using the symbols and rules as referents for *building* more abstract symbol systems" (p.335). The first two processes include building referents on students' previous experiences and manipulating referents, observing the result, and translating referents to symbol world. Their purpose is to provide symbols with meaning. The subsequent cognitive processes shift from a heavy dependence on referents to a mediation of the symbols and rules themselves.

This study examines how a teacher uses scaffolding to support the above first two cognitive processes by using abundant, similar, but condition changed slightly, examples as referents. As part of the scaffolding process, he turns over the discourse to the students.

The Study

The class described in this paper was an elective class for 7th graders (5 students) and 8th graders (13 students). Its purpose was to explore mathematical topics that were not provided in the students' "regular" math class. The teacher was a fifth-year teacher with a major in mathematics in his teacher preparation program. He has excellent rapport with middle school students and great enthusiasm for mathematics. The teacher felt that he had less time pressure and greater flexibility to design and run this class than other math classes. The classroom had a relaxed atmosphere, yet the teacher had a firm control over student misbehavior. He enthusiastically guided students to deal with numbers, especially large numbers, and made connections to real world experiences. A variety of interactions were used in the classroom: the teacher led whole-class discussions, the students worked individually and in small groups, and groups gave problem-solving presentations.

Data collection consisted of daily videotaping, field notes, collection of materials used by the students, interviews of selected students and the teacher, and an initial and a final whole-class survey of students' attitudes and perceptions of the class. The mathematical content during the 8-week period of data collection was a unit on methods of counting, including permutations and combinations. This paper focuses on the introduction of a new topic and symbol. Analysis of the discourse revealed scaffolding patterns of discourse. From that observation, a more serious focus on scaffolding evolved. In the paper, we attempt to show how the teacher introduced new content, developed students' understanding through the use of sufficiently redundant examples as referents, and then turned over the learning process to the students.

Building New Content on Existing Knowledge

Prior to this episode, the teacher introduced real-life examples of permutations related to the Rose Bowl game, phone numbers of a town, and credit card issues. The whole-class discussions were followed by two sets of exercises worked in small groups and one done individually. Toward the end of that topic, students were using symbols for permutations to present problems and calculate numerical answers. When the teacher introduced combinations into the class dialogue, he emphasized that it was important to note the difference between combinations and permutations: "If you don't understand the difference, it is going to be very, very confusing for you." He started by building the new content (combinations) on familiar content (permutations), pointing out differences:

> Teacher: Who can explain to me what a permutation is—in their own words? Yeah, TN?
>
> TN: A permutation is [pause] a, but something, like a number, like $_3P_1$, and P stands for permutations.

For TN, a permutation was two numbers associated by a symbol *P*. The teacher accepted TN's response, but then tried to help her to re-organize her understanding and connect the symbol to vivid referents. He narrowed his question to focus on the concept.

> Teacher: What does the word mean by itself? Disregard the number—[speaking slowly with emphasis] permutation?
>
> TN: Oh, the arrangement.

Again, the teacher did not reject TN's incomplete explanation of permutations. Instead, he decided to give the class a concrete example. Below, we show how the teacher used redundancy to connect the numerical component of the situation to concept development.

Use of Sufficiently Redundant Examples as Referents

The redundancy occurs in both the type and number of examples. The teacher builds an introduction to combinations based on the students' understandings of permutations:

> Teacher: All right, now, the word permutation—one key element in that idea is a very specific arrangement of things. If I were to, if we're to elect two people out of the eighteen votes, the president and the vice president in this, uh, classroom—if I elect LP as the president and LN as the vice-president, that arrangement is very important. It's totally different than if I elected LN as the president and LP as vice-president. It's the same two people, but I changed the arrangement, and I get a different situation. So, when, in a permutation, you change the arrangement and get something different, that's, that's called a permutation. You change the arrangement, you get something different. Yeah?
>
> TN: So, would it be, would it be like 18 times 17?

Although TN selected the right numbers to fit this example, her questioning response indicated that she was still struggling to attach numbers to the example. The teacher not only agreed with her, but added more verbal explanation of the meaning of those numbers. Then, he slightly changed his example situation to move to combinations:

> Teacher: That is exactly like that ${}_{18}P_2$ [writes "${}_{18}P_2$" on the chalkboard] which is how many ways I can take and arrange 2 people from 18, would be ${}_{18}P_2$. Now, that's a different problem—what if I say we're in a magazine sale and we win, uh, say candy or ice cream—and say I just want to send two people down to the office to pick up ice cream. That is not a permutation. Just say I picked LN and LP to go down to the office. If I picked LN and LP to go down to the office, that's the same thing as if I picked LP and LN to go down to the office. It doesn't matter which one I picked first, I just picked those two people. So if I switched the order, I picked LN first and then LP, that's the same as I picked LP and LN. That doesn't matter; I just sent two people down to the office to pick up ice cream for us. Who cares what order they're in, so that's not a permutation because the order is not important. That's called a combination [Writes "combination" on the chalkboard].

In this introduction, the teacher utilized numbers provided by TN, connecting the new concept to a familiar concept by example. On the left side of the chalkboard, he listed a pattern of 15 permutations using notations such as ${}_1P_1$, ${}_2P_1$, ${}_2P_2$, ${}_3P_1$, continuing to ${}_5P_5$. These notations were familiar to students. Before attaching each notation with meaningful referents, he asked the students to calculate the numerical answers for each permutation. Then they reviewed and wrote the general formula "nPr = n!/(n-r)!" on the chalkboard.

The teacher asked students to compare ${}_1C_1$ with ${}_1P_1$, using an example of choosing one person from a class with one student. He gradually increased the number of students in the imaginary class to two, three...five. He repeated similar, but condition changed slightly, examples in order to link permutations and combinations. There was a sense formed in the class that the president and vice president issue was related to permutation problems and sending students to pick up ice cream was associated with combinations. The teacher wrote combination notations and amounts on the right side of the chalkboard to leave a visual record in front of the students. As the development continued, he frequently pointed out the slightly changed conditions for the new content. They completed their list of 15 permutations and 15 combinations and students looked at the groups of symbols and numbers on the chalkboard to figure out the relationship between patterns of permutations and combinations. TN noticed that permutations and combinations could be connected by division: "So, we could do it like a permutation, but divide it by two?"

Turn Over of Dialogue to the Student

The teacher encouraged more students to participate, changing his role from a speaker who did most of the talking to one who supported and helped students to organize their own thoughts. Dialogues in the classroom turned from lecturing by the teacher to students' observation and participation.

Teacher: Check it out. For this problem, TN is saying, I can do this just like a permutation but divide it by two and I get, cause $_3P_2$ is 6, divide it by 2. I get 3. That's my answer. Let's see if it works for this [pointing to $_3C_3$] ?

Students used *picking up ice cream* to decide that $_3C_3$ was equal to one. TN's rule was not satisfied in this situation. The teacher pointed out that students held different pieces of information and he encouraged all students to participate. Many students raised their hands to show that they wanted to contribute. At this moment, students did not understand the general relationship between combinations and permutations yet, but they enthusiastically participated in the discussion as they searched for pattern. Compared to the beginning of the class, the dialogues were very different. One student, FT, presented his thoughts in fragmented phrases. The teacher patiently gave short responses to FT's comments to help him complete the statement of his discovery. Then, he shifted to recording FT's idea on the chalkboard:

Teacher: Okay, let's explain it again, so...

FT: You kind of, like on the last one, it would be 3 x 2 x 1.

Teacher: Go ahead. [FT speaks but cannot be heard.] Okay, so that equals [writing $_3P_3$ on the chalkboard]...

FT: Oh, Yeah.

Teacher: Okay, you're saying: change this to a permutation and then do what? [writes "= $_3P_3$" after $_3C_3$ = 1 on the chalkboard].

FT: Then, put this number, second, factorial.

Helped by the teacher, FT completed the numerical relationship between combinations and permutations. The teacher went one step further to give meaning, comparison, and explanation to combinations:

Teacher: And divide this by the second number factorial. [writing "3!" under $_3P_3$] That's [pause] correct. And that's, how you figure out combinations. Now, I'll explain to you why that works. When you have 3 choose 3, there is only one way to do it. Just send these 3 people down. But when you compare that to the permutations, how many different ways can I mix up these same 3 people in permutations? Yeah?

JI: Six.

Teacher: Six, so I have to divide. If I get a permutation of six, but all six are the same combination, so what I do is just like we did in the last section. You have to divide by the number of repeats. [pointing to 3! on the chalkboard] So, there is 6 repeats, or 3 factorial repeats for this.

As the teacher and students elaborate on the relationship between permutations and combinations, we notice that the teacher attempts to help students become involved in the discussion and connect meanings to each of written symbols to help students to develop understandings of the new topic.

Discussion

In this classroom, cognitive processes of connecting symbols to referents and developing symbol manipulation procedures were enthusiastically supported by the teacher. Rather than teaching in a way that directly introduced mathematical written symbols and merely computing numerical answers, which often occurred in student talk, the teacher called attention to ways of talking about concepts and ways of writing and giving meaning to symbols. He built the new content (combinations) on a familiar one (permutations) by providing redundant examples when students started this novel task. Redundancy occurred when he utilized examples as referents to attach meaning to each symbol. Similar, but condition changed slightly, examples played a significant role of making symbols meaningful. The teacher also switched smoothly from lecturing to involving students in the discussion. In this transition process, students had opportunities to organize their thoughts, gain insights about the meanings of those symbols and develop the ability to manipulate them.

References

Greenfield, P.M. (1984). A theory of the teacher in the learning activities of everyday life. In B. Rogoff, & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 117-138). Cambridge, MA: Harvard University Press.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics, 19*, 333-355.

Rogoff, B., & Gardner, W. (1984). Adult guidance of cognitive development. In B. Rogoff, & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 95-116). Cambridge, MA: Harvard University Press.

Vygotsky, L. (1986). *Thought and language*. Cambridge MA: MIT Press. (Translated and edited by Alex Kozulin)

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press. (Edited by Michael Cole, Vera John-Steiner, Sylvia Scribner, & Ellen Souberman)

Wertsch, J. V. (1985). *Vygotsky and the social formation of mind*. Cambridge, MA: Harvard University Press.

Wood, D., Bruner, J. S., and Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17*, 89-100.

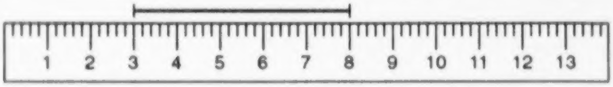
WHY IS THE USE OF A RULER SO HARD?

Constance Kamii, The University of Alabama at Birmingham

This study involved three days of teaching in two fourth-grade classes. A pretest and a posttest were given consisting of (a) a NAEP item asking for the length of a line drawn next to a ruler, (b) a Piagetian unit-iteration task, and (c) the measurement of an object with a ruler that had the "0" mark away from its edge. It was found in the pretest that 86% of the children had constructed the logic of unit iteration but that most of them could not use a ruler correctly. The posttest revealed that, although there was progress, the problems found in the pretest persisted among a third to a fourth of the children. These problems were all related to the initial unit of measurement.

The 1985-86 National Assessment of Educational Progress (NAEP) revealed that only 14% of the third graders and 49% of the seventh graders chose the correct answer of 5 cm as the length of the line shown in Fig. 1 (Lindquist & Kouba, 1989). The 1990 NAEP included a similar item and produced similar findings (Mullis, Dossey, Owen, & Phillips, 1991).

Item	Percent Responding[a] Grade 3	 Grade 7



How long is this line segment?[b]		
3cm	4	1
5cm	14	49
6cm	31	37
8cm	30	9
11cm	6	2
I don't know.	15	2

[a] The response rate was .80 for grade 3 and .97 for grade 7.

[b] An actual centimeter ruler was pictured.

Figure 1.

The purpose of this paper is to explain why the use of a ruler is difficult by describing findings from a three-day teaching experiment in two classes of fourth graders. I first gave a pretest consisting of three parts: (a) the NAEP question shown above, (b) a Piagetian task of unit iteration, and (c) a measurement task requiring the use of a ruler. I then joined two teachers in their respective classrooms as they engaged in activities that required the use of a ruler. The experiment ended with a posttest that shed new light on children's difficulty in using rulers.

Table 1

	Pretest	Posttest
	n=44	*n*=44
NAEP question		
5 cm	45	64
6 cm	48	34
8 cm	5	2
11 cm	2	0
Unit-iteration task	86	95
Tasks requiring the use of a ruler		
Ruler shown in Fig. 3		
Alignment with 0 (4 3/4 inches) or ignoring all numerals, counting intervals, and giving correct answer	18	84
Alignment with edge of ruler (3 3/4 inches)	57	11
Alignment with "1" (5 3/4 inches)	9	5
Alighment with 1/8 inch to left of 0 (4 1/2 inches)	16	0
Ruler shown in Fig. 5 (used only in posttest)		
Alignment with implicit 0 (the correct answer of 13 cm)		73
Alignment with edge of ruler (12+ cm)		14
Alignment with "1" (14 cm)		14

The Pretest (given in individual interviews)

The NAEP question

As can be seen in the preceding table, 45% of our fourth graders gave the correct answer of 5 cm, but 48% counted the numerals 3-8 on the ruler and said the line was 6 cm long. Half of our fourth graders thus demonstrated that they counted points rather than intervals.

A unit-iteration task

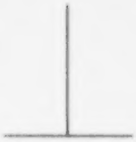


Figure 2.

This task, based on Piaget, Inhelder, and Szeminska (1948/1960) and Kamii (1991), was given to find out if our children had constructed the logic of unit iteration. Unit iteration here refers to the ability to use a small, flat block (1.25 x 1.25 inches) repeatedly to determine whether or not the two lines in an inverted T (Fig. 2) have the same length. Both lines were 4.75 inches long, but the vertical one looked longer because of an optical illusion. The logic of unit iteration is necessary for a child to use and understand conventional units (intervals) such as inches and centimeters. As can be seen in the table, 86% of our fourth graders demonstrated the logic of unit iteration.

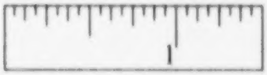


Figure 3.

A task requiring the use of a ruler

To find out how children used the left extremity and "0" point of a ruler, I asked them to measure the horizontal line of the inverted T with a ruler like the one in Fig. 3. Only 18% of our fourth graders gave the correct answer of "about 5 (or 4 3/4) inches." The most common error (made by 57%) was to align the edge of the ruler with the beginning of the line being measured, read the numeral corresponding to the end of the line, and say that the line was "about 4 (or 3 3/4) inches long." Two other kinds of errors also demonstrating the difficulty of the initial interval. One was to align the "1" on the ruler with the beginning of the line and to say that the line was "about 6 (or 5 3/4) inches long." Nine percent of our fourth graders made this error. The second type of error was to align the beginning of the line with a mark on the ruler about 1/8 inch to the left of the "0" mark. Sixteen percent of our fourth graders did this and said the line was "about 4 1/2 inches long." These children meant to align the "0" mark with the beginning of the line but thought that the point 0 was directly above the numeral 0 (see Fig. 3).

The overall conclusion drawn from the pretest was that since most of our fourth graders had constructed the logic of unit iteration, the use of a ruler was developmentally appropriate to teach. More than half of the children had trouble thinking about the first unit (an interval), but this difficulty seemed superficial compared to the deep logic of unit iteration that most of our children demonstrated.

Three Days of Teaching

The measurement activities recommended by textbooks have two major weaknesses. First, textbooks ask questions such as "How many centimeters wide is your desk?" that are irrelevant to children. Second, they ask "How many?" without giving children any reason for measuring things accurately. Our classroom activities were the following three kinds that were more purposeful and interesting.

Measuring to compare

An example of this kind of activity was inspired by *Opt: An Illusionary Tale* (Baum & Baum, 1987), a collection of pictures such as the one in Fig. 4 asking if the height of the hat is greater than the width of the brim. Throughout the three days of teaching, we asked the children to use a ruler like the one in Fig. 3.

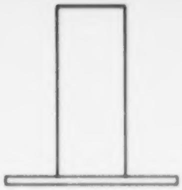


Figure 4.

Measuring to draw

We asked the children to make drawings similar to Fig. 4 but with different dimensions, to take home and amuse their families.

Measuring to make something

The intriguing object we suggested to the children to make was a "Magic Calendar." However, a different arts-and-crafts activity could also have been used necessitating the accurate use of a ruler.

A particularly important part of our constructivist teaching was to avoid direct teaching and, instead, encourage the exchange of points of view among children. As Piaget (1979) said, "The confrontation of points of view is already indispensable in childhood for the elaboration of logical thought, and such confrontations become increasingly more important in the elaboration of sciences by adults (p. vii)." When a child said, "My ruler is wrong," to another child, for example, we encouraged the second child to respond. A frequently heard response was: "The ruler doesn't make any difference because an inch is an inch. See, I'll show you. . . ."

We learned much about children's ways of thinking by interacting with them in the classroom. For example, when they had trouble figuring out how to use our ruler (Fig. 3), they asked us for help. A possible reaction in such a situation was to find out "where the child was" by saying, "Would you show me an inch—an example of an inch." Some children responded by pointing to the "1" on the ruler, suggesting that an inch to them was a point or a numeral rather than an interval. When this happened, we usually said, "I thought an inch was about this long," showing an interval between two fingers.

Many children aligned the edge of the ruler with the edge of the object being measured and counted the intervals instead of using the numerals on the ruler. When we saw this behavior, we sometimes asked, "Wouldn't it be easier to put the 0 on the edge like this (demonstrating) so you could just read the number at the other end?" Some children responded with a "No." Others slid the ruler to the left, past the 0, and aligned the edge of the object with the "1" on the ruler! As they later explained during the whole-class discussion, "Zero doesn't count," and "When you count, you don't say 'zero-one-two.' You say 'one-two-three.'" We thus learned that some children's belief that "zero doesn't count" was preventing them from thinking about the initial interval.

The Posttest

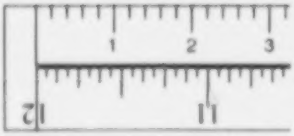


Figure 5.

As can be seen in the table presented earlier, the children did better on the posttest, but a large percentage, 34%, continued to choose the answer of 6 cm on the NAEP question.

The "acid test" required the use of an unfamiliar ruler (see Fig. 5). The marks on this ruler started about 6 mm away from the edge, and the 0 point was not numbered. Another novelty was that this was a centimeter ruler, and the children had been using only inches. Although 73% of our children gave the correct answer of 13 cm by using the unfamiliar ruler correctly, the errors described earlier persisted among the other students. Fourteen percent aligned the edge of the ruler with the edge of the object and reported a length of "a little more than 12 cm." Another 14% aligned the "1" mark with the edge of the object and said it was 14 cm long.

Conclusion

Measurement of length is introduced in kindergarten and taught repeatedly in subsequent years according to most state curriculum guides and nationally distributed textbooks. I thought that the use of a ruler would be more appropriate and easy in fourth grade because 86% of our children had constructed the logic of unit iteration. However, this logic turned out to be far from sufficient for the learning I expected.

Mathematics educators, including the authors of the *Standards* (NCTM, 1989), say that the way to build a conceptual foundation for the use of instruments is to provide experiences with concrete objects and to ask children to estimate how many units will be found by counting them. The experiment described above shows the need to examine children's thinking more deeply and precisely. The problems of the initial unit, the edge of the ruler, the "0" point, and the "1" have been observed by many teachers and some researchers such as Heraud (1989) and Bright and Hoeffner (1993). Further research is necessary to find out how best to encourage children to modify their thinking about these aspects of measurement.

References

Baum, A., & Baum, J. (1987). *Opt: An illusionary tale.* New York: Puffin Books.

Bright, G. W., & Hoeffner, K. (1993). Measurement, probability, statistics, and graphing. In D. G. Owens (Ed.), *Research ideas for the classroom*, (pp. 78-98). Reston, VA: NCTM.

Heraud, B. (1989). Case studies of children's understanding of the concept of length and its measure. In C. A. Maher, G. A. Goldin, & R. B. Davis (Eds.), *Proceedings of the eleventh annual meeting, North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 135-142). New Brunswick, NJ, Rutgers University.

Kamii, C. (1991). Children's readiness for measurement of length. In R. G. Underhill (Ed.), *Proceedings of the thirteenth annual meeting, North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 113-118). Blacksburg, VA, Virginia Tech.

Lindquist, M. M., & Kouba, V. L. (1989). Measurement. In M. M. Lindquist (Ed.), *Results from the fourth mathematics assessment of the National Assessment of Educational Progress* (pp. 35-43). Reston, VA: NCTM.

Mullis, I. V. S., Dossey, J. A., Owen, E. H., & Phillips, G. W. (1991). *The STATE of mathematics achievement: NAEP's 1990 assessment of the nation and the trial assessment of the states*. Princeton, NJ: Educational Testing Service.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Piaget, J. (1979). Foreword. In Kamii, C., & DeVries, R. (1980). *Group games in early education*. Washington, DC: National Association for the Education of Young Children.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. London: Routledge and Kegan Paul. (Original work published 1948)

EFFECTS OF DIFFERING TECHNOLOGICAL APPROACHES ON STUDENTS' USE OF NUMERICAL, GRAPHICAL AND SYMBOLIC REPRESENTATIONS AND THEIR UNDERSTANDING OF CALCULUS

Donald T. Porzio, Northern Illinois University

Research on the effects of using different forms of technology in calculus instruction has typically focused on using technology to emphasize concept understanding while de-emphasizing routine computational skills. This study investigates the impact of different instructional approaches to calculus by examining their effects on students' abilities to use and understand connections between representations when solving calculus problems. 100 participants came from intact classes from three different differential calculus courses. The first used a traditional approach to calculus instruction that emphasized use of symbolic representations. The second was similar but instruction stressed use of symbolic representations and graphical representations generated via graphics calculators. The third used the electronic course Calculus & *Mathematica* (Davis, Porta, and Uhl, 1994) where instruction emphasized use of multiple representations and solving of problems designed to establish or reinforce connections between representations.

Data were collected using pre- and posttest instruments and 36 student interviews. The pretest and posttest measured students' initial preferences for certain representations when solving problems. The interviews and posttest were used to evaluate students' use and understanding of different representations when solving calculus problems. A theoretical framework for analyzing differences in students' abilities to use and understand connections between representations was developed from the notions that (a) a concept is understood if it is part of a network of internal representations, (b) the degree of understanding is determined by the number and strength of connections between representations, and (c) through reflective abstraction, students' knowledge is constructed while solving and interpreting problems.

Results indicated Calculus & *Mathematica* students were better able to use and to recognize connections between different representations than the other students. Graphics calculator students had trouble recognizing connections between graphical and symbolic representations even though use of these representations was stressed during the course. Results suggest the addition of a technological component to the existing curriculum to provide easier access to representations may not necessarily improve students' understanding of calculus. Other implications of the study's results relative to curriculum and students' understanding will be discussed.

Reference

Davis, W., Porta, H., & Uhl, J. (1994). Calculus & *Mathematica*. Reading, MA: Addison-Wesley.

THE ROLE OF MULTIPLE REPRESENTATIONS IN LEARNING ALGEBRA

Mary E. Brenner, Theresa Brar, Richard Durán, Richard Mayer,
Bryan Moseley, Barbara R. Smith and David Webb
University of California, Santa Barbara

The transition from arithmetic to algebra is a notoriously difficult one (Booth, 1989; Herscovics & Linchevski, 1994). Success in algebra problem solving depends both on *symbol manipulation skills* for solving algebraic equations and *problem representation skills* based on conceptual knowledge about the meaning of word problems (Mayer, Lewis, Hegarty, 1992). To help students in this transition, a team of teachers and researchers developed a unit on functions using three math reform principles: (a) Instead of emphasizing symbol manipulation, we emphasize problem representation skills. In particular, students learn to construct and coordinate multiple representations of functions, including expressing functions in words, tables, graphs, and symbols. (b) Instead of teaching problem-solving skills in isolation we anchor them within a meaningful thematic situation—making the decision of which company should supply pizza to the school cafeteria. (c) Instead of focusing solely on the product of problem solving, we emphasize the process of problem solving, in cooperative groups and through modeling by teachers.

Methods

7th and 8th graders (N=157) took a series of pretests, received 20 days of mathematics instruction on functions based on the above approach (treatment group) or a traditional approach (comparison group), and took a series of posttests.

Results

The treatment group made larger gains than did the comparison group on tests of solving equations ($t(155) = 3.30$, $p < .01$). The treatment group showed significantly larger gains in correctly writing equations, completing tables and drawing graphs ($t(155) = 2.49$, $p < .01$) as well as larger gains in using these skills while working on word problems ($t(155) = 3.30$, $p < .01$). These results demonstrate qualitative differences in the learning outcomes produced by different instructional methods. The cognitive consequences of traditional instruction focusing on symbol manipulation were reflected in improvements in students' ability to solve equations. The cognitive consequences of learning-by-understanding involved improvements in students' ability to represent functional relationships in equations, tables and figures, and to translate between these. The results were consistent with the idea that problem representation skills are learnable. More importantly, students were able to transfer these representation strategies to new situations.

References

Booth, L. R. (1989). A question of structure. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 57-59). Hillsdale, NJ: Erlbaum.

Herscovics, N. & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27, 59-78.

Mayer, R.E., Lewis, A. B., & Hegarty, M. (1992). Mathematics misunderstandings: Qualitative reasoning about quantitative problems. In J.I. D. Campbell (Ed.), *The nature and origins of mathematical skill* (pp. 137-154). Amsterdam: North-Holland.

CONSTRUCTIVIST PRACTICE AND THE BOUNDS OF METACOGNITION

Marshall Gordon, The Park School of Baltimore

The constructivist perspective, whether it emphasizes the individual (von Glasersfeld, 1984) or is more oriented toward a social context (Cobb, Wood, & Yackel, 1990), holds that knowledge is constructed by the participants, not provided by others or secured by an objective reading of reality. Thus in the context of school education, constructivist practice necessarily requires giving consideration to each student's concerns regarding and effort toward making sense of their learning experience.

Metacognitive activities, intrinsic to the development and representation of a constructivist-based learning environment, have been valuable in helping students develop their mathematical problem-solving awareness (e.g., Schoenfeld, 1987; Confrey, 1990). In these reflection-in-action (Schon, 1987) studies, students are assisted in the immediacy of the problem-solving experience to share their thinking as they go and have the opportunity to interact with the teacher toward clarifying their intuitions and reasoning. In contrast, the research to be presented will explore a variation of reflection-on-action (Schon, 1987). Here students have had the opportunity to choose and act toward securing or changing some attitudes and/or behavior(s) of their own that they believe would promote their thinking more productively, critically reflect on how they are proceeding with the teacher's assistance if desired, make adjustments in their practice including the allocation of energy, and continue toward securing a self-chosen goal. This reflects this researcher's experience that an extended period of time is required to see the effects of decisions toward change and others' experience (Lester, Garofolo & Knoll, 1989) that there are affective and contextual factors associated with the learning environment which impact on students' metacognition.

Findings are that the students found this effort valuable personally and their mathematics grades increased to a statistically significant degree.

References

Confrey, J. (1990). What constructivism implies for teaching. In R.B. Davis, C.A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics*. Journal for Research in Mathematics Education Monograph No. 4 (pp. 107-122). Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., T. Wood, & E. Yackel (1990). Classrooms as learning environments for teachers and researchers. op cit. pp.125-146.

Lester, F.K., J. Garofolo, & D.L. Kroll (1989). Self-confidence, interest, beliefs, and metacognition: key influences on problem solving behavior. In D. McLeod & V. Adams (Eds.), *Affect and mathematical problem solving: a new perspective (pp. 75-88)*. New York: Springer-Verlag.

Schoenfeld, A.H. (1987). What's all the fuss about metacognition? In A.H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-216). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schon, D.A. (1987). *Educating the reflective practicioner*. San Francisco: Jossey-Bass.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17-40). New York: W. W. Norton.

THE EFFECTS OF VARIED CONCRETE OBJECTS ON SOLUTION PROCESSES FOR ARITHMETIC WORD PROBLEMS: A CASE STUDY

Yeping Li, University of Pittsburgh

This paper reports results of a study on the effects of varied concrete objects on solution processes for arithmetic word problems. The subject was a third grade pupil, who was asked to solve eight word problems. With each word problem, various concrete objects were provided as possible referents. The subject was told that either paper-and-pencil or the concrete objects could be used to solve the presented word problem, that she could take as much as time as needed, but that she was required to "think-aloud" as she solved the problems.

The purpose of assigning various concrete objects to the word problems was to identify which aspect of the concrete objects determined their utility with respect to the subject's solutions. The concrete objects and problem statements used by the subject were treated as external visual representations and verbal representations, respectively. To examine the relationship between those two types of external representations, the researcher developed a schema based on Simon's work on "informal and computational (non)equivalence" (1989) and on concrete and conceptual levels. The specified tasks (i.e., the word problems and the presented various concrete objects), the think-aloud protocols, and the researcher's direct observation of the student as she completed each task were the data sources used to analyze the subject's problem-solving behaviors according to the schema developed for this study.

Results from this study suggest that: (1) The subject tended to favor concrete objects for representing and solving word problems even when using the concrete objects tended to be less efficient than using symbolic computation; (2) Whether or not the number of presented concrete objects was sufficient for representing the subsets in word problems and their manipulations is a critical factor in determining the utility of concrete objects; (3) Once the subject used concrete objects inappropriately as visual representations for solving a particular word problem, the objects became a 'cognitive obstacle' for the subject in her attempt to understanding and solving the problem; and (4) In making use of concrete objects as visual representations in her solutions, the subject adopted the principle of visual consistency either between the two external representations or within concrete objects.

Reference

Simon, H. A. (1989). On the forms of mental representation. In H. A. Simon, *Models of thought, Volume II*, (pp. 403-412). New Haven, CT: Yale University Press.

EXAMINATION OF REPRESENTATIONAL PREFERENCES ON LIMIT TASKS FOLLOWING GRAPHING CALCULATOR IMPLEMENTATION IN CALCULUS

Nina R. Girard, University of Pittsburgh

Graphing calculators present a dramatic new challenge in the teaching and learning of mathematics. Although many factors have led to discussion of calculus reform at the collegiate level, implementation of graphing technology has been the main spark generating these discussions. As Kaput (1992) delineated, the new technologies "re-energize" age old questions regarding educational goals, appropriate pedagogical strategies, as well as beliefs about the nature of subject matter, nature of learner and their learning, and relationship between knowledge and the knower (p.516).

The multiple-representation-of-concepts view of mathematical learning has been growing in significance. This theoretical view purports that students can develop deeper, more flexible understandings of concepts. Kaput (1989) suggested that multiple representation allows for suppression of some aspects of complex concepts and accentuation of others, helping to facilitate cognitive linking of representations and creating a whole that is more than the sum of its parts. The ability of students to operate within and between different representations (graphical, numerical, and algebraic) of the same concept or problem setting is fundamental to the effectiveness of the technological approach of mathematical instruction. Powerful geometrical or graphical representations of a concept can be easily added to usual algebraic representations with the aid of graphing calculators. Opportunities to increase depths of understanding by linking algebraic representations to more graphical representations are provided, thereby enhancing visualization of concepts.

This poster will present a microscopic view of students' responses to several tasks presented on a final examination in first term calculus. Graphing calculators were implemented into the college level course and a multiple representation approach was used in instruction. The tasks presented focus upon the concept of limit, in an attempt to see with which representation (graphical, numerical/tabular, or algebraic/analytical) students chose to solve the problem, when presented limits that contained functions unfamiliar to them at a Calculus I level. Enlightenment as to the students' conceptual understanding and representational knowledge and fluency is suggested. Issues of conflict resolution and the role of the graphing calculator as an exploratory or confirmatory tool are also examined.

References

Kaput, J. J. (1989). Information technologies and affect in mathematical experiences. In D.B. McLeod & V.M. Adams (Eds.), *Affect and mathematical problem solving* (pp 89-103). New York, NY: Springer-Verlag.

Kaput, J. J. (1992). Technology and mathematics education. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). Reston, VA: National Council of Teachers of Mathematics; New York: Macmillan.

Curriculum Reform

IMPACT OF CALCULUS REFORM IN A LIBERAL ARTS CALCULUS COURSE

Patricia A. Brosnan, The Ohio State University
Thomas G. Ralley, The Ohio State University

This report describes the changes in a freshman-level calculus course, Survey of Calculus, that occurred as a consequence of adopting a reformed calculus text, *Calculus* by Deborah Hughes-Hallett, Andrew Gleason et al. (better known as the Harvard Consortium Calculus or HCC text). The perspective is that of the lecturer.

The course is intended as an introduction to calculus for liberal arts students, that is, students who will not be expected to use calculus as a mathematical tool in their area of major study. The course exists because of a faculty belief that calculus is one of the great intellectual achievements of humankind, has been a major factor in the development of western civilization and should be part of every liberal education. Prior to the adoption of the Hughes-Hallett/Gleason text however, these reasons for calculus as a part of a liberal education were nowhere apparent in the course. Rather, the course was a shadow of the mainline scientific calculus, emphasizing development of skills with computational elements of calculus. Given an audience whose interests are non-scientific and whose skills with symbolic manipulation are not strong, the course left students with a feeling that mathematics is a collection of formulas and procedures to be memorized and then forgotten. (One of the principal motivations for the calculus reform movement was concern about students learning to manipulate symbols rather than understanding concepts that form a basis for general analysis in problem solving (Douglas, 1986).

Research Questions

- What factors motivated one mathematics professor to make changes in his Survey of Calculus course?
- What changes were made in curriculum, pedagogy, and assessment as a result of these motivations?

Method

Study Participants. Students enrolled in MATH 117 A Survey of Calculus for both the Winter (n = 104) and Autumn (n=40) Quarters, faculty (n=1), graduate teaching associates (GTA) (n=2), and a random sample of students (n=15) selected for interviews across two 10-week quarters.

Data Collection. A cyclical process of questioning, observing, and hypothesis generating occurred throughout the study. Major data sources included weekly interviews, daily observations, field notes, and collected artifacts.

Results and Discussion

For successful reform, the text materials must change, the instructor must change, and the assessment must change. We will look at each of these aspects of the Survey of Calculus course and describe the changes in each.

The Hughes-Hallett/ Gleason text was selected because it is quite different from traditional texts.[1] The essence of the change is captured in the following quote from the preface of the text:

> - At every stage, this book emphasizes the meaning (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs...
> - There are few examples in the text that are exactly like the homework problems, so homework problems can't be done by searching for similar-looking "worked-out" examples. Success with the homework will come by grappling with the ideas of calculus.
> - Many problems in the book are open-ended. This means that there is more than one correct approach and more than one correct solution...
> - This book assumes that you have access to a calculator or computer that can graph functions, find (approximate) roots of equations, and compute integrals numerically. There are many situations where you may not be able to find an exact solution to a problem, but can use a calculator or computer to get a reasonable approximation. An answer obtained this way is usually just as useful as an exact one. However, the problem does not always state that a calculator is required, so use your own judgment...
> - This book attempts to give equal weight to three methods for describing functions: graphical (a picture), numerical (a table of values) and algebraic (a formula). Sometimes it's easier to translate a problem given in one form into another.... It is important to be flexible about your approach: if one way of looking at a problem doesn't work, try another. (Hughes-Hallett, Gleason, et al., 1994, p. xiii)

One of the features most appealing for the liberal arts audience is given in the last point: the text makes strong use of graphical and numerical representations and (in the portion of the book used in the course) downplays the importance of formulas. For an audience whose algebraic skills are modest, this emphasis was highly beneficial, allowing them to examine concepts without being required to carry out extensive algebraic computations.

Another important feature of the text are the examples and problems that require thoughtful application of the concepts. Below is a problem from the text to illustrate the type of thinking students were asked to do (Hughes-Hallett, Gleason, et al., 1994, p. 34).

> Values of three functions are contained in Table 1.16 (The numbers have been rounded to two decimal places.) Two are power functions and one is an exponential. One of the power functions is a quadratic and one is a cubic. Which one is exponential? Which one is quadratic? Which one is cubic?

x	$f(x)$	x	$g(x)$	x	$k(x)$
8.4	5.93	5.0	3.12	0.6	3.24
9.0	7.29	5.5	3.74	1.0	9.01
9.6	8.85	6.0	4.49	1.4	17.66
10.2	10.61	6.5	5.39	1.8	29.19
10.8	12.60	7.0	6.47	2.2	43.61
11.4	14.82	7.5	7.76	2.6	60.91

Another example, taken from the chapter on differentiation, illustrates how a traditional topic can be treated in a new and intriguing manner.(Hughes-Hallett, Gleason, et al., 1994, p. 128)

> Table 2.13 shows the number of abortions per year, A, performed in the US in year t (as reported to the Center for Disease Control and Prevention). Suppose these data points lie on a smooth curve $A=f(t)$.

Table 2.13
Abortions reported in the US (1972-1985)

Year, t	1972	1976	1980	1985
Number of abortions reported, A	586,760	988,267	1,297,606	1,328,570

> (a) Estimate dA/dt for the time intervals shown between 1972 and 1985.
>
> (b) What can you say about the sign of d^2A/dt^2 during the period 1972-1985?

Typically the second derivative is introduced as an exercise in differentiating the first derivative of a function $f(x)$ and is used to identify the concavity of the function, presumably useful in sketching the graph of $f(x)$. Those who have done this know that points where the concavity changes are very difficult to identify on the graph. Graphing calculators make this use of the second derivative obsolete. In class discussion of this example, students were asked tox construct arguments both for and against legislation limiting access to abortion. The side arguing for limitations used the fact that the number of abortions was increasing (i.e., $dA/dt > 0$). The side argu[illegible] against limitations used the fact that the number of abortions was increasing at a decreasing rate (i.e., $d^2A/dt^2 < 0$). In another example illustrating the use of the second derivative, the authors quote a member of Congress who during the 1985 Defense Department budget hearings complained, "It's confusing to the American people to imply that Congress threatens national security with reductions when you're really talking about a reduction in the increase" (Hughes-Hallett, Gleason, et al., 1994, p.127). Examples of this sort lend credence to the argument that calculus should be a part of a liberal education.

Change on the part of the lecturer was motivated by several factors. The Hughes-Hallett / Gleason text provides abundant opportunities to examine and discuss problems that probe conceptual understanding and this encouraged the lecturer to increase time spent interacting with students (a non-trivial task in large lecture sections) in examining ideas. Another motivation for change came from the NCTM reforms that encourage the use of cooperative groups. Parts of the lecture hour were regularly used to put students in groups of three or four to work together on specially prepared problems that were turned in at the end of class. It was a humbling experience for the lecturer to observe how his beautifully prepared, carefully organized lecture presentation seemed to have made no impression on the students as they struggled to construct their own understanding while working on the problems. The ease with which a graphing calculator overhead unit could be used in class and the integration of technology into the text was a third motivation for change; increased use of technology during the lecture. A TI-82 graphing calculator was used in almost every lecture. An illustration of a place where the calculator was particularly effective is in the authors' introduction to the derivative in the form of a thought experiment in which a grapefruit is tossed up into the air and then falls back to the ground with its height at time t recorded in a table. Using the calculator, it was possible to reproduce the data for height and time in one second increments as a table in the calculator. From the initial table, the class was able to calculate average velocities over one second intervals about time t_0. The table increment was then changed to present the height and time in half second increments and average velocities about t_0 were again computed. The process was repeated one more time with a table having time increments of one tenth of a second. From the three sets of calculations of average velocities, the limiting value was clear to the students. This sort of demonstration is not feasible with blackboard and chalk; the time required to generate the table so dominates the process that the calculation becomes the important issue and the use of the data is obscured.

Evaluation strategy changed in several ways. First, written homework assignments from the text replaced one of the mid-term exams, making written problem solutions approximately 22% of the total marks for the class. Second, the format of exam problems changed; they were more open-ended and asked for explanation of the reasoning used to arrive at a solution. An example of such a problem is the following:

> You're home for a long weekend and your little sister, who is taking Advanced Placement Calculus in high school, tells you that she is failing because she doesn't have a clue as to what a

> derivative is, or how to find a derivative or what the derivative does. Write a paragraph describing what you would do to help her learn calculus.

On a problem such as this, the expectation was that the student would mention something about the derivative as a way to measure the rate at which a function changes, would give an example in which different representations (numeric, graphical or symbolic) were used as a means of measuring rate of change, would give a description of the mathematical definition of derivative (along the lines of "the limiting values of the average rates of change over increasingly smaller and smaller intervals about a specified value of the variable"), and, perhaps, give an example of a problem in which the derivative would be used. As mentioned earlier, a third change was use of part of the lecture hour for problem solving by students working cooperatively in small groups. About once a week, during the final twenty-five minutes of the lecture hour, students were asked to work in groups of three or four. The problem was generally a topic from that day's lecture or something from the text that had been studied recently. The group was to come to consensus on a solution to the problem and to write a solution together. This work constituted less than 5% of the total marks for the class and was used as a bonus, added to mid-term scores.

The use of written homework assignments, the change in exam questions and the group problems in lecture all contain elements of reform efforts to develop students' facilities with written communication of mathematical ideas. The Harvard materials provide a rich variety of problems that require careful examination of concepts and the application of that understanding to new situations. It was important that students regarded this as a central part of their activity in the course, which meant including it as part of the grade.

What are the perceptions of the instructor about the global changes that occurred? The learning was different; the material was appropriate to students' needs and level of sophistication. Students came away from the course with different attitudes about mathematics. The use of the graphing calculator opens up new ways of understanding mathematics, new ways of representing mathematical objects and aids students in their studies of mathematics. The use of cooperative groups is an important technique in promoting student involvement in learning. Increased use of writing in mathematics is critical to students learning conceptually rather than mechanically.

References

Douglas, R.G. (1986). Toward a lean and lively calculus. *MAA Notes*, #6.

Hughes-Hallett, D., Gleason, A., et al. (1994). *Calculus*. New York: John Wiley & Sons, Inc.

INTERNATIONAL INFLUENCES ON THE NCTM STANDARDS: A CASE STUDY OF EDUCATIONAL CHANGE

Douglas B. McLeod, San Diego State University
Robert E. Stake, University of Illinois
Bonnie Schappelle, San Diego State University
Melissa Mellissinos, San Diego State University

The NCTM *Standards* have multiple origins. In part they developed out of concerns from NCTM committee members about textbook adoption policies that favored traditional texts. They also constituted a response to the public furor caused by *A Nation at Risk*, and instantiated *An Agenda for Action's* 1980 recommendations on curriculum. International curriculum and research projects also influenced the NCTM *Standards*. The initial NCTM emphasis on standards as accountability criteria shared certain similarities with the National Curriculum effort in England and Australia, where reform is reportedly stalled. The more recent view of the *Standards* as aspiration may help extend the duration of their influence.

The publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) was the culmination of a series of important events in mathematics education in North America. The development of these *Standards* is usually described in the context of conference recommendations from the US (Crosswhite, Dossey, & Frye, 1989; Romberg & Webb, 1993), but international forces were also at work. The purpose of this paper is to describe the origins of the NCTM *Standards* and to analyze how international forces helped shape the *Standards* and the reform movement in mathematics.

This case study has focused on understanding the origins of the NCTM *Standards*, as well as their development, dissemination, and impact in K-12 classrooms. Our methods followed the recommendations of Stake (1994). Main sources of data included interviews with NCTM leaders and state mathematics supervisors in the US. One of eight studies of educational change in the US (see Romberg & Webb, 1993), our project is part of an international effort coordinated by the Organisation for Economic Cooperation and Development in Paris. One goal of our project is to explain to policy makers from abroad how a professional organization like NCTM could provide direction for educational change, a task that is usually left to government officials.

Origins of the NCTM *Standards*

The decline of test scores was one of several issues that were influential in the push for educational reform. For example, *A Nation at Risk* (NCEE, 1983) noted the decline in SAT scores from 1963 to 1980. NCTM leaders, however, put more emphasis on the results from the Second International Mathematics Study (SIMS). Although the NCTM *Standards* were being planned before the SIMS data were reported, the preliminary results were known to NCTM leaders. These leaders did not see their task as organizing mathematics education for an international competition; they were concerned about the weakness in the US curriculum that was reflected in the data. As one NCTM leader put it:

> We weren't being motivated by "world class standards" at that point. [But] we did have comparative data, especially in terms of the Japanese curriculum, which showed so much more intensity than ours did.

Although the SIMS data were important in the thinking of NCTM leaders, reports of the Second International Study (e.g., McKnight et al., 1987) were not cited in the list of references in the NCTM *Standards* (1989). That omission caused some concern, but an NCTM leader described their reasoning this way:

> I think that if you base your argument for [reform] on a temporal research result, you're being reactionary rather than proactive. . . . The focus was to take the negative, the competitive statements out of the document, and make the document a proactive, positive statement. Let's say what we believe and then act on it.

The Instructional Issues Advisory Committee (IIAC)

The quality of the US curriculum was related to IIAC's concerns about accountability. The earliest talk of professional standards in NCTM circles probably occurred in IIAC, after that committee and the Research Advisory Committee (RAC) received a request to help one of NCTM's affiliated groups with criteria for evaluating textbooks. "There was some concern [from] several places that textbooks, and therefore curricula, were being driven by non-professional considerations, political log rolling, and so on." An RAC member recalled:

> RAC had a request for information about research on the efficacy of John Saxon's algebra. Our discussion quickly broadened to the general question of evaluating curriculum materials in the absence of standards by which to measure "success" or desirability...I recall that we were acutely aware that...we were asking NCTM to abandon its long-standing and explicit policy not to pass judgment on various curriculum efforts.

IIAC had also considered the issue and took on the task of developing standards for textbook selection in early 1983, before the appearance of *A Nation at Risk* (NCEE, 1983). A committee member recalled how the notion of standards got extended:

> Somehow we got onto the idea that maybe what IIAC ought to be about was defining professional standards in general—not just for selection of textbook material but for content of the curriculum, for teaching, and so on.

The recommendations from IIAC had a strong accountability emphasis:

> There was talk about something comparable to the Good Housekeeping Seal of Approval. We would have standards that could be applied to textbooks [and] tests. The ones that were judged to meet the standards then would be given the Seal of Approval and the ones that weren't would not.

The recommendations of RAC and IIAC began to coalesce in the spring of 1983 at a meeting of the NCTM Board of Directors:

> It was interesting that not only RAC was asking the Council to take a proactive stand, but [also IIAC. We] had already seen the raw data from SIMS, which weren't known yet by the other board members . . . and [data from] the National Assessment of Educational Progress, too. So issues were coming together. IIAC said that we needed to look at setting some goals to stop . . . this fad and that fad from affecting our curriculum.

Meetings of Leaders

Shortly after the publication of the *Agenda*, the Reagan Administration eliminated all funding for K-12 mathematics and science education from the budget of the National Science Foundation (NSF). To the dismay of those who worked in the Education Directorate at NSF, some NSF leaders capitulated easily to the Reagan Administration and made the preservation of research programs in science and engineering their main priority. Meanwhile, *A Nation at Risk* (NCEE, 1983) received "unprecedented" media attention. All America heard that "Our Nation is at risk. . . . the educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people" (NCEE, 1983, p. 5). Many leaders give credit to *A Nation at Risk* for helping establish a climate that would support change:

> I think *A Nation At Risk* (NCEE, 1983) served primarily as a spark plug, a starting point for people…States were requiring a third year of mathematics and some other things and making political decisions without ever talking to the math ed community. So that…started a lot people talking about the need for reform.

In the wake of *A Nation at Risk*, two meetings were particularly central to the development of the NCTM *Standards*. In the words of one participant:

> After *A Nation at Risk* came out, the Conference Board of the Mathematical Sciences organized a retreat at Airlie House in Virginia [funded by NSF]. It was at that meeting that Joe [Crosswhite] introduced a motion…that there should be a set of standards for school mathematics at NCTM.

In December, 1983, a month after the meeting in Virginia, the Department of Education sponsored a meeting at Wisconsin: "School Mathematics: Options for the 1990s." The report of that meeting (Romberg, 1984), with its recommendations for new K-8 and 7-14 curriculum guidelines, also shows a direct link to the *Standards*.

Development of the NCTM *Standards*

When the writers of the NCTM *Standards* gathered in Utah in 1987, they were provided with a rich set of resources to help stimulate their thinking. These materials were mainly written in English, so the number of foreign countries that were represented was small. But the materials did include the Cockcroft report and "a library of SMP [School Mathematics Project] materials" from England. As one leader put it:

> We tried to organize materials from other countries—England, the Netherlands, Australia. Some of us spent a fair amount of time down at Chicago looking at some of the [Wirszup] materials...The Math Curriculum Teaching Project [from Australia] had a lot of interesting examples.

Other work from England was a significant influence at the 9-12 level:

> At the time that we were beginning to start on the *Standards*, there was some interesting work being done over at the Shell Centre in England in terms of more qualitative applications of mathematical thinking, for example, the work on the language of functions and graphs.

As the staff member who was responsible for materials noted, "We just flooded them with stuff." There were materials by D'Ambrosio (of Brazil) dealing with ethnomathematics, and the writings of Freudenthal (of The Netherlands), whose work on "didactical phenomenology" was thought to be "a little hard for most people" to get through. Writers rarely mentioned these works, but the leadership was clearly influenced by them, and saw them as compatible and supportive:

> There was a sense that kids ought to experience mathematics—that they're reinventing some of the important ideas. And then teachers negotiate with them the language in terms of signs and symbols that we commonly use.

Other international researchers who had the eye of the leadership included G. Vergnaud of France, especially his work on multiplicative conceptual fields, and J. de Lange of The Netherlands, with his "realistic mathematics education." The work of these researchers demonstrates some of the international influences on the *Standards*, especially in terms of changes in theories of learning. Steffe and Kieren (1994) have noted the influence of constructivism on the NCTM *Standards*. A leader comments:

> The term that we did not use in the write up of the *Standards*, but we certainly talked about, is...the social constructivist notion of learning.... One of the arguments that people have made is, "Why didn't you call yourselves social constructivists?" But that would have put off people who didn't understand that set of notions.

Another leader reported: "I don't remember a constructivist approach being that hot at the time that the *Standards* were being developed." But some writers were definitely being encouraged to think along constructivist lines:

> I remember coming back and talking with some of my [constructivist] colleagues [in 1987], and they thought the idea of standards was very authoritarian—they were pretty negative toward it. When we met again [later in 1987], even then we were drifting toward the *Standards* as more of a vision—a less authoritarian perspective.

The debates over the substance and the wording of the standards was often intense. As an example, consider the case of a "standard" that was suggested by one group but did not garner enough support to survive until the final draft. That proposed standard was concerned with the way that history and culture influence mathematics and its teaching. An early version of the standard, entitled "Historical and Cultural Significance," follows:

> In Grades 5-8 the mathematics curriculum should foster an historical and cultural awareness of mathematics so that students are able to:
>
> - Explore mathematics in relation to the arts, humanities, and sciences.
> - Appreciate that mathematics is an invention of the human mind.
> - Appreciate the potential of mathematics as an enjoyable activity.
> - Appreciate mathematics as a powerful, creative human activity.

The elaboration that was outlined for this proposed standard included mathematics and music, history of mathematics, recreational mathematics, and numeration systems. As some writers look back on it now, the standard would have fit very nicely with the current interest in ethnomathematics, a topic of increasing importance in research (D'Ambrosio & D'Ambrosio, 1994). At that time, however, the topic was seen as difficult to communicate and not central to the content emphasis of the *Standards*. In the words of one source:

> The middle school group came up with the standard on culture [but] the *Standards* were conceived as focusing entirely on con-

> tent, and culture was not perceived to be content. The reaction was, "Well, this is too touchy-feely." . . . The frustration that I had, and still have with that rejection, is that in fact there is a whole philosophy of mathematics that was developing at that time that looks at mathematics as a cultural creation. [But] the members of the working groups really hadn't had a chance to look at that literature and think about it.

One of the writers had a slightly different view, noting that "we wanted to show that kids . . . had things back in ancient history" that connected them to mathematics, but their arguments were not convincing:

> The more interesting thing is what is in the personal culture of each child that is mathematical. Certainly part of that is their history, whether it be racial or ethnic or whatever. We enunciated that ethno-cultural part, but we didn't have anything very strong on the personal-cultural part.

Comparing the NCTM *Standards* to Reform Efforts in Other Countries

One common interpretation of the term *standards* is the notion of accountability expressed by IIAC, which wanted to set standards that would then be used to judge textbooks and tests. This "accountability" approach to educational change has characterized much of the thinking in the US, as well as in England and Australia. The National Curriculum in England is reported to have stalled over accountability issues (Atkin, 1994), including the high cost of producing better tests that are then rejected by politicians because they don't look like traditional tests. There are also problems in Australia (Ellerton & Clements, 1994), where the debates over assessment have divided the mathematics education community. In the US, where the original focus on standards as an accountability tool has been transformed in part to an emphasis on standards as aspiration, will the NCTM *Standards* be more likely to endure?

References

Atkin, J. M. (1994). Developing world-class education standards: Some conceptual and political dilemmas. In N. Cobb (Ed.), *The future of education* (pp. 61-84). New York: CEEB.

Crosswhite, F. J., Dossey, J. A., & Frye, S. M. (1989). NCTM Standards for school mathematics: Visions for implementation. *Arithmetic Teacher, 37* (3), 55-60.

D'Ambrosio, U., & D'Ambrosio, B. (1994). An international perspective on research through the *JRME*. *Journal for Research in Mathematics Education, 25*, 685-696.

Ellerton, N. F., & Clements, M. A. (1994). *The national curriculum debacle*. West Perth, Australia: Meridian Press.

McKnight, C. C., et al. (1987). *The underachieving curriculum: Assessing US school mathematics from an international perspective.* Champaign, IL: Stipes.

National Commission on Excellence in Education. (1983). *A nation at risk: The imperative for educational reform.* Washington, DC: US Government Printing Office.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Romberg, T. A.. (1984). *School mathematics: Options for the 1990s.* Washington, DC: U.S. Department of Education.

Romberg, T. A., & Webb, N. L. (1993). The role of NCTM in the current reform movement in school mathematics in the USA. In *Science and Mathematics Education in the United States: Eight Innovations* (pp. 143-182). Paris: Organisation for Economic Cooperation and Development.

Stake, R. E. (1994). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 236-247). Thousand Oaks, CA: Sage.

Steffe, L. P., & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education, 25*, 711-733.

RADICAL CONSTRUCTIVISM AS A BASIS FOR MATHEMATICS REFORM

Grayson H. Wheatley , Florida State University
Steven Blumsack, Florida State University
Elizabeth Jakubowski, Florida State University

This paper describes the use of radical constructivism as a basis for curriculum reform in university mathematics courses and reports on research conducted on two of the courses developed. The theoretical underpinning of the project is described along with the implications for course design and instruction. Finally, results from qualitative research conducted on two of the courses is presented. The courses were found to foster intellectual autonomy, challenge students to rethink mathematics from a conceptual rather than procedural perspective, promote confidence in their mathematics knowledge, become more positive mathematics learners and make connections among algebra, geometry, and calculus concepts.

It is often said that we teach as we are taught. Undoubtedly, the nature of instruction in mathematics courses taken in college greatly influence the teaching styles and practices of teachers. "Very few teachers have had the experience of constructing for themselves any of the mathematics that they are asked to teach. " (National Research Council, 1989). In designing a middle school mathematics teacher education program, we recognized the importance of having mathematics taught in a manner compatible with the goals of their pedagogical courses. We have much experience with methods courses following the recommendations of the National Council of Teachers of Mathematics and the National Research Council being offered to students who take mathematics courses based on logical positivism and behaviorism - mathematics courses which are lecture based and emphasize practicing taught procedures. This conflict has not been lost on prospective teachers. They struggle with the question, "Why am I being asked to teach in a way I have never experienced in a mathematics class?" Beginning teachers will instinctively use teaching methods like those experienced in the many mathematics courses taken in high school and college. We have not always been able to overcome the impact of many hours listening to lectures and practicing procedures which are then marked right or wrong. Thus, we recognized the importance of mathematics courses for prospective teachers which emphasize sense making, encourage collaboration and promote intellectual autonomy.

The purpose of this paper is to describe a theoretical basis for mathematics instruction and report findings from analyses of courses based on the theory. As one component of our four-year teacher education project funded by the National Science Foundation, four mathematics courses for prospective middle school mathematics teachers were designed: Number Theory, Algebra, Geometry, and Problem Solving. Each course was created by a course development team (one for each course) composed of two or more mathematicians, a mathematics educator and several mathematics education graduate students. We were most fortunate in having a mathematician who understood the reform movement and believed in opportunities for students to construct their own knowledge as a member of each team. A team met for a year planning the geometry course and a semester for each of the other courses. Our thinking was influenced by the NCTM Professional Standards (1991) and a constructivist epistemology (von Glasersfeld, 1995a). Examples of the mathematics and instruction will be drawn from the geometry and the problem solving courses; analyses of the other courses are in progress.

Epistemology

In designing courses and planning lessons, it is useful to have a clearly defined epistemological theory. For this project, radical constructivism as described by von Glasersfeld (1995a, 1995b) served as the theoretical orientation. In this theory of knowing, which has been used for other mathematics educational reforms, it is assumed that knowledge cannot be transmitted but must be constructed by the learner. Students have only their personal experiences upon which to rely in this constructive process and each person has unique experiences. Of course a person's experiences include other persons and thus it is not a 'lonely voyage.' Thus activities in which students are encouraged to work together in solving a problem, to listen, explain and challenge peers provide rich potential learning opportunities.

A second principle of radical constructivism has to do with the nature of knowledge. For the logical positivist, knowledge is out there, out there for the behaviorist to observe. For the radical constructivist, knowledge is an individual construction which results from attempting to make sense of our experiences. Knowledge is not true or false but viable or not viable. As von Glasersfeld states,

> [We must] Give up the requirement that knowledge represent an independent world, and admit instead that knowledge represents something that is far more important to us, namely what we can *do* in our *experiential* world, the successful ways of dealing with the objects we call physical and the successful ways of thinking with abstract concepts. (pp. 6-7)

A radical constructivist epistemology places importance on constructing models of student's thinking. As von Glasersfeld states,

> In the endeavor to arrive at a viable model of the student's thinking, it is important to consider that whatever a student does or says in the context of solving a problem is what, at the moment, makes sense to the student. It may seem to make no sense to a teacher, but unless the teacher can elicit an explanation or generate a hypothesis as to how the student has arrived at the answer, the chances of modifying the student's conceptual structures are minimal. (p. 15)

Certain classroom practices are suggested by radical constructivism. First, we must negotiate a set of social norms in which emphasis is on making sense rather than following procedures specified by an instructor. The goal is for each individual to develop a rich network of schemes which are viable. The social norms might include

- A task requires time and investigation; we should not be expected to know how to do a task but instead develop our own procedures for accomplishing the task. An exploratory mind-set is essential.
- Students are expected to explain their reasoning to peers; viability is established by convincing others. An assertion (proof) which stands the test of time is said to be viable.
- Collaboration is an accepted environment for learning.

The implications for the instruction flowing from this theory are:

1. The mathematics to be studied must be analyzed to determine the major concepts and relationships.
2. It is important to build models of students thinking.
3. Based on these first two practices, tasks are designed which have potential learning opportunities.
4. All activities must be potentially meaningful to the students.
5. Meaning must be negotiated; it cannot be transmitted or legislated.
6. A major responsibility of the teacher is to facilitate classroom discourse.
7. This entire process is recursive.

Procedure

Each session of the two courses were video recorded for the full semester. Field notes were collected and each instructor reflected on lessons after each class session. The geometry course was taught by a mathematics education doctoral student and the problem solving course was taught by a mathematics education professor. The principles on which these courses were designed were:

1. The courses should focus on central ideas in mathematics and promote progressive schematization rather than specified procedures.
2. Activities must be interesting and potentially meaningful to the students. In each case an effort was made to approach the subject from a different perspective than they had seen previously. For example, many of the properties of plane geometry were developed from a study of spherical geometry, e.g., straightness.
3. Students are to be encouraged to become intellectually autonomous rather than simply doing what the instructor said whether it made sense to them or not.

4. A problem-centered instructional model (Wheatley, 1991) was adopted.

5. Collaboration was encouraged.

6. Students were required to justify the viability of their solutions. Rather than the teacher judging responses as right or wrong, students presented their solutions to the class and the class had to be convinced of the validity of the solution.

7. Technology was to be used whenever feasible. For example, two computer microworlds were developed for the geometry course and spreadsheets were used extensively in the number theory course.

8. Assessment was based, in large part, on informed professional judgment and portfolios.

In teaching the courses, considerable attention was devoted to negotiating social norms conducive to inquiry and intellectual autonomy. Students often entered the courses with a belief that mathematics is a set of facts and procedures to be explained by the teacher and remembered by them. The courses were designed to foster the view that mathematics is the *activity* of constructing patterns and relationships. Students were encouraged to take responsibility for their knowledge construction in conjunction with other members of the class.

Because the instructor rejected the role of mathematical authority, the students began to assume responsibility for justifying their actions. These justifications took the form of students presenting their solutions to problems they had solved and responding to questions raised by their peers or the instructor. At times students who disagreed or had an alternative solution went to the board and began explaining their point of view without any action by the teacher; none was required. In both courses the mode of instruction utilized was problem centered learning as described by Wheatley (1991).

Approximately one-fourth of the geometry course was devoted to a study of spherical geometry. The decision to study spherical geometry in a course for prospective middle school students was based on our belief that interesting and significant questions could be raised which would deepen the meaning given to plane geometry concepts such as straightness, angle, and quadrilateral. The topic was also potentially meaningful and interesting since we do not live on a flat surface and NASA activities have raised our consciousness of the earth as a sphere. The statement, "The sum of the angles of a triangle is 180 degrees" takes on a richer meaning once triangles have been drawn on a beach ball with marking pens and the sum of the angles determined. In addition to spherical geometry, topics in plane geometry and measurement were studied through problem solving.

The students became quite interested in the study of spherical geometry as evidenced by observations, their journal entries, interviews and written evaluations at the end of the course. Students were thrown into a state of disequilibrium by some of their findings as they engaged in these activities. Of particular value were the two computer microworlds in which students could explore paths on a sphere. Not only were the students motivated to study spherical geometry but they made significant geometric (mental) constructions.

In the problem solving course, the nature of solutions became increasingly more organized and sophisticated. Initially, students attempted to identify formulas and substitute numbers but soon realized this approach would not work on the nonroutine problems they faced. But as they participated in the negotiation of a different way of doing mathematics, they became more thoughtful about their activity. For example, in week six of the course students presented a variety of solutions to the following problem.

> A column of soldiers 25 miles long marches 25 miles a day. One morning, just as the day's march began, a messenger started at the rear of the column with a message for the man at the front of the column. During the day he marched forward, delivered the message to the first man in the column and returned to his position just as the day's march ended. How far did the messenger walk?

This problem required rather sophisticated problem solving strategies and considerable power in thinking in terms of rates. Additional information about the problem solving course can be found in Trowell (1994).

Analysis of the courses indicated that 1) students were challenged to rethink mathematics concepts previously studied but not understood; 2) students developed confidence in their mathematics knowledge; 3) students became more positive as mathematics learners; and 4) students increased their competence and made connections among algebra, geometry, and calculus concepts.

Summary

In a *Call for Change: Recommendations for the Preparation of Teachers of Mathematics*, the Mathematics Association of America states that, ". . . collegiate mathematics classrooms must become a place where students actively do mathematics rather than simply learn about it" (p. 2). This statement could be interpreted as embracing a constructivist approach to mathematics teaching. In this study, evidence for the power of university mathematics courses based on radical constructivism was obtained. While certainly not the only viable theoretical orientation for successful mathematics teaching, radical constructivism, growing out of Piagetian theory, provides a sound basis for facilitating mathematics learning.

References

Mathematics Association of America (1991). *Call for Change: Recommendations for the Preparation of Teachers of Mathematics*. Washington, D. C.: Mathematics Association of America.

National Council of Teachers of Mathematics (1991). *Professional Standards for Teaching Mathematics*. Reston, VA: NCTM.

National Research Council (1989). *Everybody Counts.* Washington, D. C.: National Academy Press.

von Glasersfeld, E. (1995). A constructivist approach to teaching. In L. Steffe and J. Gale (eds.), *Constructivism in education.* Hillsdale, NJ: Lawrence Erlbaum and Associates.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning.* Washington, D. C.: The Falmer Press.

Trowell, S. (1994). The negotiation of social norms in a university mathematics problem solving class. Unpublished doctoral dissertation. Florida State University, Tallahassee, FL.

Wheatley, G. (1991). Constructivist perspectives on science and mathematics learning. *Science Education, 71(1),* 9-21.

STUDENTS' CONCEPTIONS OF WHAT IT MEANS TO DO MATHEMATICS

Linda Dager Wilson, University of Delaware

What is the response of students to the reform efforts in mathematics education? A survey taken in September of 59 eighth grade mathematics students showed that their conceptions of what it means to do mathematics were predominantly traditional in nature. The survey was repeated in the spring, when these students had experienced nine months of a reform-oriented class. The spring survey showed that students were in many respects more open in their acceptance of alternative activities. The majority of students, however, still felt strongly that "listening to the teacher explain" should be included in a conception of school mathematics. There were also strong negative opinions among a majority about writing journals in a mathematics class.

Gabe:	I think explaining your thinking is stupid. If I know the answer to a problem and I get it right, why should I have to explain how I got it? It's a waste of time.
Mustafa:	When I hear other students explaining how they got a problem, that's when I really learn, and when I have to tell someone else how I got the answer it helps me think more clearly about where that answer came from.

These are just some of the conflicting opinions of eighth graders, talking about what it means to do mathematics. This range of opinion reflects some of the resistance and some of the openness to the changes that are taking place in some mathematics classrooms in response to the reform movement in mathematics education. The NCTM Standards documents (NCTM, 1989; NCTM, 1991; NCTM, 1995) call for shifts in mathematics education towards a "rich variety of mathematical topics and problem situations," "active student learning," "environments that support learning," and "assessment that is ongoing and based on multiple sources of evidence" (NCTM, in press). One implication from these documents is that students in "reform" classrooms will experience a much wider range of activities involved in doing mathematics. One could hypothesize that these students would then have an expanded conception of what it means to do mathematics, so that, for instance, writing a journal in mathematics class would not be a foreign idea, or explaining how you got an answer might be just as important as getting the right answer.

The Setting

The study was conducted with two classes of eighth grade Algebra I students in a public middle school in the Mid-Atlantic. There were 30 students in one class and 29 in the other. Both classes were taught by the same teacher. The gender breakdown was 39 male, 20 female, and there were 8 nonwhite and 51 white students in the two classes combined.

The teacher, Ms. Vincent, has twenty years' experience in mathematics teaching at secondary and middle levels. She has been at this school for ten years. She is actively involved in several reform-oriented initiatives at the school, state and national levels. Ms. Vincent's teaching could be characterized as moving in the directions called for in the three NCTM Standards documents. The major emphasis in any of Ms. Vincent's classes was on the importance of explaining your thinking, either orally or in writing.

Methods

A Likert scale instrument was administered to the students in both classes on the second day of school in September, 1994. Each item described an activity that might occur in a mathematics classroom, such as "using manipulatives," and students were asked to respond according to whether they strongly disagreed, disagreed, agreed, or strongly agreed that this activity was a part of what it means to do mathematics. The same instrument was administered to the students in the spring of 1995 during the last week of school.

After each survey in both fall and spring, four students were selected to take part in group interviews (Ms. Vincent was not present). Students were chosen on the basis of their responses, with the goal being that students with diverse opinions would have an opportunity to talk with each other. These interviews were audiotaped and transcribed. The eight students who were interviewed in the fall made it clear that their responses to the items reflected not only the activities they had experienced in past mathematics classes, but also the activities that they personally thought "ought to be" part of a mathematics class.

Results

The chart that follows (Table 1) shows the results of the survey for fall and spring. The items, which were in a random order on the survey, have been reorganized here according to those that might be considered "traditional" mathematics activities and those that might occur in a "reformed" classroom. Results are combined for the two classes, and the "strongly disagree" and "disagree" responses have been combined, as well as the "strongly agree" and the "agree," so as to show the overall percentage of students who either agree or disagree that a particular activity is part of what it means to do mathematics.

If we consider the first 9 items in the table to be "traditional" activities, we see that there was strong support for these activities, especially in the fall, when no activity had a majority disagreeing. At that time, the activities that students most often agreed with were *Listening to the teacher explain* (96), *Getting the right answer* (92), *Doing problems on worksheets* (88), and *Practicing computational skills*(85). All but the last of these activities dropped in percent agreement in the spring by at least 10 percentage points. By the spring, those "traditional" activities that drew the most agreement were *Practicing computational skills* (83), *Listening to the teacher explain* (82), and *Memorizing basic facts*(78).

In the fall, the lowest level of agreement among the "traditional" activities was for *Using a textbook*(60) and *Getting an answer quickly* (53). Both of these activities lost support in the spring, and they were again the two that had the least support at that time.

Every "traditional" activity went down in percent agreement from fall to spring, with the exception of *Showing all your work*. The activities that dropped the most were *Getting an answer quickly* (53 to 33), *Using a textbook* (60 to 42), and *Getting the right answer* (92 to 75). The "traditional" activities that changed the least from fall to spring were *Practicing computational skills* (85 to 83) and *Showing all your work* (75 to 77).

Consider the remainder of activities on the list as "reform" activities. In the fall, the activities that most students agreed were part of doing mathematics were *Trying different ways to solve a problem* (100), *Using manipulatives* (96), *Working in groups* (94), and *Using a calculator* (90). These are the same activities that had

Table 1.
Percent of Students (n=59) Who Agree With Each Activity

Activity	Fall	Spring
Getting the right answer	92	75
Practicing computational skills	85	83
Drill and practice	75	65
Doing problems on worksheets	88	75
Memorizing basic facts	83	78
Getting an answer quickly	53	33
Using a textbook	60	42
Listening to a teacher explain	96	82
Showing all your work	75	77
Working in groups	94	85
Explaining your thinking orally	73	68
Explaining your thinking in writing	63	78
Making an educated guess	85	82
Testing hypotheses	85	77
Trying different ways to solve a problem	100	88
Presenting solutions to the class	77	67
Using a calculator	90	85
Using manipulatives	96	87
Listening to other students explain	79	72
Doing projects	76	80
Judging other students' work	32	64
Writing journals	19	28
Having a conference with the teacher	68	80
Putting together a portfolio	54	63

the most agreement in the spring, though all had slightly lower levels of agreement.

Students in the fall most often disagreed with *Writing in journals* (19) and *Judging other students' work* (32). By the spring, *Writing in journals* had found increased support (28), but was still well below the 50% mark. However, *Judging students' work* (64) had gained a majority. Overall, 6 of the "reform" activities gained support, while 9 lost. The categories that lost the most were *Trying different ways to solve problems* (100 to 88) and *Presenting solutions to the class* (77 to 67). The activities that gained the most were *Judging other students' work* (32 to 64), *Explaining your thinking in writing* (63 to 78), and *Having a conference with the teacher* (68 to 80). The activities that changed the least from fall to spring were *Making an educated guess* (85 to 82), *Explaining your thinking orally* (73 to 68), and *Using a calculator* (90 to 85).

Conclusions

In general, there was more uniform agreement for the "traditional" activities than for the "reform" activities in the fall. By the spring, there was more uniformity in support for the "reform" activities. In the fall the percent agreement for "traditional" activities is clustered between 53 and 96, while the "reform" activities range from 19 to 100. By the spring, the percent agreement for "reform" activities is clustered between 63 and 88, while the "traditional" activities have agreement levels from 33 to 83.

The greatest changes from fall to spring illustrate an increase in support for "reform" activities and a decline in support for "traditional" activities. That is, *Judging other students' work* rose from 32 to 64 and *Explaining your thinking in writing* rose from 63 to 78. At the same time, *Getting an answer quickly* fell from 53 to 33, *Using a textbook* fell from 60 to 42, and *Getting the right answer* fell from 92 to 75. Overall this would seem to indicate that students in these classes were more open to the idea that critiquing each other's work and explaining their thinking were important to doing mathematics, and less inclined to feel that quick and accurate answers and dependence on a textbook were important.

When the results are broken down into the 4 separate categories that appeared on the survey and changes are tracked from fall to spring, there is only one item that showed a marked change in the shape of the distribution of scores from fall to spring. That is, for most items there were more responses of "agree" and "disagree" than there were of "strongly agree" and "strongly disagree," so that the decline in agreement usually indicated a distribution that was merely shifted downward between fall and spring. However, the activity *Explaining your thinking in writing* was unique in this regard. In Figure 1 the double bar graph shows that there was not only an increase in support for this activity, but more students chose to "strongly agree" with this activity in the spring.

Would I make the claim that these students were in general more open to "reform" activities in the spring than they had been in the fall? The answer, based on this data, would have to be a "yes, but...". On the one hand, the five categories

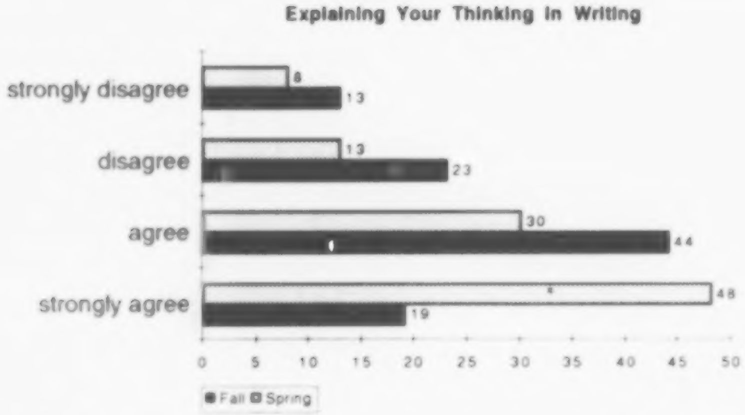


Figure 1

that changed the most would indicate a more open set of opinions (as pointed out above). Also, *all* the "traditional" activities lost support during the year, while 6 out of 15 "reform" activities gained support. However, a closer look at those "traditional" activities that continued to have strong support in the spring indicate that students still strongly believed that *Listening to the teacher explain* is very much a part of doing mathematics. In addition, these students began the year with a strong negative attitude toward *Writing in journals*. While this activity gained some support (from 19 to 28), it was still the most negative activity on the list. In the interviews, many students described experiences with writing in journals during prior mathematics classes. This was not an activity that Ms. Vincent used during the year I observed. Some students had never done this kind of activity and couldn't imagine that it could be part of a mathematics class, while others had done it and had a negative experience.

Implications

This study is based on a simple survey of two classes of eighth graders in one school. Obviously it is not possible to generalize these findings very far beyond this classroom. Yet there are some implications of this small study that may be worth considering. The first is that, in our zeal to reform mathematics classrooms we may find benefits in listening to students and what sense they are making of the changes being asked of them. Second, student opinions may be important indicators of the impact of reform-oriented actions. Finally, surveys such as this one may give some clues as to which opinions might be most difficult to change.

References

National Council of Teachers of Mathematics (in press). *Capturing the vision.* Reston, VA: Author.

National Council of Teachers of Mathematics (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Epistemology

FACT FAMILIES AS SOCIALLY CONSTRUCTED KNOWLEDGE

Betsy McNeal, University of Pennsylvania

This paper explores the process by which mathematical knowledge is socially constructed. Interactional analysis of a lesson on fact families shows how one third grade mathematics class negotiated the meanings of writing a number sentence for a picture and of a fact family. In the course of classroom interactions, teacher and students shift the lesson's focus from number sentences that represented physical images to permutations of 3 numerals around 2 operation symbols.

There is a large body of theoretical work on the social construction of knowledge as it applies to mathematics teaching and learning. Some studies focus on individuals' construction of mathematical knowledge while participating in classroom interactions, others describe the development of communal definitions of what it means to do mathematics, and still others focus on the influence of cultural symbols on knowledge development.

Bauersfeld, Krummheuer, & Voigt (1988) apply the theory of symbolic interactionism to the analysis of interactions in mathematics classrooms. They argue that the meanings of objects, words, and actions lie in the meanings that individuals attribute to them in the course of social interaction. Voigt (1992) argues that, "In classroom life the meanings of mathematical concepts and the validity of mathematical statements are socially accomplished. . . . (E)specially in introductory situations, we cannot presume that the learner would ascribe specific meanings to the topic by themselves — meanings which are compatible to the mathematical meanings the teacher wants the student to ascribe" (p. 5). As teacher and students work toward mutual understanding of a mathematical idea, they may reach what Krummheuer calls a "working interim" where both parties come to believe that they understand each other while, from the observer's perspective, they have created consistent, but not completely compatible, understandings of the topic at hand. In studying classroom interactions, the observer could therefore infer a particular individual's knowledge of, say, fact families, from observations of his/her interactions with the objects or with other individuals, and similarly, one could infer the collective knowledge of fact families that is constructed by the group through their attempts to communicate. The collective understanding that emerges may differ from that of individual participants.

Building on this work, this paper describes the dynamic process by which collective mathematical knowledge in a 3rd grade classroom community is constructed. Through analysis of one mathematics lesson, this paper further attempts to provide an example of how the students as well as the teacher influence the nature of the knowledge developed.

The objective of the lesson examined here, according to the required textbook, was "to use fact families to recall addition and subtraction facts" (Eicholz et al., 1985, p. 10). However, as teacher and students interact, the collective meaning of "fact family" and the purpose of the lesson change. As the class moves through the 4 phases of the lesson, introduction, practice activities, written seatwork, and a final challenge problem, the lesson intended to focus on relationships among facts becomes a lesson in symbol manipulation.

Analytic Technique

This lesson was selected from data collected for a larger project that provided qualitative descriptions of the interaction patterns that emerged in a 3rd grade textbook-based mathematics class (McNeal, 1991). This particular lesson seemed to be a striking illustration of the theories currently under discussion among researchers in mathematics education and educational psychology. No claims are made that this textbook lesson is typical.

Data from the larger study included field notes, video recordings and transcripts of 28 mathematics lessons over the first 8 weeks of instruction. Based on the work of Bauersfeld, Krummheuer, and Voigt (1988), individual transcripts were analyzed line by line, in chronological order, for patterns that would illuminate the mathematical meanings and communicative practices of this community. Assertions developed from each lesson were then compared with those from each of the previous lessons. Exceptions to emerging patterns were also tested against the entire body of data following analytic procedures of Erickson (1986). Interpretation of the following transcript is thus based on analysis of the entire corpus of data, rather than on the one episode alone.

The Lesson

The following mathematics lesson occurred on September 1 during the 6th class session for the year. After about 20 minutes of problem solving, the class began the textbook portion of the lesson. The actions described took 42 minutes, and were followed by afternoon recess.

Following the suggestion in the textbook, Mrs. Rose (all names are pseudonyms) used pictures of dominos to elicit from the class the definition of a fact family.

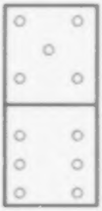


Figure 1

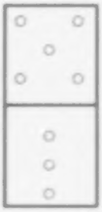


Figure 2

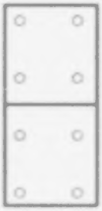


Figure 3

1 Mrs. R: Notice the domino boys and girls. [pointing to Figure 1 on the overhead projector] How many spots do you see on the top of the domino here?

2 Students: 5.

3 Mrs. R: [pointing] How many do you see on the bottom?

4 Students: 6.

5 Mrs. R: OK. I would like for someone just to give me, ah, an addition number sentence for these, for this domino right here. An addition number sentence. Who can give me one. [calls on one student whose hand is raised]

6 Student: 5 plus 6 equals 11.

7 Mrs. R: All right. [writes 5 + 6 = 11, then makes a side comment] 5 + 6 equals 11. Who can give me another addition number sentence for this? Chris? [no response] Up there. We have one number sentence, 5 + 6 equals 11, what else could we do? What else could we use? Use those numbers up there.

8 Chris: 6 plus 5?

9 Mrs. R: Wonderful. 6 plus 5 equals 11. [writes 6 + 5 = 11] Who can give me a subtraction number sentence using these dominos? Betty.

10 Betty: 6 take away 5.

11 Mrs. R: How many do we have altogether, Betty?

12 Betty: [after a short pause] 11.

13 Mrs. R: 11.

14 Betty: Take away 5.

15 Mrs. R: 11 take away 5 equals what, Betty?

16 Betty: 6.

17 Mrs. R: 6. Very good. [writes 11 - 5 = 6] Who can give me another one? Another subtraction number sentence? Karl.

18 Karl: 11 take away 6 equals 5.

19 Mrs. R: Now, look here [pointing to number sentences], boys and girls. How many, How many facts do we have there?

20 Students: 4. Oh! 4.

21 Mrs. R: 4 facts. How many numbers, Chris, did we use? How many numbers?

22 Chris: 3...2.

23 Mrs. R: How many numbers did we use?

24 Chris: 3.

25 Mrs. R: We used 3. We just made what we call a fact family.

26 Student: A [fact or fat?] family. [laughs]

27 Mrs. R: A fact family is 4 facts made out of 3 numbers. [shows Figure 2] Let's look at this domino right here. Let's see if we can think of 2 addition number sentences for it. [calls on Jennie whose hand is raised]

28 Jennie: [starts to go to the board] Um, I know one for the top one [Figure 1].

29 Mrs. R: Just, you just tell me. Just tell me. For this right here.

30 Jennie: 16 minus 5. Equals 11.

31 Mrs. R: OK. Jennie, did we have 16 . . . dots?

32 Jennie: [makes a face] No.

33 Mrs. R: [laughs] All right. Who can give me a number sentence; who can give me two addition number sentences for [Figure 2]?

Mrs. Rose focused the class on the important features of the domino (lines 1-5), and they quickly produced the first addition fact (line 6). When she called for a second, Chris seemed unsure what she meant, but made the expected interpretation, and no discussion was warranted. Mrs. Rose therefore did not realize that the domino representation might produce multiple interpretations until Betty (line 10) indicated her understanding that the task required using the 2 numbers shown in any number sentence. Although Betty's interpretation was consistent with her classmates' responses, it was not compatible with the intended task. This prompted Mrs. Rose to give the class more information, implying that students should use the *total* number of dots (line 11). As she started to move on (line 27), Jennie volunteered another fact for Figure 1, having misunderstood both the definition of the mathematical task and the social cue that the group had finished collecting facts for this domino. Her sentence included more than the number of dots shown, and suggested that she understood the task to mean: Create a sentence using numbers made from the two given. (This was confirmed later when she explained how she had come up with her numbers.)

In the remainder of the introduction, Mrs. Rose led the class through a similar sequence for Figure 2, and then used Figure 3 to illustrate the special case of a family with only two facts. She then gave individual students some practice activities. These exercises required students to make fact families for three numbers given without a picture. When students produced inappropriate number sentences, Mrs. Rose prompted them to check that they had used only the given numbers. For example, she wrote 1, 5, and 6 in a circle and called two students to the board, "Make a fact family out of these numbers. Quick as you can. (to the class) You boys and girls see if they're correct." When Nan wrote 11 + 6 = 17, Mrs. Rose stopped her as she wrote 17, "What's the number you just wrote?" She then asked, "Do you see 17 on here?" and reminded her, "Using these three numbers." Finally, only John and Annie were still working: They had found three facts for 2, 6, and 8, but were struggling to find the fourth. Mrs. Rose wanted to move on so the class would have sufficient time to complete their written assignment, so she came to assist them.

> Mrs. R: You've got 8 - 2 is 6 then you have 6 + 2 = 8, 2 + 6 = 8, what do we still need? We've got 2 pluses, we've got one minus, what do we still need? Do we need another plus or do we need a minus? We've already taken away 2, now what are we gonna take away? Good. Very good.

Mrs. Rose then quickly reviewed the instructions for each problem on the assigned textbook pages. For example, the first problem was: 8 + 5 = __, 5 + 8 = __, 13 - 5 = __, and 13 - 8 = __. When Mrs. Rose called on Mike for 5 + 8 = __, he replied, "5 + 8 equals . . . 15?" Pointing to something in his book, Mrs. Rose reminded him, "We're only gonna use these three numbers, 13, 8, and 5." His response was inaudible to the observer, but Mrs. Rose went on, "These are the 3 numbers we're using for the fact family. We just said 8 + 5 is 13, now what's 5 + 8?"

Mrs. Rose asked the students to copy the number of the problem and then write answers only. When they began to work, most of the students spoke with Mrs. Rose about the format or instructions for their written work. When she and the students talked about fact families, she assisted them by saying, "Only use these three numbers," or "Remember a family has four facts." In one case, the following exchange occurred.

> Mrs. R: See these 3 numbers you're gonna use? OK, you got 8 + 5, right? There's 8, there's a 5, what number do we need now? [response is inaudible] 13. Put 13 right there. Now look at these numbers, there's 5, there's 8, what other number do we need? [no response] There's the 5, there's the 8. 13. Put 13 right under there. OK, now there's the13, there's the 5, now what number do we need here? [response is inaudible] Very good.

Mrs. Rose then put a "challenge problem" on the blackboard that she had taken from the book for the students to try when they had completed their work: "Use the 'Addition on Venus' symbols shown to write four fact-family number sentences: $\Delta + \Sigma = \Omega$" (Eicholz et al., 1985, p. 10). Mrs. Rose copied only the three symbols, Δ, Σ, Ω, and asked the students to create a fact family for these figures. The task therefore looked like the triples of numbers presented during the practice activities. Only one student, John, challenged the teacher's task saying, "Those aren't numbers. You can't make a fact family." No other students joined his protest, showing that his interpretation of the task differed from theirs.

Discussion

This lesson illustrates the social construction of knowledge in two ways. First, the group negotiated what it meant to write a number sentence for a picture. Seeing the domino according to the conventions of school mathematics involved learning to see only what was in the picture ($16 - 11 = 5$ was inappropriate for Figure 1), and learning that the entire quantity shown must be maintained ($6 - 5$ was also inappropriate). Assuming this interpretation to be self-evident put students in the position of guessing what the teacher had in mind. This helped reproduce the elicitation pattern (Voigt, 1985) seen here.

Second, as the lesson about related facts referred to as fact families proceeded, the purpose and meaning of these changed. In the introduction, Mrs. Rose intended building on the students' contributions, using the domino as a concrete representation of the relationships that she believed a fact family described. The students, however, translated the number of spots on each half of the domino into the numbers to be used in composing arithmetic facts. For them, this first part of the lesson was not about related facts, as they did not know the term "fact family" until they were done. It was about figuring out what the task was. Their unexpected responses and her desire to avoid stating the definition in turn obligated Mrs. Rose to point to features of the picture that implied the meaning she had in mind. Although several students offered number sentences that were deemed inappropriate ($6 - 5$, $16 - 5$, $3 - 8$ for Figure 2, and $4 -4 = 0$ for Figure 3)) as they tentatively tried to make sense of her expectations, at no time did they challenge her constraints or ask for explicit clarification of the task. At this stage of the lesson, there were at least three different under-standings of the representation and hence of the task, but the collective understanding was that the lesson was about creating number sentences from given numbers.

During the practice activities, students continued to test their understanding of the task and of fact families against the teacher's. Having seen several examples completed by this time, they became less tentative as they received immediate feedback. The exercises that presented three numerals without a picture caused Mrs. Rose's response to inappropriate number sentences to shift from references to a picture ("How many do we have altogether?") to references to a list of numbers ("Is that one of the 3 numbers?"). The collective understanding of the lesson also shifted: Fact families were permutations of three numbers around the + and - symbols. When the students' confusion persisted in a lesson that seemed simple to her, and none of her previous forms of assistance were sufficient, Mrs. Rose suggested that John and Annie check the number of addition and subtraction facts they had.

During seatwork, instructions to provide only answers further separated the definition of a fact family from relationships among the facts. This was compounded by the exchange, audible to the whole class, in which Mrs. Rose effectively shifted Mike's attention from computing $8 + 5 = \_\_$, to filling in the bllank by process of elimination.

Despite John's protest that creating fact families from symbols with no conventions for relating them was unreasonable, no one, including the teacher, recognized the validity of his claim. This was the final phase in the evolution of the meaning of fact families from a set of useful relationships to a set of permutations of three numerals around two operation signs.

References

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), *Foundations and methodology of the discipline of mathematics education* (pp. 174-188). Antwerp: Proceedings of TME Conference.

Eicholz, R. E., O'Daffer, P. G., Fleenor, C. R., Charles, R. I., Young, S., & Barnett, C. S. (1985) *Addison-Wesley mathematics, Teacher's edition, Book 3*. Menlo Park, CA: Addison-Wesley.

Erickson, Frederick. (1986). Qualitative methods in research on teaching. In Merlin C. Wittrock (Ed.), *Handbook of Research on Teaching* (3rd ed.), (pp. 119-161). New York: Macmillan.

McNeal, B. (1991). *The social context of mathematical development*. Unpublished doctoral dissertation, Purdue University, W. Lafayette, IN.

Voigt, J. (1992). *Negotiation of mathematical meaning in classroom processes*. Paper presented at the Seventh (Quadrennial) International Congress of Mathematics Education, Quebec.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathematiques*, *6*(1), 69-118.

A THEORY OF SECOND-GENERATION CONSTRUCTIONS

Terry M. Price, Washington State University
Verna M. Adams, Washington State University

When a student begins to appropriate an idea from the classroom discourse, the idea is likely to be perceived incompletely because the speaker's understanding of the idea cannot be conveyed in its entirety through the discourse. Under the guidance of the teacher, the discourse serves to stimulate further development of the idea itself, the development of connections to existing knowledge, and its use in constructing new content. The way in which students appropriate ideas presented to them by another individual and make them their own are what we call *second-generation constructions*. In this study, all students, including the student who presented the idea to the class because he acquired the divisibility by 8 rule from his father, were creating second generation constructions. The theory we propose is a substantive theory (Glaser & Strauss, 1967; Glaser, 1978), not a formal theory.

In this study, we attempt to describe students' constructions of mathematics in a seventh-grade mathematics class as they talk about dividing by 8. The focus of the investigation is not students' constructions in a teaching experiment. Instead, we focus on the "everyday activity" (Lave, 1988) in the practice of doing mathematics in a class taught by the "regular" teacher. We look at the ways in which students appropriate an idea presented to them by another individual and make it their own. Because context is integral to the cognitive events involved in constructions (Rogoff, 1984), the phenomenon is likely to have important characteristics related to the context.

The data discussed in this paper was collected during a unit on number theory and is part of a larger study focused on the relationship between classroom discourse and problem solving. Because the idea was presented by a student but did not originate with him, the understandings and connections he developed are what we call *second-generation constructions*; that is, second-generation constructions occur when a student appropriates an idea from the discourse and constructs connections to her/his existing knowledge base. The student who presented the rule in this study acquired it from his father.

A student idea is not essential to a theory of second-generation constructions and we do not claim to present a full-blown theory. We focused on a single case that occurred naturally in the classroom as a result of the teacher's decision to promote discussions of student thinking and justification. Although the idea of second-generation constructions emerged from our data (that is, it described the development of a student idea), the idea could have been introduced by the teacher. The importance of student ideas is emphasized by the National Council of Teachers of Mathematics (1991) in their description of the teacher's role in discourse. We believe an advantage to following the development of a student idea is increased student ownership of the content.

Preparation of this paper was supported in part by National Science Foundation, Grant No RED-9254922. Any opinions, conclusions, or recommendations are those of the authors and do not necessarily reflect the views of the National Science Foundation.

The Study

Data collection consisted of a combination of classroom observations (video tapes, audio tapes and field notes), whole-class surveys, and interviews. We draw on the grounded theory method of Glaser and Strauss (1967) and Glaser (1978) in which, after identifying a phenomena of interest from the data, additional data is collected and coded. Coding of data began as field notes were taken and continued through the analysis. Categories began to emerge during the analysis through incident-to-incident comparisons. The analysis progressed to comparisons of incident to properties of a category. As new categories emerged, subsequent data collection was influenced by the results of the previous data. Analysis continued after data collection was completed to further develop the properties of the categories. Finally, data from all sources was coordinated and sorted into groups based on students' responses to the final survey.

A Description of Blayne's Participation

In this section, we describe Blayne, the student who gave the rule to the class. The following timeline shows the amounts of time between data-collection points. Daily observations of the classroom began before the start of this study and continued beyond the time frame of interest to this study.

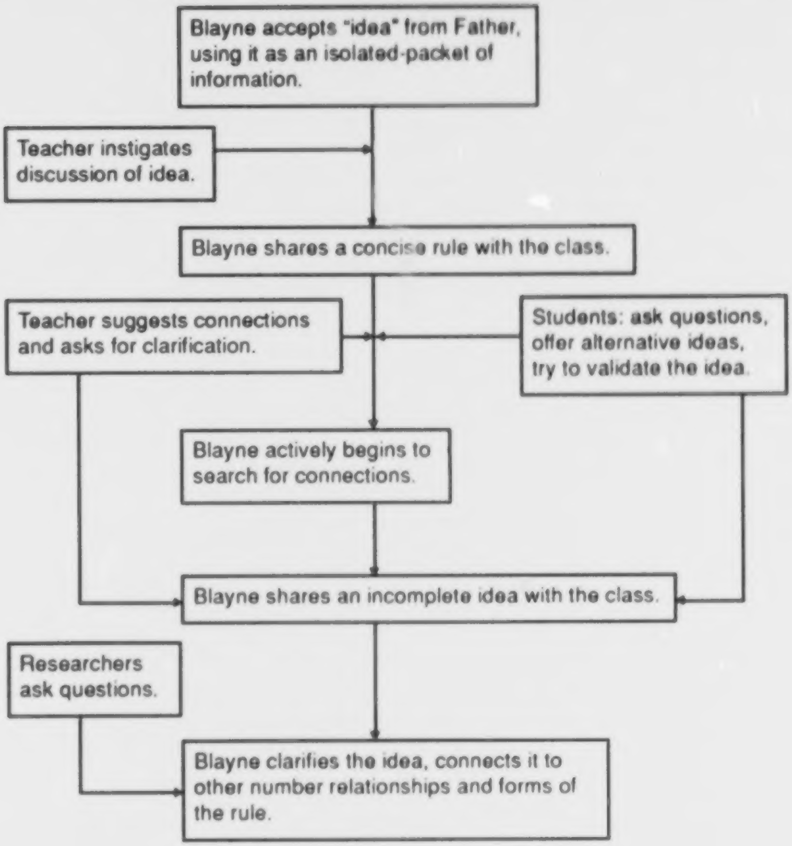
Data Collection Points and Timeline

Initial Presentation	Presentation	Presentation	Initial Class Survey	Blayne's Interview	Second Survey
Sept. 21	Sept. 23	Sept. 26	Sept. 29	Oct. 6	Oct. 25

——> 2 days ——> 1 day ——> 3 days ——> 7 days ——> 19 days

In the interview Blayne explained that when his father told him the rule the year before, he had been trying to determine what numbers would divide into other numbers. His father gave him a little "trick" for 8 where you divide by 2 three times. Blayne was motivated to remember and use the rule. When Blayne shared the rule with the class on September 21, it followed a class discussion of other divisibility rules. In spite of Blayne's familiarity with the "eight rule", his initial statement was garbled: "You divide by 2 six or eight times." Blayne's responses to questions from the discourse suggested that his understanding of the rule was not connected to other knowledge. He had initially treated his father's idea as an isolated-packet of information to be called upon when working divisibility problems. The following diagram shows the structure of what happened.

At each data-collection point, Blayne gave a more concise statement of the rule and what it meant, but not without glitches. On the second day, for example, he began to show on the overhead how the rule applied to the number 56. Before he completed the example, he shifted from a written and oral form of communica-


Blayne accepts "idea" from Father, using it as an isolated-packet of information.
Teacher instigates discussion of idea.
Blayne shares a concise rule with the class.
Teacher suggests connections and asks for clarification.
Students: ask questions, offer alternative ideas, try to validate the idea.
Blayne actively begins to search for connections.
Blayne shares an incomplete idea with the class.
Researchers ask questions.
Blayne clarifies the idea, connects it to other number relationships and forms of the rule.

tion to an oral form only and, at the same time, switched the number to 16. Initially, Blayne indicated that the results of the successive divisions should be "even". By the interview he talked about the division process as not having either a "remainder" or a "decimal" result and he stated that a decimal result after the second division meant that eight would not go into the number. We were surprised that on the initial survey Blayne indicated that the eight rule would not always work, stating his justification: " I think that because in class we tried it and the number that didn't work was 12345678." In the interview, he revealed that he had made an error on the survey. His confusion stemmed from the fact that while in class, he thought that the two- rule gave a whole number answer, but dividing by eight did not. Later he used a calculator and obtained a decimal remainder for both calculations.

On September 23, Blayne clearly did not know that dividing by 2 three times could be related to 2^3. By the time that he responded to the second survey, however, he had made the connection, not only to the different symbolic representations of 8, but also to different forms of the rule that included: (a) If a number has 3 factors of 2, then it is divisible by 8. (b) If a number is divisible by 8, then you can divide it by 2, divide the answer by 2, and divide that answer by 2 again, getting a whole number answer each time you divide. If any of the answers is not a whole number, then the original number is not divisible by 8. (c) If 2^3 is a factor of a number, then the number is divisible by 8. We believe that the discourse with the teacher, other students, and researchers sustained his attention and oriented it toward number relationships and different forms of the rule.

Blayne's responses on the second survey indicated that he had formed connections to other mathematical knowledge and the rule was no longer just a trick. He had not, however, generalized the rule to division by powers of other numbers. When he was asked to determine if 675 was a multiple of 27 without dividing by 27, he summed the digits on both numbers and divided the results. He, however, was not alone; only three students attempted to generalize the rule to powers of three.

Toward a Theory of Second-Generation Constructions

From our observations of Blayne and other students, we begin to formulate a description of the characteristics of the influence on student constructions (e.g., acceptance-nonacceptance) and the characteristics of the construction process (e.g., connections, type of justification they use) that result from the events in the classroom.

Acceptance/Nonacceptance of the Rule

This characteristic of the influence on student constructions was evident in the discourse by the questions and comments of the students. For the following discussion, acceptance/nonacceptance was determined by the connections that students made to other forms of the rule and to their spontaneous use of the rule on the surveys. That information was then coordinated with the information from the videos and field notes about student participation in class discourse. The students generally fell into the categories of either accepting, exploring, or resistant to the rule, containing 9, 16, and 2 students, respectively.

Acceptance: The 9 "accepting" students revealed their acceptance through spontaneous use of the rule and understandings of other forms of the rule. All of them spontaneously used the rule as justification on the second survey for the question: Suppose you divide a number by 2 and get an even number. Then you divide the answer by 2 and get an odd number. Is the original number a multiple of 8? On the first survey, 5 of these students spontaneously divided by 2 three times when they were asked: Is 2000 a multiple of 8? Furthermore, these students had made connections to other forms of the rule and, for the most part, believed that the rule would always work. They relied heavily on ***example-based justification***; that is, they used a larger number of examples than other students to convince themselves the rule worked. Their classroom participation was minimal and their construction process was silent. Their thinking was not evident in the classroom discourse. Generally, when they did offer ideas to the class, the ideas consisted of examples of numbers not divisible by 8 and language clarification.

Exploration: This group of students was less accepting of the rule than the first group. About half of this group spontaneously used the rule in a calculation, but no one spontaneously used the rule as justification. Only 6 out of the 16 students in this group made connections to other forms of the rule. In general, these students were undecided with respect to the rule, but were more inquisitive than other students. They were more actively engaged in the dialogue, offering interpretations of the results of the discussions, exploratory conclusions about the workability of the rule and alternate revisions of the idea. More than half of the students who participated in the classroom discussion fell into this group and were clearly actively trying to construct an understanding of the rule. This group did not make up their minds about the rule as quickly as the other two groups and gave a mixed pattern of responses on the surveys. "Failures of context" (Edwards & Mercer, 1987) in the discourse affected these students more than others.

Resistance: Neither student in this group spontaneously used the rule on the surveys or made connections to other forms of the rule. They thought the rule was time consuming and inefficient. One stated: "I don't understand why you go through the trouble. Why don't you just divide by 8 to begin with?" He had what might be considered a healthy skepticism about proof via examples, stating that he did not believe the rule would always work because "Someone will prove him wrong somehow." The other student considered the rule to be "undependable." This group maintained their resistance to construction, in spite of social interaction, because they valued efficiency. We believe that they could be persuaded to pursue an active construction if given acceptable justification.

Perception of Value

Perception of value had two properties: value attached to the rule and value attached to people. Unlike other students in the class, the two who were resistant to dealing with the rule did not place any value on its use. Considerable value was given to Blayne, and his confidence in his own abilities was affected in a positive manner. The teacher created a positive climate where Blayne felt comfortable expressing his idea. He was perceived by the teacher and some of his peers as having a higher level of understanding than was actual fact, and some students began to perceive themselves as less competent than Blayne. Because of the teacher's perception, Blayne was allowed more "floor time" than other students for the exploration of their ideas. This floor time was significant because he benefited more from the discussions than other students. His idea was given value and, during the investigation, was referred to as a "theory". That language implied that it had importance, perhaps more importance than others.

Discussion

Much of what we want to say is left for further development in a longer paper. In the natural environment of the classroom, the ways in which students appropriate mathematical knowledge from the discourse is a nontrivial process. In general, students acquire mathematical ideas introduced to them by someone else (e.g., the teacher, peers, parents). This appropriation requires precious classroom time and special attention to the discourse. In this study, time allotted to discussion sustained the interest in the idea. In addition, the research itself influenced the perception of value. The role of the perception of value should not be taken lightly. Blayne had a full year to develop ideas related to his rule, which he did not do without the sustained interest of the teacher and other students, their questions and their comments. An implication for planning instruction is that ideas should be revisited over time and the discourse is an important component of the construction process.

References

Edwards, D., & Mercer, N. (1987). *Common knowledge: The development of understanding in the classroom.* New York: Routledge.

Glaser, B. G. (1978). *Theoretical sensitivity.* Mill Valley, CA: Sociology Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine De Gruyter.

Lave, J. (1988). *Cognition in practice.* New York: Cambridge University Press.

National Council of Teachers of Mathematics (1991). *Professional Standards for Teaching Mathematics.* Reston, VA: Author.

Rogoff, B. (1984). Introduction: Thinking and learning in social context. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 1-8). Cambridge, MA: Harvard University Press.

Functions and Graphs

GRAPH, EQUATION AND UNIQUE CORRESPONDENCE: THREE MODELS OF STUDENTS' THINKING ABOUT FUNCTIONS IN A TECHNOLOGY-ENHANCED PRECALCULUS CLASS.

Armando M. Martínez-Cruz, National University of Mexico

Our area of research is aimed on developing a conceptual knowledge of functions in technology-enhanced classes. In this paper we report on the first stage of our research, documenting how students who use graphing technology think about functions. In this paper we report three models of thinking about functions (graph, equation and unique correspondence) that we found among eight high school students in a precalculus class enhanced with graphing calculators. Models emerged from their function images observed during a period of nine months.

Current efforts to reform mathematics education advocate the use of technology at all levels. In these efforts, an area of inquiry that has attracted the attention of mathematics educators is the teaching and learning of functions through technology. In general, it is expected that computers, and more recently graphing calculators are the kind of media that might help students to visualize appropriate representations of functions (Goldberg, 1987). Hence, it is conjectured that graphing capabilities of computer technology might have a positive impact on the teaching and learning of functions. These claims are supported by Dunham and Dick's review of early reports on graphing calculators (1994). Our area of research seeks to contribute to a better understanding of how students who use graphing calculators think about functions. Three models of students' thinking about mathematical functions in a technology-enhanced precalculus class are presented here, a brief discussion of the relationships between them.

Theoretical framework

The theoretical framework developed for the research incorporated historical (cf Kleiner, 1989) and psychological contributions (processes and objects) (Sfard, 1989) to the development of functions; concept images and concept definitions (Tall, 1989); and multiple representations. We accept a constructivist view on mathematical knowledge.

The study and its methodology

Data reported in this paper belong to a larger project aimed to contribute to the teaching and learning of mathematical functions through technology. This paper involves data collected during nine months in the scholastic year 1991-1992. This initial part of the study investigated students' knowledge and development of functions in a technology-enhanced precalculus class. Students in the Calculator and Computer Precalculus Project (C^2PC, Demana & Waits, 1988) use graphing technology as an integral part of their class. Eight students from a class participating in the C^2PC were selected for case studies of their knowledge and development of functions. In particular, we investigated "*What are the concept images and the* *concept definition of function that students in this technology-enhanced precalculus class have?"* We relied on the interpretivist tradition of ethnographic research for it provides methodologies for studying the evolution of change in mathematics teaching and learning. Collection of data for each each case study involved a practice test on functions (Markovits, Eylon, and Bruckheimer, 1988) at the beginning of the study, five interviews, daily classroom observations, researcher's journal, testing materials used in class, and a student handout for extra credit.. Consideration of criteria related to the trustworthiness of the study (credibility, transferability, dependability, and confirmability) were taken into consideration as well (Lincoln & Guba, 1985).

Procedures

We discuss here only about students' protocols, since they provided the most useful information on sketching students' thinking about functions. Five protocols for interviewing students were selected or developed in the course of the study. Items were suggested by the cascading design of the study to investigate working hypothesis. Pertinent literature on functions was consulted to design the protocols (Dreyfus & Vinner, 1989; Even, 1989; Ferrini-Mundy & Graham, 1991; NCTM, 1989; Tall & Vinner, 1981). Items asked the students about the relationship between equations and functions, about the relationship between graphs and functions, to decide if some given graph was a function, to decide about the existance of a function with given algebraic features, or to provide examples of functions Items involved discrete and continuous sets and piecewise functions.

Discussion of findings

A domain analysis (Spradley, 1979) and a coding paradigm (Lincoln & Guba, 1985) was used to analyze the interviews and testing materials. Such analysis identified nine function images that students in the study associated with the concept of function (Martínez-Cruz, 1993). Resulting images were used to build a network of the concept. Links and emphases on the network (see fig. 1, 2, 3) suggested categories (graph, equation and univalence) in students' thinking. We present the categories as models of students' thinking about functions.

The models

Each model is made up of the images that emerged from all the students, however, not all images were detected on each student. Hence, these models do not state that a student can be identified as thinking about functions as one single model. On the contrary, the facts that the concept image may be incoherent, contain conflictive parts with the concept image itself or with the concept definition, or contain potential seeds for future conflict even in the learning of a formal theory (Tall & Vinner, 1981) are evident here.

Graph

The graph model refers to the graphical representation of functions. Students associated several ideas to this representation.

1) Functions can be represented by graphs.
2) Graphs can be functions (if they pass the vertical line test or the univalence criterion).
3) A graph is an intermediate step to decide whether or not an equation is a function,
4) Functions are graphs.
5) All functions can be graphed.
6) Graphs are functions.
7) Graphs come from equations.

Students' networks of functions images allowed us to identify connections and missing links among their images. One of the students, Tyler, showed a strong tendency to have a graphical representation to deal with functions (Fig. 1) (although he could talk about the equation representation or the unique correspondence criterion). This significant difference with other participants (see figures 2 and 3) is reflected on his network with a thicker line around his graph image and with an unconnected network. A second difference is how anchored his familiarity image was. He recognized a function when he has seen or graphed a similar or identical graph. Otherwise, he would reject a function based on his experience. The networks suggests also a use of the vertical line test (but not as an equivalent statement to the unique correspondence criterion).

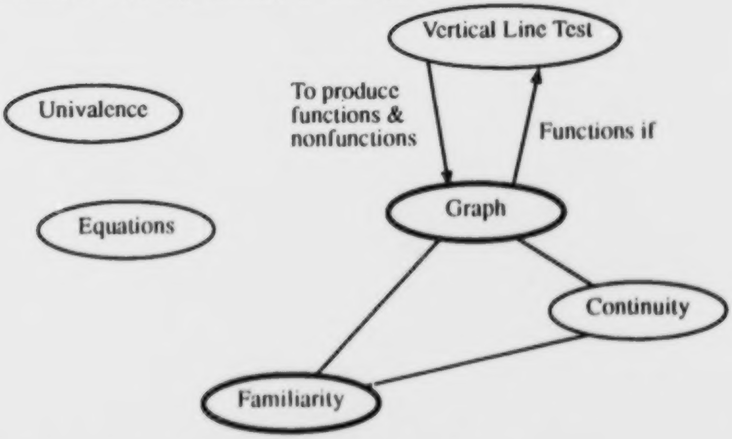


Figure 1. A student with a graph model of functions.

Equation

The equation model refers to the symbolic (algebraic) representation of functions. It appeared as a "chain" (formula) of variables and numbers. Students associated six ideas with this model.

1) A relationship between x and y.
2) Functions come from equations.
3) A means to represent functions.
4) Functions are equations.
5) Not all functions are equations.
6) Not all equations are functions.

Sara's network (Fig. 2) is a representative of an equation thinking. She relied more on an algebraic representation than on other images to deal with functions (as represented with a thicker line). A connected network is a main difference with Tyler's network and which suggests a progress on her thinking about functions. Six students showed similar networks (except for the existence of the regularity image or for their consistency on recognizing the equivalence between the unique correspondence criterion and the vertical line test). Such consistency plus a reliance on the unique correspondence criterion is a characteristic of the unique correspondence model.

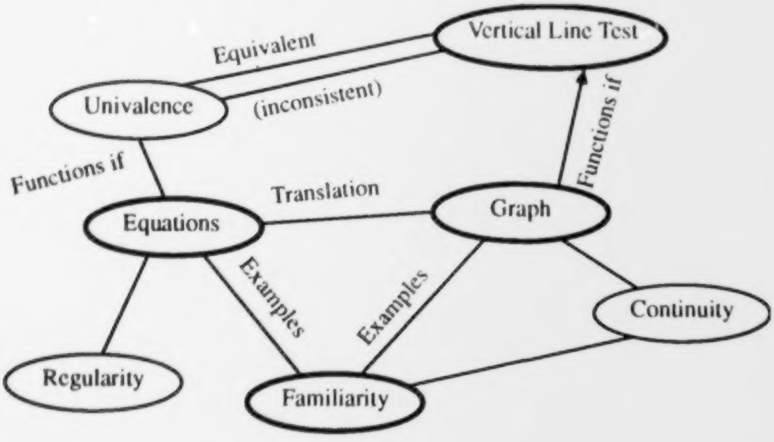

Vertical Line Test
Equivalent
(inconsistent)
Univalence
Functions if
Functions if
Translation
Equations
Graph
Examples
Examples
Continuity
Regularity
Familiarity

Figure 2. A student with an equation thinking of functions.

Unique correspondence

The unique correspondence model refers to the formal definition for a function introduced in this class (and at times stated as "one output for every input"). Students attached four images to this model.

1) A property of functions.
2) An implicit equivalence to the vertical line test.
3) A definition of a function.
4) A means to decide if equations or graphs (continuous or discrete) are functions.

Figure 3 shows the network of the single student who relied more strongly on the unique correspondence model than on any other model. This network also shows consistency on recognizing the unique correspondence criterion and the vertical line test (notice the thickness of both boundaries).

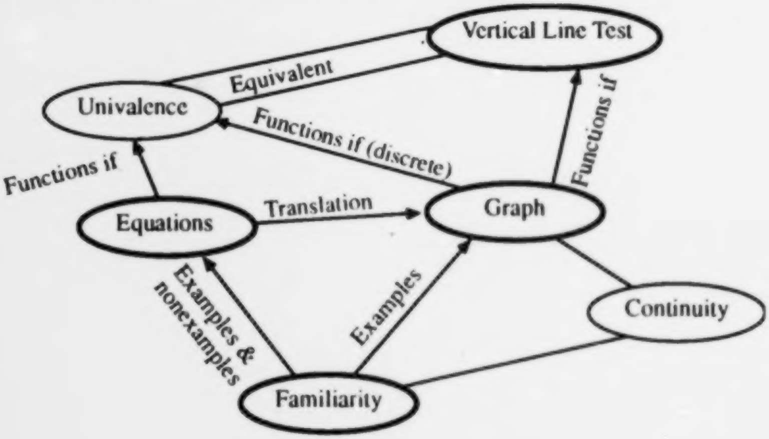

Vertical Line Test
Equivalent
Univalence
Functions if (discrete)
Functions if
Functions if
Translation
Equations
Graph
Examples & nonexamples
Examples
Continuity
Familiarity

Figure 3. A student with a unique correspondence thinking of functions.

A difference between the equation and the unique correspondence models is recognizing the vertical line test and the unique correspondence as equivalent and using this recognition consistently to apply the appropriate one in a given task.

Links between the models

The vertical line test is one of the links (among others) that differentiates the networks. Students recognized the vertical line test as:

1) A means to decide whether or not a graph is a function.
2) A means to decide whether or not an equation is a function.

3) A property (condition) of functions.
4) An equivalent statement to the univalence criterion.
5) A means to produce graphs of functions or non-functions.

A second difference among the networks is the link given by translating from a given representation (algebraic usually) to another representation (graphic) to recognize functions.

Implications

Although a student may have images belonging to all three models as presented, it is noted that our data suggest that for some students one single model was more anchored in their mind than others, and they acted accordingly. Hence, they could not cope with some of the tasks presented during the interviews. Our second part of the research deals with interpreting results in the classroom. In this case, we apply the findings to teach explicitly "the knowledge and procedures of each succeeding stage of development" (Carpenter & Fennema, p. 5).

References

Carpenter, T., & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. Carpenter, & S. Lamon (eds.). Integrating research on teaching and learning mathematics. Albany, NY: SUNY.

Demana, F. & Waits, B. (1988). The Ohio State University Calculator and Computer Precalculus project: The mathematics of tomorrow today! *The AMATYC Review, 10(1)*, 46-55.

Dreyfus. T., & Vinner, S. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, 356-366.

Dunham, P. & Dick, T. (1994). Research on graphing calculators. *The Mathematics Teacher 87*, 440-445.

Even, R. (1989). *Prospective secondary mathematics teachers' knowledge and understanding about mathematical functions*. Unpublished doctoral dissertation. Michigan State University. Lansing, MI.

Ferrini-Mundy, J. & Graham, K. (1991). *Research in calculus learning: understanding of limits, derivatives, and integrals*. Paper presented at the Joint Mathematics Meeting, Special Session on Research in Undergraduate Mathematics Education, January, 1991. San Francisco, CA.

Goldberg, E. P. (1987). Believing is seeing: how preconceptions influence the perception of graphs. *Proceedings of the International Conference on the Psychology of Mathematics Education, XI*. pp. 197-203. Montreal, Canada.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Newbury Park, CA: Sage Publications.

Kleiner, E. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal, 20(4)*, 282-300.

Markovits, Z., Eylon, B. S., & Bruckheimer, M. (1988). Difficulties students have with the function concept. In A. F. Coxford and P. Shulte (Eds.) The ideas of algebra. 1988 Yearbook. pp. 43-60. Reston, VA: NCTM.

Martínez-Cruz, A. (1993). *Knowledge and development of functions in a technology-enhanced high school precalculus class: a case study.* Unpublished doctoral dissertation. The Ohio State University. Columbus, Ohio.

NCTM. (1989). *Curriculum and evaluation standards in school mathematics.* Reston, VA: The Author.

Sfard, A. (1989). Transition from operational to structural conception: the notion of function revisited. *Proceedings of the Psychology of Mathematics Education*, v. 3, pp. 151-158. Paris, France.

Spradley, J. P. (1979). *The ethnographic interview.* New York: Holt, Rinehart and Winston.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Tall, D. (1989). Different cognitive obstacles in a technological paradigm or A reaction to: "Cognitive obstacles encountered in the learning of algebra". In S. Wagner & C. Kieran (eds.) *Research issues in the learning and teaching of algebra. Research agenda for mathematics education*, v. 4. pp. 87-92. Reston, VA: Lawrence Erlbaum Associates and NCTM.

A GROWTH-ORIENTED ROUTE TO THE REIFICATION OF FUNCTION

David Slavit, Washington State University

This paper presents an alternate perspective for utilizing the action/process/object framework when discussing student conceptions of function. This perspective extends previously used frameworks by incorporating student conceptions that deal with functional properties and situational contexts, but differs in the analysis of the nature of the functional properties that are considered. The hypothesis is that students not only use notions of set-based correspondences and transformations to think of functions as mathematical objects, but functions are also conceived as mathematical objects that possess (or fail to possess) growth properties of specific functions. These properties include symmetry, linearity, continuity, etc. It is argued that all students who develop a structural view utilize functional properties, but a focus on different kinds of properties produce different conceptions of function. Empirical evidence is used to support the theory.

Theories Of Student Conceptions Of Function

> I am astonished when I see some of the analyses which purport to be the scientific foundations on which school curriculums are to be built and find no mention of these general ideas of order and arrangement and precision. I am told that the school should teach children how to make change and how to measure wall paper and how to tell time and that sections of arithmetic should be devoted to these specific tasks, but I look in vain for any appreciation of the fact that the school ought to lead pupils who have only a hazy and unsystematic notion of the world to see the value of arrangement and order in all thinking and to cultivate the general ideas of regularity and precision.
>
> Charles Judd, 1928

Operational Views of Function

The action/process/object theory of conceptual development has received considerable attention in recent studies of functional understanding. Evidence exists that students initially acquire an *action* or *operational* view of function (Briedenbach et al., 1992; Sfard, 1989). This involves function as an operation that exists *dynamically*. An action view involves understandings pertaining to the computational aspects associated with functions, such as an arithmetic process or a "function machine." It is important to note that an action view, by definition, is concerned only with local functional properties. It is only when one can see relationships between sets of inputs and outputs that a more structural view of function can occur. But making relationships between sets of input-output ordered pairs is not an easy task (Sfard and Linchevski, 1994). The problem is compounded when functional symbolisms are confronted in very different forms (such as graphs and equations) and when functional ideas are confronted through purely mathematical symbols that lack a contextual basis (Filloy and Rojano, 1985). For example, when a student is asked to factor a difference of squares, what kinds of functional understandings are being built? What structural understandings of function can be acquired when a student solves a linear equation of the form ax + b = 0? In the latter example, if the student is allowed an opportunity to reflect on the solution, what can be acquired is a relationship between the x- and y-values of the x-intercept of the function. While reflections on this relationship support the action view of function, they can do little to promote more structural views of the concept unless this relationship is seen as a specific, local property that the entire function possesses. Otherwise, intercepts can become local phenomena of a function or set of functions, providing very specific information about one aspect of growth behavior or situational context.

Structural Views of Function

I will define a structural view of function as an extension of the action view. This will first be done by reexamining existing theories, especially those of Sfard and Dubinsky. An alternate theory for a structural view of function will then be presented that, while borrowing heavily from previous ideas, discusses student development in a different context. Sfard (1991, Sfard and Linchevski, 1994) defines a structural conception as the reification of an operational view. For example, considering the expression 3(x+5) + 1 as a certain number rather than a computational process illustrates the beginnings of a structural view since the expression is considered to be a fixed value of an unknown. Generalizing this conception to involve the notion of variable, where the above expression simultaneously denotes several processes, represents thinking in line with functional algebra and is considered to be a structural conception.

My use of the word structural incorporates this perspective, but also includes a different domain of student thinking as well as a different context for the use of the phrase "functional property." This different context deals less with direct relationships between input-output pairs and more with specific growth properties that functions can possess or fail to possess. As will be illustrated, the connections between the two different perspectives lie in the fact that both involve understandings of functional properties. Further, some functional properties (e.g., intercepts) are tied to individual input-output pairs, while other functional properties (e.g., monotonicity, symmetry) are tied to more global views of function that relate to numerous input-output pairs. Other properties are harder to classify (e.g., continuity). Hence, the two structural views outlined below are not meant as contrasting or opposing viewpoints, but rather complementary ways of thinking about the concept of function as a mathematical object.

Relational conception of function. Sfard (1989; Sfard and Linchevski, 1994) and Dubinsky (Briedenbach et. al., 1992) describe routes through which students may *reify* the function concept (Dubinsky uses the term *encapsulate* to denote the case when students develop object-oriented conceptions of function). I will define *relational conceptions* as those which deal with relationships between sets of inputs and outputs, conceptions in line with Sfard's previously discussed use of the term *structural*. Briedenbach et al. (1992) provide evidence to suggest that instruction which utilizes computer programming can promote relational conceptions. They define a *process conception* of function as a complete understanding of a given transformational activity which can be performed on a function. Students are more able to comprehend notions such as 1-1, onto, and invertibility once a process conception is achieved. They state that the process conception is a precursor to an object-oriented view of function. Several people, including Confrey and Smith (1995), argue that most textbooks and curricula support the development of function as a correspondence relation rather than as a covariance relation.

Growth conception of function. A second view of function beyond an action conception involves an understanding of functional properties specific to different kinds of functions. In previous writings I have referred to this conception as a *growth-oriented view*, but I have changed the terminology to highlight the connections between the two object-oriented conceptions that are described here. Briefly, a growth conception of function deals with the gradual awareness of specific functional growth properties of a local and global nature, followed by the ability to recognize and analyze functions by identifying the presence or absence of these growth behaviors. Global properties include symmetry and periodicity, and local properties include intercepts and points of inflection. There is evidence (Slavit, 1994) that the multi-representational capabilities of graphing technologies can lead students to develop a growth-oriented approach to the reification of function. Through an intensive look at elementary functions, students may understand function to be a related set of procedures and functional properties in a variety of functional representations. In essence, the procedures performed on functions give rise to an understanding of functional properties, other than those more familiar with the relational view (e.g., invertibility). These properties can be specific to a function class (such as linear slope) or generalize to several function classes (such as symmetry). A student can then conceive of function as an object possessing or not possessing these properties (Slavit, 1994). Particular contexts elicited by the semantic domain of the functional situation can also enrich the meanings in the conceived functional properties.

Prior work in this area (Slavit, 1994; Ruthven, 1990) has led me to hypothesize that the growth-oriented view develops in three stages. First, the growth-oriented view involves an ability to realize the equivalence of procedures which exist in different representations. Noting that the processes of symbolically solving $f(x) = 0$ and graphically finding x-intercepts are equivalent (in the sense of finding zeroes) demonstrates this awareness. Second, students develop the ability to generalize procedures across different classes and types of functions, particularly in algebra courses which introduce the elementary polynomials in a traditional sequence. Students at this second stage can translate procedures across representations (Stage 1), but are also beginning to realize that some of these procedures have analogues in other types of functions (such as finding x-intercepts of linear and quadratic polynomials). Third, once a student has become familiar with various functional properties, he or she can "see" a function as an object either with or without these properties. For example, a quadratic function could be viewed as a continuous function with exactly one extrema, at most two zeroes, and which is symmetric about a vertical line (with, of course, second-degree growth). Just as a relational view is convenient when dealing with set-based operations on functions (such as composition), a growth-oriented conception helps to relate specific examples of functions to their corresponding growth behaviors.

The development of a growth-oriented conception of function will take time since it is dependent on knowledge of several different functional properties, representations, and classes of functions. If linear and quadratic functions are studied almost exclusively, as is the case in many high school Algebra I courses, then a student's "library" of functional properties will be quite small. To this student, functions are certainly well-behaved and continuous, have very simple growth behaviors, are either monotonic or change the direction of growth once, and are zero at most twice. When other polynomial (usually up to fourth- or fifth-degree), exponential and logarithmic, trigonometric, radical, and absolute value functions are added, as is the case in many high school Algebra II courses, the student's library of functional properties will increase, and a growth-oriented conception of function could deepen. By seeing examples of different types of functions that share functional properties (linear and simple radical functions sharing monotonicity, quadratic and simple absolute value functions sharing a unique extrema, quadratic and some trigonometric functions sharing symmetry), and by noting similarities and differences in the functions, the student can relate specific properties to a variety of exemplars. But not all radical functions are monotonic, nor are all trigonometric functions symmetric. Further delineations relative to specific function classes continually need to be made in order to classify functions in regard to functional properties, but this is an exercise that could only strengthen a growth-oriented view. This situation suggests that the question of the sufficiency of the kinds of functions that are currently most often studied in today's high schools be reexamined in regard to allowing students access to a broader array of functional properties. An obvious question concerns the adequacy of the functional properties normally encountered in regard to a student's development of the growth-oriented view. Perhaps other types of functions should be introduced to allow for a broader range of functional properties to be introduced. These could include discontinuities, finite domains, multivariable functions, or non-functional relations. One dimension on which this question should be addressed is the types of functions which give rise to contexts and situations that support investigation of algebraic and functional ideas. The connection of a functional property to a situational meaning can help strengthen an understanding of that property. It is also interesting to note that some functional properties, such as cusps and points of inflection, are not usually studied until ideas and techniques of calculus can help make them more explicit. This suggests the consideration of the role of function in advanced mathematics as another dimension to address the above question.

Synthesis of the Views

The previous views of function are not presented as disjoint avenues of student development, nor are they intended to completely describe all of the ways in which students can develop a concept image of function. Further, the above discussion of the growth-oriented conception is in line with characteristics of algebra courses which utilize multi-representational instructional approaches, as well as student responses to this instruction (Ruthven, 1990; Slavit, 1993, 1994; Teles, 1989). It must also be stated that the intent of this paper is not to develop a stage theory for the development of function. It is quite possible that a growth-oriented conception of function may be the first structural view that the student possesses, but it is also possible that it may be the last. Instruction is likely to influence the sequencing of this development.

Students who acquire a structural view can investigate functional contexts without a reliance on procedures and local function behaviors. Instead, emphasis can be placed on global behaviors such as growth rate or on relationships between specific local properties. The structural view is needed in order for a student to establish a proceptual understanding of functional notations (Gray and Tall, 1994), an understanding that transcends the action/object duality. Students who see functional notations proceptually can have the flexibility to think about function as an action, an object, or both. Further flexibility arises when the student is able to consider functions as mathematical objects in more than one way, such as the above relational and growth-oriented views. A structural view also allows the student to better understand actions performed on a function, such as a "shift" translation (e.g., changing $f(x)$ to $f(x + 3)$) or taking a derivative. It would be quite difficult for a student to completely understand an action he or she performed on a function if a structural view of function was not yet achieved.

Most importantly, we must remember the comments of Schoenfeld et al. (1993):

> Saying when a student actually "has" the object perspective is not a simple matter. It is not a yes/no kind of knowledge, but one of degrees, and the process of learning is not of simple monotonic growth, but one that includes a fair amount of oscillation (p. 88).

Empirical Support

Prior studies provide some empirical support for the theory. However, it should be clearly stated that a theory of a growth-oriented conception of function should be supported by more data before any curricular decisions are made which have the theory as a basis.

But there is evidence that students obtain a growth-oriented view of function, particularly when exposed to instruction that makes uses of graphs and graphing technologies. Ruthven (1990) found that students who used graphing calculators were better able to describe a given graph in symbolic terms, and that their ability

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to do this relied heavily on their knowledge of functional properties. These students were also better able to identify and distinguish between classes of functions. Confrey and Smith (1995) argue that students are more likely to think of functions in terms of the covariation of two variables rather than in terms of the relational view which currently dominates the curriculum. However, they fall short of investigating student conceptions of functional properties as they relate to overall growth behavior, focusing only on general covariation relationships (e.g., in the linear equation y=2x+3, the established relationship could be "twice x plus three").

Slavit (1994) provides long-term data from a year-long study conducted in a high school Honors Algebra II course which made extensive use of graphing calculators. Questionnaires, test items, and case study interviews were recorded. Data indicated that some students reified function using growth-oriented notions. The strongest evidence came from an analysis of translation tasks, in which students used functional properties to relate different function representations, particularly when the graphic representation was involved. Further, because the instruction primarily focused on elementary polynomials, the students made false generalizations in regard to the properties which functions can possess, such as the need to be continuous. This led to misconceptions in their overall concept image. In addition, several examples of a growth-oriented view were found to "naturally" occur during problem solving episodes.

References

Briedenbach, D.E., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247-285.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26(1)*, 66-86.

Filloy, E., & Rojano, T. (1985). Operating on the unknown and models of teaching. In S. K. Damarin & M. Shelton (Eds.), *Proceedings of the Seventh Annual Meeting of PME-NA* (pp. 75-79). Columbus: The Ohio State University.

Gray, D.O., & Tall, E.M. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25(2)*, 116-140.

Judd, C. (1928). The fallacy of treating school subjects as "tool subjects." In J.R. Clark & W.D. Reeve (Eds.), *Selected Topics in the teaching of mathematics*. New York: Columbia University, Teachers College.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics, 21(5)*, 431-450.

Schoenfeld, A.H., Smith, J.P., & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in Instructional Psychology* (Vol. 4). Hillsdale, NJ: Erlbaum.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. In G. Vergnaud, J. Rogalski, and M. Artigue (Eds.), *Proceedings of the Thirteenth International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 162-169). Montreal, Canada: University of Montreal.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification — The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Slavit, D.B. (February, 1993). *Structural Understanding via Instruction Integrated with the Graphing Calculator: The Case of Function.* Paper presented at the annual meeting of the Eastern Educational Research Association, Clearwater, FL.

Slavit, D.B. (April, 1994). *The Effect of Graphing Calculators on Students' Conceptions of Function.* Paper presented at the meeting of the American Educational Research Association, New Orleans.

Teles, E.J. (1989). Numerical and graphical presentation of functions in precalculus. *Dissertation Abstracts International, 51*, 777A.

IMPOSSIBLE GRAPHS

Tracy Noble, TERC
Ricardo Nemirovsky, TERC
Jesse Solomon, Brighton High School
Joe Cook, Paige Academy

Graphs without a time axis, such as velocity vs. position graphs, offer interesting possibilities for exploring graphing and motion. Relations depicted by these graphs are not limited to functions. In this paper, we describe interviews with a high school student named Olivia who uses a motion detector to create such graphs. While exploring which graphs are logically impossible, she encounters the constraints of the velocity vs. position graphing environment, which we argue are a crucial part of learning about this type of graph and about the relation of velocity to position.

Introduction

In any mathematical representation, there are things that are impossible to do. In a graph of distance vs. time, for instance, the graph line cannot come back to the left, because time cannot go backwards. This constraint arises from the formal properties of distance vs. time graphs, and from the way we understand distance and time. Students of mathematics are generally discouraged from considering the impossible cases in any representation; they are encouraged, instead, to consider cases of possible graphs, since those are the cases they will likely encounter. However, we argue that for any type of graph, considering the impossible graph shapes and trying to understand why these graphs are impossible is an important aspect of learning about the constraints of the graph, and thus the logic that governs how one moves in that graphical space.

In this paper, Olivia, a senior in high school, uses a motion detector to create graphs of velocity vs. position (v vs. p). These graphs are an important part of dynamical systems modeling (Tufillaro, Abbott, and Reilly, 1992), in part because they represent the state of a moving object (given by its position and velocity) as a single point on the graph, creating a compact representation of a system's behavior, in which the relationship between the velocity and position of an object determines which graph shapes are possible. For example, the simple harmonic motion of a weight bouncing up and down on a hanging spring, assuming no damping for simplicity, could be represented as a sine wave on a position vs. time graph that evolves to the right for as long as the motion lasts, or as an ellipse on a v vs. p graph, drawn over and over as the weight continues to bounce:

The work reported in this paper has been supported by NSF Prime Grant # RED-9353507. All opinions and analyses expressed herein are those of the authors, and do not necessarily reflect the views of the funding agency. The authors thank Paul Wagoner, Tracey Wright, and Mark Ogonowski for their editorial feedback.

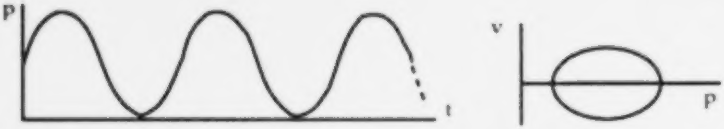


Figure 1

As we see in Figure 1, the constraints on graph shapes in velocity vs. position space are different than those we are accustomed to in temporal graphs. A velocity vs. position graph can double back, because position can decrease as well as increase. However, if the position increases, the velocity must be positive, so the graph line must lie above the x-axis; similarly, if distance decreases, the line of the graph must lie below the x-axis. Thus it is possible to create the ellipse in Figure 1, but it can only be created in a clockwise direction, so that the line is above the x-axis while it extends to the right, and below the x-axis as the line comes back to the left. The realm of impossible graphs is thus more complex than in the case of temporal graphs, where any figure that doubles back on itself or has a perfectly vertical line in it is impossible to make. It provides a rich territory for exploring the distinctions between possible and impossible graphs, and for investigating motion from a new perspective.

As Olivia determines which graphs are possible and which are impossible in velocity vs. position space, we learn about how she distinguishes between the two. In some cases, Olivia does "thought experiments," in which she traces out a graph shape on the computer screen at the same time as she describes the physical motions needed to create the graph. In this way, she finds that some graphs are not physically realizable. Olivia also uses physical experiments with the car to find out what is possible, but experimentation doesn't always provide the final answer for her. We have found that the relationship of her physical experiments to the rules she constructs about what are possible graphs is more complex than the relationship described in textbooks as "the scientific method". Olivia's thought experiments and her physical experiments both help her to distinguish possible from impossible graphs, and she uses both of them in unusual ways that can help us understand how to make sense of the realms of both impossible and possible graphs.

The Interviews

Olivia was in 12th grade at a Boston-area public high school at the time of the interview, had a strong background in science and math, and felt competent in these subjects. Olivia was one of five students in this study, each of whom was interviewed for five hour-long sessions, using individual teaching experiments (Cobb and Steffe, 1983). The interviewer, Tracy Noble, posed some pre-determined problems to the students, but also encouraged them to explore questions of their own whenever possible.

Tracy and Olivia spent the first interview playing a game in which they made drawings to represent their motions of a hand-held toy car. At the start of Olivia's second interview, Tracy introduced the motion detector to her, with the minimal explanation necessary for Olivia to start using it. The motion detector senses the distance from itself to the nearest object in its path, and the software (MacMotion™) uses this information to compute the velocity of the object in real time (See Figure 2).

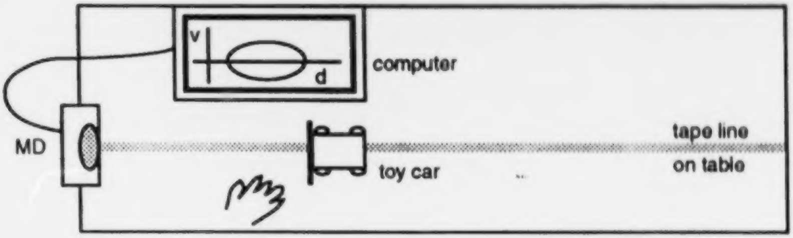


Figure 2

Episode 1 - Direction of Motion on a Graph

Tracy and Olivia spend about 20 minutes using the motion detector to make velocity vs. position graphs of the car's motion before Olivia makes a graph which is a large oval, half above and half below the x-axis (See Figure 3), and tries to determine where on the graph her motion started. [In this figure, Olivia's gestures with the cursor are represented in the left column, and her associated utterance is shown in the right column]

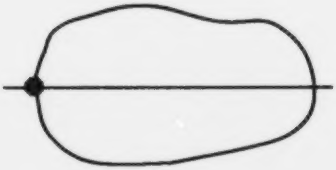


Figure 3.

Notes on Episode 1: In trying to determine how she started her motion, Olivia does a thought experiment in which she imagines moving from left to right along the bottom half of the graph: "Did I start here [left-most point of the oval] and zoom off [tracing with the cursor the bottom half of the oval, from left to right]". Olivia "fuses" the graph and the motions of the car on the table in her language, speaking about "zoom[ing] off," while moving the cursor on the computer screen and also referring to her motion of the car on the table (Ochs, Jacoby, and Gonzales, 1994; Nemirovsky, Tierney, and Wright, 1995). She uses her movement of the cursor along the graph to try to imagine the physical situation that would create the graph, and finds a contradiction, "because to go further away you have to be going [tracing from left to right along top half of oval] a positive velocity away from it [the motion detector]," but she had been moving the cursor from left to right along the bottom half of the oval, where the velocity is negative. Olivia's thought experiment allows her to imagine creating a graph in a direction that would not have been possible, and to understand why it is not possible.

Episode 2 - The Vertical Line, an Impossible Graph

After a few minutes more of discussion, Olivia and Tracy make several more velocity vs. position graphs, and Tracy suggests organizing the shapes in a table with three columns. Olivia and Tracy fill in the "Easy" column with "oval," "waves," and "crazy shapes." In the Difficult column, they place "circle" and "horizontal line" (See Figure 5). They leave the third column blank.

Olivia asks, "Vertical line was pretty easy, wasn't it?" and she finds a way to make two vertical lines show up on the computer screen, by removing the car from the range of the motion detector, seemingly creating a huge velocity peak, but actually creating a graph that does not represent the car's motion. Olivia quickly realizes this, and attempts to make a vertical line while keeping the car in the motion detector's range, moving it toward and away from the motion detector quickly, creating the following graph:

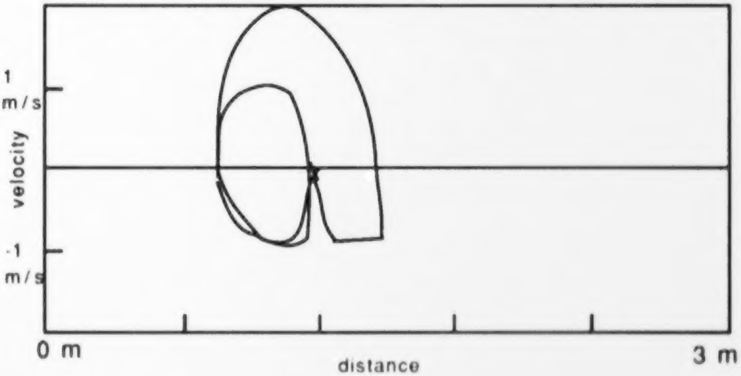


Figure 4

Olivia: Its—The problem is you have to go very quickly in no, very quickly in no [pause] distance, which is impossible unless; no, it's impossible.

Tracy: [pointing to the nearly-vertical sides of the ovals] What about this—these? What would you call these guys here [the nearly-vertical sides]?

Olivia: They're pretty close to vertical but they're not actually [vertical], I mean, they can't be [vertical]. They're just very quick.

Tracy then adds the title "Impossible" to the third column of the table, and Olivia places "vertical line" as the first entry in this column.

Notes on Episode 2: This is the first discussion in which Olivia uses the term "impossible" to describe the making of a particular shape. She has encountered graphs which were difficult to create before, but the vertical line has a new quality: it disobeys a rule she constructs for this graphical space: the rule that you can't go "very quickly in no distance."

When Tracy asks Olivia about some of the nearly-vertical lines of her graph, Olivia responds by saying that "they can't be" vertical lines. Even if Olivia saw a vertical line produced at this point, this statement suggests that she would not believe that it was both truly vertical and truly representing the motion of the car. She has determined that an actual vertical line would be impossible, trusting her sense of the logic of this graphical space more than she trusts the mechanics of the motion detector and graphing program. This is a case which does not fall into the typical model of trusting an experiment to determine the validity of an idea or theory.

Episode 3 - Table of Shapes

Throughout the rest of this interview, and part of her next interview, Olivia fills in the table of figures even further. The final table is represented below in Figure 5:

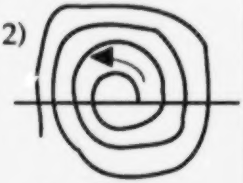
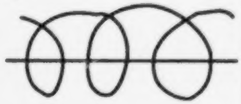
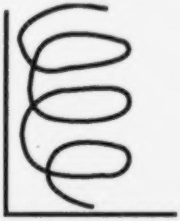
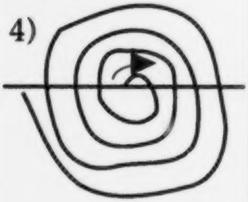
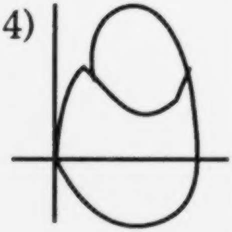
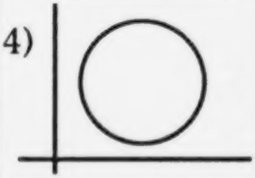
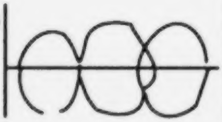
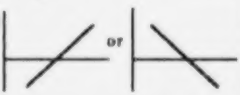
EASY	DIFFICULT	IMPOSSIBLE
1) oval	1) circle	1) vertical line
2) waves	2) Horizontal line	2) 
3) crazy shapes "looks absurd"	3) 	3) 
4) 	4) 	4) 
	5) 	5) 

Figure 5

Olivia uses her rule that an object has a positive velocity going away from the motion detector and a negative velocity going toward the motion detector to deem the counter-clockwise spiral impossible ("Impossible" #2). She also develops a new rule that "anytime you want to look at something [a graph] which doubles back on itself, it has to go below the horizon [x-axis]," that is, no doubling back (coming back to the left) without crossing the x-axis first. Olivia uses this rule to classify the vertical spiral (#3) and the circle above the x-axis (#4) as impossible.

Conclusions

Finding the ways in which she cannot move causes Olivia to articulate some rules about how she *can* move, which help her to understand the logic of this graphical space and how to move within it. Far from limiting Olivia's exploration, her encounter with the constraints of this graph leads her to imagine a group of several "Impossible" figures that she could never create using the motion detector.

Olivia uses a number of resources to make sense of the possible and impossible graph shapes she imagines. Her experiences of moving in front of the motion detector and trying to understand the resulting graphs, have helped allow her to describe a graph in a way that "fuses" the graph shape and the motion needed to create it. Thus, Olivia's tracing out of a graph shape works as a thought experiment involving both her tracing a particular graph shape on the screen and her imagining a particular motion of the car. This ability to "try out" a graph shape without actually performing the motions becomes an important tool for distinguishing between possible and impossible graphs. Olivia's fusion of graph shapes and the motions needed to create the shapes also helps her to create a set of rules that govern how one can move in the graphical space, based on the logic of physical motion. Olivia's confidence in these rules is sometimes stronger than the confidence she shows in physical experiments themselves, contrary to some textbook descriptions of the scientific method, in which theories "are accepted only so long as they are consistent with all observed facts" (Shortley and Williams, 1971, p. 2). Olivia's experience of exploring impossible graphs is valuable to her in large part because of the way she experiences a graph shape, the quantities that are graphed, and the motion that would make the graph, all as parts of a whole experience that blurs the distinction between physical event and representation.

References

Cobb, P. & Steffe, L. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, 14(2), 83-94.

Nemirovsky, R. ; Tierney, C.; and Wright, T. (1995). Body Motion and Graphing. *Paper presented at the annual meeting of the American Educational Research Association*, San Francisco, April.

Ochs, E.; Jacoby, S.; and Gonzales, P. (1994). Interpretive Journeys: How physicists talk and travel through graphic space. *Configurations*, 2(1).

Shortley, G and Williams, D. (1971). *Elements of Physics*, Englewood Cliffs, NJ: Prentice-Hall.

Tufillaro, N.; Abbott, T.; & Reilly, J. (1992). *An Experimental Approach to Nonlinear Dynamics and Chaos*, Redwood City, CA: Addison-Wesley Publishing Company.

COLLEGE CALCULUS STUDENTS' USE OF VERBAL AND GRAPHICAL REPRESENTATIONS TO INTERPRET RATE OF CHANGE MODELS

Maria L. Blanton, North Carolina State University
Wendy N. Coulombe, North Carolina State University
Sarah B. Berenson, North Carolina State University

While the calculus reform movement has seen a variety of new approaches to teaching, there has been general agreement that both conceptual understanding and connections between calculus and the real world need to be emphasized. As a result of the reformed pedagogy, the study of the concept of rate of change has become a growing field in mathematics education.

The purpose of this study was to explore students' ideas about rate of change models developed from linear and nonlinear functions. Verbal and graphical representations were used to gain insights about the ideas concerning rate of change that students bring to a first semester college calculus course. These representations were embedded in real-world contexts to enable the researchers to explore the connections students make between calculus and rate of change models in real world situations. In addition, we were interested in how these students make a transition between graphical and verbal representations.

A total of ten students were individually interviewed on videotape. Students were selected for the interviews primarily on a volunteer basis, although some effort was made to insure diversity among the participants with regard to gender, race, and high school background. A clinical interview format was used in the spirit of Piaget.

Some interesting themes have emerged from the data. For example, students seem to have difficulty controlling for two variables in both nonlinear functions and functions of zero slope. Also, students appear to be able to construct graphs from given verbal scenarios more easily than they are able to interpret graphs using words.

FOSTERING CONNECTIONS BETWEEN CLASSES OF POLYNOMIAL FUNCTIONS

Judith Curran Buck, Plymouth State College

The "units" pertaining to the classes of polynomial functions within the algebra curriculum have ordinarily separated linear functions, quadratic functions, and polynomial functions of higher degree into disjoint topics. NCTM (1989) advocates that connections be promoted between the topics in the mathematics curriculum to increase the potential for retention and transfer of mathematical ideas.

This report highlights results of a recent qualitative study (1994) that used clinical interviews to probe into the connections made between the classes of polynomial functions by a group of Algebra II students. "Connections" here refers to knowledge that relates to *all* classes of polynomial functions; as an example, for all polynomial functions, the zeros can be determined graphically from the points of intersections of the graph with the x-axis. The interviews were followed by "teaching episodes" (Steffe, 1984) designed to contribute to the students' formation of connections by presenting linear functions as the "building blocks" of other classes of polynomial functions. Computer software was used in both the interviews and teaching episodes to aid in the graphical investigations.

Analysis of the clinical interviews reveals that graphical exploration of polynomial functions made the connections across classes more salient for the students. The students were often inhibited from making connections algebraically because of the sequence and the content of instruction. The method of building polynomials from linear expressions used in the teaching episodes not only fostered connections between the classes of polynomial functions, but also between the graphical and algebraic representations of these functions. In particular, the relationship between the zeros of a polynomial, $f(x)$, the roots of the equation, $f(x)=0$, the factors of the polynomial, and the intercepts of the graph became more evident across classes.

References

NCTM (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.

Steffe, L. P. (1984). The teaching experiment methodology in a constructivist research program. In M. Zweng (Eds.), *Proceedings of the fourth international congress on mathematical education* (pp. 469-471). Boston: Birkhauser.

GRAPH KNOWLEDGE: UNDERSTANDING HOW STUDENTS INTERPRET DATA USING GRAPHS

Susan N. Friel, University of North Carolina-Chapel Hill
George W. Bright, University of North Carolina-Greensboro

There are abundant opportunities for research on how children learn and understand both the process of statistical investigation and the statistical concepts related to using this process. This paper discusses a research study that focused on middle school students' abilities to read and to move between different graphical representations (i.e., line plots, bar graphs, stem-and-leaf plots, and histograms) before and after instruction. The representations that are discussed were selected for a number of reasons. First, the representations are common in the school mathematics curriculum. Second, understanding the process of data reduction and the transitions from displays of raw data to those which present grouped data is an inherent part of developing graph knowledge, as is understanding the structure of the graphs themselves. Third, considering the ways students respond to questions involved in interpreting data by reading graphs provides insights into students' graph knowledge.

During Fall, 1994, we conducted a study of the ways that students in grades 6, 7 and 8 made sense of information presented through graphical representations and made connections between related pairs of graphs. Students were tested both before and after an instructional unit developed specifically to highlight a particular sequence of graphs that took into consideration increasing degrees of data reduction and building connections between pairs of graphs. Small samples from each grade were also interviewed before and after the unit. Data from the interviews and the tests of these samples of students are used to illustrate the difficulties and successes that students experienced in attempting to understand the material from this unit.

Using Curcio's (1989) three components of graph comprehension (reading the data, reading between the data, and reading beyond the data) as an organizing framework for reporting results, specific attention in this session will be given to the nature of the responses students made to selected written problems presented by pre- and post-instruments. In addition, we also will look at students' building transitions between pairs of graphs (i.e., line plots and bar graphs; stem plots and histograms) and ways in which they use information from one graph to construct its paired representation. We have sought to address both the need for context and an awareness of the process of statistical investigation within the assessment environment. This research provides a framework both for looking at students' knowledge of graphing (in the statistical sense) and for developing a research agenda related to this area.

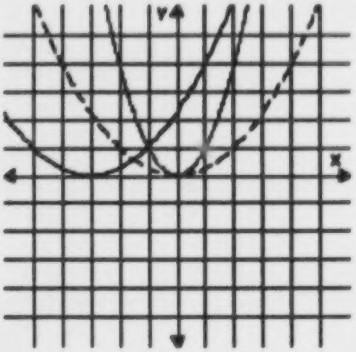
Reference

Curcio, F. (1989.) *Developing Graph Comprehension.* Reston, VA: NCTM.

A NEW STRATEGY FOR EQUATIONS AND GRAPHS

Susan Hagen and Harold Mick, Virginia Tech

Our research investigates a new strategy that students use to construct a knowledge of equations and graphs based on transformations. Major factors in the construction process are connections between geometric and algebraic representations. Our hypothesis is that the key to making these connections is using ordered pairs to serve as a "conceptual bridge" between graphs and equations. With the connections established, transformations from one graph to another correspond to changes of coordinates that correspond to changes of graphs.



To illustrate one direction, from graph to equation, suppose we write the equation of the solid parabola shown to the right with our new strategy. We let (r,s) be a generic point on the solid parabola. Our task is to find a relationship between r and s.. Since we know the relationship between coordinates of points on the parent parabola, we move the solid parabola to the parent by translating right 3 units followed by a vertical scale by a factor of 4. This transformation corresponds to the change of coordinates $(r,s)\rightarrow(r+3,4s)$. Since the point $(r+3,4s)$ lies on the parent, we know that the second coordinate is equal to the square of the first. In symbols we write, $4s = (r+3)^2$. Therefore r and s satisfy the equation

$$Y = \frac{1}{4}(X+3)^2.$$

Our data have been collected in university precalculus classes, high school precalculus classes, and from secondary mathematics teachers enrolled in an independent study course. Preliminary results suggest that the definition of a graph is difficult for students to apply both to writing equations and sketching graphs. The students who develop an understanding of the point connection in both directions use the point's coordinate addresses to determine the direction.

STUDENTS' CONSTRUCTIONS OF POSITION AND VELOCITY DURING CBL™ EXPLORATIONS

Kathy M. C. Ivey, Western Carolina University

To mathematically empower students, reform documents have emphasized the need for contextually based mathematics problems (NCTM, 1989; NCTM, 1980). Yet in undergraduate calculus, students seldom encounter any real quantitative data. Even "reform calculus" materials concentrate on data supplied to students through graphs or tables. Prior studies have looked at high school students' confusion of the graphs of a function and its derivative (Nemirovsky & Rubin; 1992) and at college physics students' development of the concept of velocity from observing rolling balls (Trowbridge & McDermott; 1980). This study considers what students come to know about position and velocity functions when they gather and analyze data of their own motion.

Using a Texas Instruments CBL™ (Calculator Based Laboratory™) System, college students in first term calculus gathered and interpreted quantitative data. A CBL is a hand held unit which collects physical data when paired with a TI-82 or TI-85 graphing calculator and a sensing probe—in this instance a Vernier Ultrasonic Motion Detector. Lab days were videotaped, with informed consent from the students, and individual students and faculty members were interviewed using a semi-structured interview format.

Analysis of videotapes revealed several categories of understandings and misunderstandings about calculus concepts. One observation is the effect that actual physical enactment had on students' concepts of the relationship between position functions and velocity functions. This paper examines how students' constructions of the position function and velocity function changed during an early CBL lab activity and considers the robustness of their altered understandings. The influence of physical enactment of position functions on students' understanding of the concepts of position and velocity appears to be related to the impact of manipulatives use on young children's understandings of basic arithmetic. This line of research holds promise for understanding students' constructions of basic calculus concepts.

References

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1980). *An agenda for action.* Reston, VA: Author.

Nemirovsky, R., & Rubin, A. (1992). *Students' Tendency to Assume Resemblances between a Function and Its Derivative.* (Working Paper 2-92) Cambridge, MA: TERC.

Trowbridge, D. E., & McDermott, L. C. (1980). Investigation of student understanding of the concept of velocity in one dimension. *American Journal of Physics, 48*, 1020-1028.

THE FUNCTION CONCEPT IN COLLEGE ALGEBRA

Robert Mayes, University of Northern Colorado
Larry Lesser, University of Northern Colorado

The curriculum development project ACT in Algebra was instituted to address pedagogical and curricular concerns in college algebra. The project focuses on applications, concepts, and technology. The preliminary text for the project took its name from these three foci, *ACT in Algebra*. The project has several goals. First, the curriculum materials and pedagogy are based on current research on the most effective techniques for improving student concept and problem solving ability. Second, applications are used to motivate and introduce new topics. Students analyze discrete data in numerical and graphical form, then apply discrete techniques to find a mathematical model. Third, technology is used as a tool in the exploration of mathematical concepts and applications. Fourth, the use of small group learning is used to promote communication, connections and conceptual understanding in mathematics. The use of group projects in the classroom and in the computer lab promote active student learning, since students construct knowledge through communication with other students. Assessment of higher level cognitive abilities requires a change in how student success is measured. *ACT in Algebra* assesses conceptual understanding and problem solving through group written reports.

In the spring semester of 1995 a study was conducted to study the effects of the preliminary *ACT in Algebra* curriculum and pedagogy. The study addressed the following questions: How does the ACT Curriculum enhance a student's conceptual understanding in mathematics and a student's ability to model and solve realistic problems? Does the ACT Curriculum improve student attitude towards mathematics and the use of technology in the learning and teaching of mathematics? What is the effect on student learning of a pedagogy stressing cooperative learning, active student construction of knowledge, and computer laboratory experiments?

The focus of the poster session will be to report on qualitative research into the effects of the ACT curriculum on the student's concept of function and attitudes towards mathematics. The powerful qualitative research software, *Atlas/ti*, was used to analyze transcriptions of videotaped student interviews. The results of this analysis will be reported.

STUDENT ACTIVITY AND FUNCTIONS: A PROPOSED FRAMEWORK

Georgianna T. Klein, Grand Valley State University

Research literature has proposed several models for describing understandings students have when studying functions. A dominant theme in several models is to characterize functions as being understood operationally or structurally, or as processes or objects (Harel & Dubinsky, 1992; Sfard, 1991). A second theme addresses wholeness of the function concept—whether a function is seen as a single entity, includes domain and range, is reversible, can be operated on, etc. These models are closely related to curricular goals mathematics teachers have for high school students' understanding and often reflect a teacher's perspective, rather than actual student activity. Built into many existing models is the assumption that if students do not have at least partially coherent conceptions of functions per se, they are considered not to have conceptions of functions at all.

This session draws on research in a high school class on functions and on themes existing in the literature to propose an alternative framework. The model separates issues related to the function concept per se from student activity with functional relationships in problems. It consists of two complementary parts, one for perspectives students take when working with functional relationships in problems and a second that addresses whether students see problems as being about functions and the extent to which students understand the function concept as a whole. Following Confrey and Smith (1994), the first part of the model will define correspondence and covariation approaches, but will distinguish dynamic and static versions of each. The second part will distinguish three levels of wholeness, which are then crossed with two dimensions of awareness of functions in the problem. The poster will explore how these two parts are related and what the framework can contribute to discussing students' understanding of functions.

References

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Harel, G., & Dubinsky, E. (Ed.). (1992). *The concept of function: Aspects of epistemology and pedagogy*. Washington, D.C.: The Mathematical Association of America.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

STUDENT UNDERSTANDING OF THE EQUIVALENCE BETWEEN ALGEBRAIC AND GRAPHICAL REPRESENTATIONS OF A FUNCTION

Eric J. Knuth, University of Colorado at Boulder

Multiple representations of functions are an important topic in first year high school algebra, yet many students leave high school lacking an understanding of the equivalencies between these representations. The importance of this ability to use multiple representation systems is emphasized in the NCTM *Curriculum and Evaluation Standards*, "[students should be able to] translate among tabular, symbolic, and graphical representations of functions" (p. 154). Many studies have been conducted that have investigated students' conceptions of functions, interpretations of graphs, or understanding of different representation systems of functions. A fewer number of studies though, have specifically examined how students understand, and to what depth they understand, the equivalence between the algebraic form of a function (i.e., its equation) and its graphical representation.

Research results are presented from a study which examines student understanding of this equivalence between the algebraic and graphical representations. The study looks specifically at how students understand the following question: *What does it mean to have the graph of an equation?* Using a matrix sampling technique, seven teachers presented two hundred eighty-four high school students, varying in level from first year algebra through calculus, with a series of questions designed to provide insight into their understanding. An examination of the students' responses shows an overwhelming reliance on the algebraic representation of a function, even on tasks where the graphical representation would seem more appropriate.

The study also includes in depth analysis of interviews of two students, a first year algebra student and a pre calculus student. The focus of the analysis is mainly concerned with students' inadequate or lack of connections between the algebraic and graphical representations of a function characterized by a treatment of the domains as essentially independent. The students predominantly view the tasks from a process ritualistic perspective. The discussions with the two students also suggest that the flexibility in moving between and within representations is no easy task. The analysis of the students' thinking points to gaps in the understanding of fundamental connections between the representations.

Reference

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: author.

Geometric Thinking

GEOMETRIC UNDERSTANDING IN GIFTED STUDENTS PRIOR TO A FORMAL COURSE IN GEOMETRY

Marguerite M. Mason, University of Virginia

This study investigated whether the van Hiele model accurately described the geometric thinking of students in the 6th through 8th grades identified as academically gifted. The results from 120 students who completed a 25-item multiple choice test and 64 students who participated in 30-45 minute individual interviews were analyzed. The gifted students demonstrated higher overall van Hiele levels than the usual student entering a high school geometry course. Many lacked correct basic definitions, but they would attempt to deduce the definitions from contextual clues. Once they established a definition, correct or incorrect, most students reasoned consistently from it. However, gifted students, particularly the 35.8% that skipped levels, need Level 2 and 3 experiences to provide a foundation for their reasoning.

Dutch educators P. M. van Hiele and Dina van Hiele-Geldof proposed a linearly-ordered model of geometric understanding which asserts that a successful learner passes through five hierarchical levels of geometric thinking in order. The purpose of this study was to investigate whether the van Hiele model accurately describes the geometric thinking of students in the 6th through 8th grades who had been identified as academically gifted.

The van Hiele Model

Levels of Geometric Thought

According to the van Hiele model of geometric understanding (van Hiele, 1959/1985; van Hiele, 1986; van Hiele-Geldof, 1984), students progress through five sequential, hierarchical levels of thought as their understanding of geometry develops: visualization, analysis, abstraction, deduction, and rigor. The learner cannot achieve one level without mastering the previous levels. While a teacher can reduce content to a lower level and it may appear to be mastered because the student has rotely memorized, a student cannot skip a level and still achieve understanding (Clements & Battista, 1992). Progress from one level to the next is more dependent on educational experiences than on age or maturation.

Previous Research

Burger and Shaughnessy (1986) found mainly Level 1 thinking for subjects in grades K-8. Fuys, Geddes, and Tischler (1988) found entry levels of 1 and 2 with 6th and 9th graders, but several students, especially those deemed above average in mathematics ability prior to instruction, exhibited Level 3 behavior by the completion of the six hours of clinical interviews and instruction. Usiskin (1982) found a hierarchy of levels existed in the 2,699 students enrolled in 99 high school geometry classes that he examined. Almost 40% of his students finishing high school geometry were below Level 3. Mayberry (1983) found sufficient evidence among

19 undergraduate preservice elementary teachers to support the hierarchical aspect of the theory, but she rejected the hypothesis that an individual demonstrated the same level of thinking in all areas of geometry included in school programs. These results were replicated with preservice teachers in Spain for Levels 1 through 4 (Gutiérrez & Jaime, 1987; Gutiérrez, Jaime, & Fortuny, 1991).

This study examines whether the van Hiele model accurately describes the geometric thinking of gifted students prior to a formal course in geometry and makes comparisons with what has been found with other populations.

Method

The present study focuses on students in the 6th through 8th grades who have been identified by their school districts as gifted. The subjects had mathematics percentile ranks of 97 or above on the Iowa Test of Basic Skills or the Stanford Achievement Test and teacher recommendations indicating other distinguishing characteristics relevant to mathematics achievement. The population consists of 120 students, drawn from over 50 different school districts, who participated in a National Science Foundation sponsored Young Scholars Program targeted for gifted youth from rural areas during 1990-94. None of the students included in this study had taken a formal course in geometry.

Paper-and-Pencil Tests

To enable comparisons with a large general population of students enrolled in high school geometry classes, the 25-item multiple choice paper-and-pencil test developed by the Cognitive Development and Achievement in Secondary School Geometry Project (CDASSGP) (Usiskin, 1982) was administered to all 120 gifted students. The test, with 5 proposed answers per item and 5 items per level, was originally developed to test the van Hiele theory. Answering 4 of 5 questions correctly at a level in this test indicated mastery of that level. If a student met the criterion for mastery of each level up to and including level *n* and failed to meet the criterion for mastery of all the levels above level *n*, the student was assigned to level *n*. If the student could not be assigned to a level in this manner, the student was said to "not fit."

The Interviews

Sixty-four randomly selected subjects participated in a 30 - 45 minute individual interview, conducted by the researcher. The questions used as a starting point in the interview were a subset of the instrument developed by Mayberry (1981). The entire Square Strand and other questions of interest were employed.

Results and Discussion

Paper-and-Pencil Tests

The distribution of the CDASSGP test scores in this study appear in Table 1.

Table 1
% of Subjects at Each van Hiele Level as Determined by the CDASSGP Test

			van Hiele Level					
		not mastered						
Grade	n	1	1	2	3	4	5	no-fit
8	46	0.0	15.2	10.9	26.1	2.2	10.9	34.8
7	36	2.8	8.3	8.3	25.0	5.6	2.8	47.2
6	38	7.9	23.7	23.7	7.9	7.9	2.6	26.3
Total	120	3.3	15.8	14.2	20.0	5.0	5.8	35.8

Table 2
% of Gifted Students and Students Entering High School Geometry at Each van Hiele Level on the CDASSGP Test Excluding "No-Fits"

			van Hiele Level				
		not mastered					
Grade	n	1	1	2	3	4	5
8	30	0	23	17	40	3	17
7	19	5	16	16	47	11	5
6	28	11	32	32	11	11	4
Total	77	5	25	22	31	8	9
High School*	241	27	51	15	7	0	0

* Data for students entering high school geometry were reported by Senk (1989, p. 315). Other data is from gifted students in the current study.

Despite their younger age, these gifted students demonstrated higher overall van Hiele levels than the usual student entering a high school geometry course. Senk (1989), using this same CDASSGP instrument with students beginning a high school geometry course, found 241 students who "fit the model". The distribution of their levels is shown in Table 2. Of the 77 gifted students who "fit the model" in the current study, only 5% had not mastered Level 1 and 17% were classified as having attained van Hiele Levels 4 or 5. In Senk's study, only 22% were above Level 2 compared to 49% of the gifted students in the current study. However, as seen in Table 1, over 35% of the gifted subjects tested did not fit the model. This is in contrast to the CDASSGP study in which only 12% of the over 2,600 students about to take high school geometry did not fit the model.

In comparison to students closer to their own age, Fuys, Geddes, and Tischler (1988) found no one functioning above Level 2 in interviewing 6th and 9th grade average and "above average" subjects while Burger and Shaughnessy (1986) found mainly Level 1 thinking in grades K-8.

Proof Readiness. To examine the predictive power of a student's van Hiele level, Senk (Usiskin & Senk, 1990) compared the levels indicated by the CDASSGP test prior to a high school geometry course with their performance in proof writing as measured by the CDASSGP Proof Test at the end of the course. Applying her findings to this study, only 5% of the gifted students, all 6th and 7th graders, have not mastered Level 1 indicating a probability of success in proof writing of less than .35. 25% have mastered Level 1 with a probability of successful proof writing between .35 and .60. The remaining 70% have van Hiele levels 2 or greater and so have probability of proof writing success greater than .75.

Interviews

The percentage of subjects at each van Hiele level as determined by the interviews is given in Table 3 for the Square and Right Triangle Strands. Guided by the findings of Burger and Shaughnessy (1986) and Fuys, Geddes, and Tischler (1988) of the levels that students of these grades might be expected to attain, no Level 5 questions were administered and Level 4 was administered only in the Square Strand.

Table 3. *Percent of Subjects at Each van Hiele Level as Determined by the Interviews in the Square and Right Triangle Strands*

			van Hiele Level					
			not mastered					
Grade	n	Strand	1	1	2	3	4	no-fit
8	20	square	0.0	0.0	35.0	25.0	20.0	20.0
		rt. Δ	0.0	20.0	20.0	45.0	n/a	15.0
7	24	square	0.0	0.0	25.0	8.3	45.8	20.8
		rt. Δ	0.0	20.8	12.5	50.0	n/a	16.7
6	20	square	0.0	5.0	30.0	10.0	35.0	20.0
		rt. Δ	10.0	30.0	10.0	25.0	n/a	25.0
Total	64	square	0.0	1.6	29.7	14.1	34.4	20.3
		rt. Δ	3.1	23.4	14.1	40.6	n/a	18.8

Analysis of the interviews indicated that the van Hiele levels are hierarchical in gifted subjects. Excluding the 15 subjects who exhibited mastery of the highest levels of both the square and right triangle strands administered, only 8 of the remaining 49 subjects were deemed to be thinking at the same level in the two content areas.

Concepts and Logical Reasoning

Burger and Shaughnessy (1986) have characterized the van Hiele levels as very complex structures that involve the development of both concepts and reasoning processes. This dual nature of geometric understanding in gifted students is particularly evident in the portions of the interviews dealing with isosceles and equilateral triangles. Even though isosceles and equilateral triangles are a standard part of the school mathematics curriculum prior to the end of sixth grade, as shown in Table 4, the subjects provided a wide range of definitions for the term "isosceles triangle."

When questioned, 14 subjects of the sixty-three admitted that they were unsure of the correct definition of isosceles triangle, but 12 of these students provided definitions such as no sides congruent, all sides congruent, all angles less than 90°, or containing one angle greater than 90°. Only two subjects answered "I don't know" when asked for a definition of an isosceles triangle. Many of these definitions appear to be deductions based on the structure of previous questions.

Table 4. *Number of Students At Each Grade Level Providing Specific Definitions of "Isosceles Triangle"*

	Grade			
	6	7	8	Total
At least 2 sides =	5	8 (5)[a]	3 (3)[a]	16
2 sides =, interpreted as "at least 2"	2 (1)[a]	7 (1)[a]	7(1)[a]	16
2 sides =, interpreted as "exactly 2"	3	1	4(1)[b]	8
2 sides =, inconsistent interpretation	2	2	0	4
2 sides = with 3rd side different	0	1	4(2)[a]	5
No sides @	2	2	2	6
All sides @	2	0	0	2
2 = <s	2 (1)[c]	8 (8)[c]	4 (3)[c]	14
All <s < 90°	2 (1)[b]	0	0	2
One < > 90°	0	2	0	2
I don't know	1	0	1	2

Notes. [a] Number in parentheses refers to number of subjects who also referred to 2 angles or at least 2 angles being the same. [b] One subject defined an isosceles triangle as being "a 3-sided figures with all angles < 90° and with 2 sides of the same length and 2 angles that are the same" with the exactly two interpretation. [c] Number in parentheses refers to number of subjects who also referred to two sides or at least two sides being the same.

Once they gave a definition, most students reasoned consistently from it. For example, one 8th grade girl who defined an isosceles triangle as "The sides are all different sizes", answered the question "Are some right triangles isosceles triangles?" by saying "Yes. A triangle could have a 90° angle and have two different angles for the rest." Only four students were inconsistent is applying their stated definitions.

Conclusions and Recommendations

The reasoning ability of these gifted subjects was far beyond what may have been anticipated, given their lack of knowledge of basic definitions and concepts. Many of these gifted subjects had not been exposed to or did not remember what the critical defining attributes of various figures were. However, they tended to look for similarities and differences in figures and deduce what the defining attributes might be. Many of the students lacked correct basic definitions of terms such as congruent and similar, but they would attempt to deduce the definitions from contextual clues. Once they established a definition, correct or incorrect, most students reasoned consistently from it. Deduction is a strength of most of the subjects. However, they have not been exposed to the "rules of the game" and so do not know how to construct an acceptable formal geometric proof. It should be noted that deductive reasoning is a skill which can be developed outside the context of geometry, as it apparently has with many of these subjects.

Despite their younger age, these gifted students demonstrated higher overall van Hiele levels than the usual student entering a high school geometry course. Using probabilities developed in the CDASSGP study (Usiskin & Senk, 1990), 70% of the students, who were able to have a level assigned, have a probability of proof writing success greater than .75. However, gifted students, particularly the 35.8% that have skipped levels and do not fit the model, need Level 2 and Level 3 experiences in order to provide a foundation for their reasoning. Provided with this additional foundation, gifted middle school students should be capable of a proof oriented geometry course.

References

Burger, W., & Shaughnessy, J. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education, 17*, 31-48.

Clements, D., & Battista, M. (1992). Geometry and spatial reasoning. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). New York: Macmillan Publishing Co.

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education Monograph, 3*.

Gutiérrez, A., & Jaime, A. (1987). Estudio de las caracteristicas de los niveles de van Hiele. In J. Bergeron, N. Herscovics, & C. Kieran (Ed.), *Proceedings of the Eleventh Conference of the International Group for the Psychology of Mathematics Education*, III (pp. 131-137). Montreal: University of Montreal.

Gutiérrez, A., Jaime, A., & Fortuny, J. (1991). An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education, 22*, 237-251.

Mayberry, J. (1981). An investigation in the van Hiele levels of geometric thought in undergraduate preservice teachers. *Dissertations Abstract International, 42*, 2008A, (University Microfilms No. 80-23078).

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education, 14*, 58-69.

Senk, S. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20*, 309-321.

Usiskin, Z. (1982). Van Hiele levels and achievement in secondary school geometry. (Final Report of the Cognitive Development and Achievement in Secondary School Geometry Project). In Chicago, IL: University of Chicago.

Usiskin, Z. & Senk, S. (1990). Evaluating a test of van Hiele levels: A response to Crowley and Wilson. *Journal for Research in Mathematics Education, 21*, 242-245.

van Hiele, P. (1959/1985). The child's thought and geometry. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele* (pp. 243-252). Brooklyn, NY: Brooklyn College, School of Education. (ERIC Document Reproduction Service No. 289 697).

van Hiele, P. (1986). *Structure and insight*. Orlando: Academic Press.

van Hiele-Geldof, D. (1984). The didactics of geometry in the lowest class of secondary school. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele* Brooklyn, NY: Brooklyn College, School of Education. (ERIC Document Reproduction Service No. 289 697).

THE ROLE OF BUILDING 3-D MODELS IN THE SENSE MAKING OF GEOMETRY

Alejandro Solano, Florida State University
Grayson H. Wheatley, Florida State University
Norma Presmeg, Florida State University

Current research in the area of geometry (Stigler, Lee, & Stevenson, 1990; Clements & Battista, 1992) shows a substantial number of students are failing to learn geometry meaningfully. Investigating in greater depth how students create images that help them make sense of geometry seems to be a viable way to become more knowledgeable in this area. The goal of this research was to construct explanations of the ways in which students create imagery and how they use it in geometrical situations. For the purpose of this paper, a focus was placed on the construction and use of geometrical models by participants, and how this experience helped them to create useful images for their learning of geometry.

Methodology

The qualitative emphasis on process has been particularly beneficial in educational research (Bogdan & Biklen, 1992). Hence, given the focus of this research, a qualitative approach was taken as the research methodology. A constructivist perspective (von Glasersfeld, 1987) was used as the umbrella under which this research was conducted. During data collection, the main strategies used to gather information were participant observations as they build models of cubes, truncated tetrahedron and prisms, formal and informal interviews, and participants' reflections about their experiences during the building of three-dimensional models.

The participants in this research were two prospective high school mathematics teachers during their junior year of study. The selection was based on the scores they obtained in the Wheatley Spatial Ability Test (WSAT) (Wheatley, 1978). One participant had a high score on the WSAT and the other one near the mean on the WSAT.

Findings

Building three-dimensional models seems to be very powerful in facilitating students' understandings of geometry. In this study it was found that participants valued the experience of building three-dimensional models prior to becoming engaged in geometrical tasks. This experience helped them solve problems involving area and volume of geometric figures. Participants improved drastically their construction of relationships as a result of being able to build three-dimensional models. As would be expected, the transformations they used to solve proposed tasks were heavily influenced by their previous experiences. Overall, the building of three-dimensional models proved to promote richer quality of imagery in participants that helped them to construct meanings and thus, to solve tasks not only more accurately but also more creatively.

References

Bogdan, R. & Biklen, S. (1992). *Qualitative research for education: An introduction to theory and methods* (2nd ed.). Boston: Alynand Bacon.

Clements, D. H., & Battista, M. T. (1992). Geometry and Spatial Reasoning. In D. A. Grows (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (1 ed.). New York: Macmillan.

Stigler, J. W., Lee, S. Y., & Stevenson, H. W. (1990). *Mathematical knowledge of Japanese, Chinese, and American elementary school children.* Reston, VA: National Council of Teachers of Mathematics.

von Glasersfeld, E. (1987). *The construction of knowledge: Contributions to conceptual semantics.* Salinas, CA: Intersystems Publications.

Wheatley, G. H. (1978). *The Wheatley Test of Spatial Ability.* West Lafayette, IN: Purdue University.

Language and Mathematics

THE DEVELOPMENT OF A STUDENT THEORY: THE ROLE OF DISCOURSE

Verna M. Adams, Washington State University
Terry M. Price, Washington State University

This study examines the role of the discourse in the development of students' understandings of a rule for determining if a number is divisible by 8. The rule was suggested by a student in a seventh-grade mathematics class. Its validity was investigated in whole-class discussions that occurred on 3 consecutive days. In addition to making field notes and videotaping classroom interactions, what students learned was investigated through the use of whole class surveys and interviews with 9 out of 29 students. The discourse served to sustain the investigation, to assist students' development of the idea and to confuse students. Confusion occurred as a result of "failures of context" when the discourse failed to deal with the complexity of the language structure involved in a rule and to discriminate meanings of words such as *even* and *evenly*.

> Learning always takes place in everyday activity, whatever that activity might be. (Lave, Smith, and Butler, 1988, p. 79)

Current perspectives on the teaching and learning of mathematics suggest that the everyday activity in mathematics classrooms should include students talking about mathematics. As a result of that talk, students will be introduced to ideas they have not previously developed on their own. It follows then that a student's understanding can be expected to evolve under conditions of systematic cooperation with the teacher and other students. Moreover, because students' mathematical understandings are anchored to the contexts in which they are learned (Lave, 1988), students understandings will be anchored to the classroom discourse. According to Vygotsky (1978), the introduction of a new concept into the discourse in the classroom initiates a long and complex process in which the student eventually appropriates the concept. He argues that the deliberate introduction of new concepts into the dialogue, rather than precluding the spontaneous development of those concepts, charts new paths for their development and "may influence favorably the development of concepts that have been formed by the student himself" (p. 152).

The Study

The students in this study were 29 "average" seventh-graders in one mathematics class; the teacher was the regular classroom teacher. At the beginning of almost every class, 10 to 15 minutes were used for an activity called mental math. The remainder of the 45-minute class consisted of whole-class discussion, individual seat work, or small-group work. This paper focuses on a theory about divisibility by eight that was presented in a whole-class discussion at the end of the

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Preparation of this paper was supported in part by National Science Foundation, Grant No. RED-9254922. Any opinions, conclusions, or recommendations are those of the authors and do not necessarily reflect the views of the National Science Foundation.

sixth week of school. We identify the student who presented the theory by the fictitious name of Blayne. Our investigation followed the recommendation of Lave, Smith, and Butler (1988) to focus on the activity of learners, with an emphasis on what is actually occurring and what is being learned. In this report we discuss the role of the discourse in these components of the investigation.

Method

Our method of investigation draws on the grounded theory method of Glaser and Strauss (1967) and Glaser (1978) in which, after identifying a phenomena of interest from the data, additional data is collected and coded. Selection of this new data is determined by the emerging theory in order to maximize the information relevant to the theory.

Initial data collection, consisting of daily videotaping and recording of field notes of the classroom activities and the collection of curriculum materials, began in the fourth week of school before the start of this study and continued beyond the scope of the study to the end of a unit of instruction on number theory that was 4 weeks long. The topic of interest emerged from the data when Blayne suggested a rule for dividing by eight. Although we use the word *rule* in this paper, the teacher, the students, and the researchers during the data collection referred to it as Blayne's theory. Blayne, with the support of the teacher, led the class in an investigation of the validity of the procedure that he proposed.

In order to investigate the extent that Blayne's theory was appropriated by other students, several stages of data collection occurred. Blayne discussed his rule about dividing by eight on three consecutive class days. Videotaping and field notes of these presentations were followed by an initial whole-class survey and then by individual interviews of 9 students. These interviews were evaluated at the end of each day and new or modified hypotheses about student understandings were considered. Upon completion of the interviews, a second whole-class survey was constructed and administered in order to determine if students had incorporated the ideas into their understanding to the point where they would be able to generalize the rule to dividing by 27.

Overview of the Analysis

The key players in this study were (a) Blayne, who presented the theory of divisibility by eight, (b) the teacher and (c) other students who participated in the class discussions. The teacher orchestrated the discussion without telling the students exactly what to think. She made attempts at getting students to explain their understandings of the theory and gave positive feedback to students by telling them that she liked their ideas. She occasionally corrected student statements about language or the procedure.

What Occurred? Discussions of divisibility by eight occurred in this class on 3 different days consisting of 2 minutes of discussion, 13 minutes, and then 17 minutes. Blayne's first explanation was garbled and the teacher asked him to think about it overnight. The next day, she asked him to present his theory at the overhead to the whole class. On the third day, she asked him restate it so the class could determine if the rule would always "work." On the second day, Blayne stated his theory as "Anything that you are trying to find out if eight goes into, you have to divide by 2 three times, and if the answer to those three are even, then it will go into eight." Later his language became "If all three of those answers are even, then you can divide it by eight." and "If the answers come out even." Even was clarified with the help of the teacher to mean that it would not have a decimal in the answer.

In the class discussions, students tried to validate Blayne's theory using their own numbers. The magnitude of the numbers used by Blayne was limited to 2-digit numbers. Other students, however, used numbers with up to 8 digits. The interview data indicated that students generally had a comfort zone with numbers up to about 100. Thus, through the use of numbers outside their comfort zone, students made the discussion into a true investigation aimed at deciding if the procedure would always work. The teacher made three attempts to get students to draw conclusions from their examples and three attempts at closure by making statements such as "it seems to work." Students puzzled over whether or not they were talking about "even numbers" or "dividing evenly." They asked if Blayne's ideas were related to 2^3 and if you could construct similar divisibility rules for other numbers. One student wanted to know if the answers were the same when one divided by 2, three times and when one divided by 8. At the end of the 3-days of discussion, some students appeared to be convinced that Blayne's method of determining whether or not a number was divisible by eight was valid. Others were skeptical, offering hypotheses such as "maybe the theory works if all the digits are even," conclusions such as "I don't think it works, not all even numbers work," and questions such as "Why not just divide by eight to begin with?" The teacher left the discussion open by telling students to go home and try some examples in order to answer some of their questions.

The surveys and interviews revealed that students who did not believe the theory would always work had different reasons for rejecting the idea. One student said "no" because other theories also worked. Two students were skeptical of the justification process. One explained: "I don't know why I think that, I guess because there must be a number out there that can fool him." Two types of justifications were used by students to either accept or reject the theory. *Problem-specific justification* was an explanation based on a particular problem, rather than several examples. Another type of justification, *example-based justification*, was characterized by comments such as "because I have used it a lot," and "because we have worked on problems in class and out of class and it always worked." Some of the students' justifications were either missing or uninterpretable.

What was learned? The classroom discourse revealed gaps in Blayne's understanding of his own theory. For example, when a student asked if the theory had anything to do with 2^3, he answered, "I don't know." An interview with Blayne revealed that he had learned the procedure from his father. In effect, he had accepted the idea but had not developed an understanding of it. By the time of the second survey, Blayne's responses indicated that he had filled in some of the gaps in his understanding. He recognized that "A number is divisible by 8 if it has 3 factors of 2" and "If 2^3 is a factor of a number, the number is divisible by 8." He, however, did not generalize the procedure to create a rule for dividing by 27.

Ten students actively participated in the discussions, interjecting comments, conjectures and questions. These students either (a) indicated that they did not understand the theory, (b) believed that it would work, (c) did not believe it would work, or (d) felt that it was too long or too much trouble. In the initial survey, 11 students spontaneously used Blayne's theory in response to the question: Is 2000 a multiple of 8? Another ten students divided 2000 by 8, one student guessed, and four students based their answers on misconceptions that can be connected to other divisibility rules. One student reasoned: "8 goes into 200, so it should go into 2000." Two students did not complete the survey.

On the second survey, several questions investigated language issues from the class discussions. For example, the words *divided by* were often misused. Only 8 students had an understanding of how to interpret these words. Nearly half (13) divided the small number into the large number, and 3 always interpreted from left to right. Thus, for many students, the information that influenced their interpretation was the size or the order of the numbers. Students also were less secure with the language "3 factors of 2" than with "2 to the third power" as a meaning for the notation "2^3." An examination of the discourse revealed that the language "3 factors of 2" was not explicitly connected to the notation.

Students' interpretation of what it means to say that the rule worked was different from the researchers' interpretation. Students considered the rule to "work" if the number was divisible by 8 and "not work" if the number was not divisible by 8. One student concluded that Blayne's "theory isn't always going to work." We took this to mean that she thought the rule was not valid. In the interview, a different interpretation emerged and we concluded that the student understood the theory, but not the language of the question that had been posed:

Interviewer : Is 86 divisible by 2?

Student: Yes. [She showed that 86 divided by 2 was 43.]

Interviewer: Is 43 divisible by 2?

Student: No.

Interviewer: So what does that tell you?

Student : That it's not divisible by 8.

Other issues related to the students' understanding surfaced in the interviews. Some students, for example, did not connect the theory to what they already knew. One student , explained: "I don't use it. It doesn't make sense and I think its just too long." Yet when she was asked to factor 64, she created a factor tree by dividing by 2 to get 32, and then dividing 32 by two to get 16, and so on. When the interviewer attempted to see if she would connect her technique to Blayne's theory she responded: "I don't know, mine just seems a little bit easier than Blayne's. . . . maybe because I've been doing mine and I haven't been doing his." The student claimed that dividing an even number by two "only works really if you are trying to make a factor tree." Although she used the method in the context of the factor tree, she refused to utilize it in the context of divisibility by eight.

The Role of Discourse

The discourse made a difference in the way the student rule developed. One example of this influence was that the effect of using large numbers in student examples was to push the ideas outside the students' comfort zones to create a problem-solving situation for which they did not have a clear method of solving. The discussions, however, were sprinkled with instances of language errors and procedural errors related to division, factors, and multiples that appeared to create some confusion. These difficulties point to a need to practice "talking mathematics" in order to coordinate thoughts and words in ways that communicate the thoughts. Thus the classroom discourse provided an opportunity for students to develop their skills in using mathematical language.

Overall, the discourse functioned in a positive way for students, especially for Blayne. He began to consider his ideas in new ways, to refine the language that he used, and to connect the rule to concepts such as factors and powers. The discourse also created some confusion for Blayne. On the initial survey he indicated that he did not believe that his own theory was always true. In an interview he explained that this response was a result of one of the examples given in class, but that he had since changed his mind.

Student misunderstandings that occurred seemed to be attributable to a lack of closure; that is, the discourse did not provide a definitive conclusion about what dividing by 2 three times implied. Failures to resolve some of the issues raised in the discussions can be related to what Edwards and Mercer (1987) refer to as learning failures related to a failure of context. They suggest that "'learning failures' are not necessarily attributable to individual children or teachers, but to the inadequacies of the referential framework within which education takes place. In other words, they are failures of context" (p. 167). Failures of context occurred because the discourse failed to adequately differentiate ideas, particularly to differentiate meanings of some of the words. For example, in the class discussions, the word *even* had two different meanings. It was first used to narrow the set of numbers to "even numbers." To be divisible by two a number must be even. It was also used to mean that there was no remainder when one divided. Confusion of these meanings leads to different conclusions only on the third division when testing Blayne's rule for divisibility by eight.

Differentiation of meaning was important for the word *worked* because the rule works in two ways: The rule tells you that a number is divisible by eight or that a number is not divisible by eight. Many of the students used only the first meaning. A complex discourse structure is implicit in the rule; three different if-then statements, for example, are relevant to understanding the rule. In this study, the rule was explained via examples. The complexity of the implicit discourse structures was not part of the discussion. We conclude that more direct attention to language structures is needed in order for students to participate effectively in classroom discourse that is focused on validating mathematical ideas. If students are going to discuss each other's theories and validate the correctness of the mathematics on their own then thev need to begin to have some understanding of mathematical language structures.

Finally, as noted by two students, *efficiency* is a weak purpose for the study of divisibility by eight or any other divisibility rule. The context for the discussion of divisibility by eight in this classroom was that of validating the rule, not just using it. Within this context, students could connect ideas to factors and powers of numbers, to generalize the structure of the rule to other numbers, and to develop an understanding of mathematical language. A strong purpose for including divisibility rules in the curriculum is to build a greater understanding of numbers and number relationships.

References

Edwards, D. & Mercer, N. (1987). *Common knowledge: The development of understanding in the classroom.* New York: Routledge.

Glaser, B. G. (1978). *Theoretical sensitivity.* Mill Valley, CA: Sociology Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine De Gruyter.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life.* New York: Cambridge University Press.

Lave, J., Smith, S., & Butler, M. (1988). Problem solving as an everyday practice. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 61-81). Reston, VA: Lawrence Erlbaum Associates and The National Council of Teachers of Mathematics.

Vygotsky, L. (1978). *Mind in society: the development of higher psychological processes.* Cambridge, MA: Harvard University Press. (Edited by Michael Cole, Vera John-Steiner, Sylvia Scribner, & Ellen Souberman)

THE EFFECTS OF WRITING TO LEARN MATHEMATICS ON THE TYPES OF ERRORS STUDENTS MAKE IN A COLLEGE CALCULUS CLASS

Mary K. Porter, St. Mary's College
Joanna O. Masingila, Syracuse University

This study examined how engaging calculus students in Writing to Learn Mathematics affected the types of conceptual and procedural errors that the students made on their examinations. Students in two sections of an introductory college calculus course in Fall 1994 were the respondents in this study. We used Hiebert and Lefevre's (1986) characterization of conceptual knowledge as a framework to guide our examination of students' conceptual knowledge. To analyze the errors the students made, we developed a classification system and used some of the ideas and methods of Movshovitz-Hadar, Zaslavsky and Inbar (1987).

Many college students experience difficulty with doing mathematics (Kolata, 1988). It is not unusual to find students that use mathematical procedures with little or no understanding of the concepts behind these procedures (Hiebert & Lefevre, 1986; Schoenfeld, 1985). Some research (e.g., Oaks, 1988) has suggested that a student's difficulty in mathematics can be related to that student's beliefs that mathematics consists only of meaningless symbols and operations. Such students do not realize that there are concepts behind their procedures. They have a rote conception of mathematics that encourages them to learn only by memorizing, which ultimately prevents them from succeeding in mathematics (Oaks, 1990).

Some mathematics educators have suggested that students may be encouraged to change their conceptions of mathematics through the use of Writing to Learn Mathematics (WTLM) (e.g., Oaks, 1988), and that WTLM may benefit students' development of conceptual understanding (e.g., Gopen & Smith, 1990; Rose, 1989). However, no comparative research has been done to determine whether WTLM's proposed benefit to conceptual understanding is an actual benefit. Two comparative studies (Guckin, 1992; Youngberg, 1990) have investigated WTLM's proposed benefit to procedural ability; both of these studies focused on students in an algebra course.

Aim of the Study and Guiding Frameworks

The purpose of this study was to examine the effect of WTLM on the conceptual understanding and procedural ability of students in an introductory college calculus course. Hiebert and Lefevre (1986) characterized conceptual knowledge as that which is part of a network comprised of individual pieces of information and the relationships between these pieces of information. We are using Hiebert and Lefevre's characterization as a framework to guide our work in examining students' conceptual knowledge. To determine if WTLM helps students improve their conceptual understanding and affects their procedural ability, we are developing and will use an error classification system, based on the work of Movshovitz-

Hadar, Zaslavsky, and Inbar (1987), whose work serves as a guiding framework for our data analysis.

Methods and Data Sources

Students in two sections of an introductory college calculus course in Fall 1994 were the respondents in this study. Both classes were taught by the same instructor (the first author) with an emphasis on the mathematical concepts relevant to the course. One class, the WTLM group, participated in writing activities both inside and outside the class. The other class, the comparison group, were not assigned any writing activities. However, whenever the WTLM group was given a writing activity, the comparison group was given an activity that involved the same concepts as the WTLM's activity. Activities from both classes were discussed in class and assessed by the instructor.

Students in the WTLM group participated in a variety of writing activities. Occasionally, the students were given impromptu writing prompts during class time, to which they were asked to respond in writing. However, because class time is limited, the students were also given writing activities that were completed outside of class. The WTLM students were asked to write about topics related to course concepts and procedures. In these writing activities, the students were asked to explain course ideas in their own words, to discuss the relationship between course concepts, and to think, on paper, about concepts and procedures of the course. Students were also asked to reflect, in writing, on their study habits and performance in the course, and about the beliefs they hold about mathematics.

Some examples of writing activities are as follows:

- Explain to a friend, in writing, what a function is.
- What is a derivative?
- Why would someone want to find a derivative?
- How are Rolle's Theorem and the Mean Value Theorem related?
- Explain the First Derivative Test. Why does it work?
- What is the best way to study for a mathematics class? Why?
- Discuss your reaction to your performance on the test. Discuss ways in which you could improve your preparation for and performance on the next test.

The comparison group did not participate in the writing activities. However, whenever the WTLM group was given a writing activity (generally twice a week), the comparison group was given an assignment or quiz (graded or not, depending on whether the WTLM group's activity was graded) that will involve problems of the same content as the WTLM group's writing activity. For example, when the WTLM group was asked to describe, in writing, all of their thoughts and actions as they attempted to solve a certain homework problem, the comparison group was asked to solve the same problem and be prepared to discuss their thoughts and actions. Both groups received feedback on their work through written comments and discussion in class.

The data for the study consist of student responses from both classes on three in-class examinations and one final examination. All examinations were identical for both classes. The examinations included both routine exercises and nonroutine problems. We used ideas from the error classification system developed by Movshovitz-Hadar et al. (1987) as a basis for our data analysis. Movshovitz-Hadar et al. developed a classification system for errors in secondary mathematics (not including calculus). They classified student errors according to the following six categories: (a) Misused Data, (b) Misinterpreted Language, (c) Logically Invalid Inference, (d) Distorted Theorem or Definition, (e) Unverified Solution, and (f) Technical Error (Movshovitz-Hadar et al., 1987). We used this model, and Hiebert and Lefevre's (1986) framework of conceptual understanding, as a starting point, and described categories that emerged from our data for classifying students' errors in calculus, which has not been previously done. We analyzed the students' errors in a qualitative manner that Movshovitz-Hadar et al. called *constructive analysis*.

Findings

Discussion of the Categories that Emerged

By analyzing three midterm examinations and the final examination at the end of the semester, we collected 1,241 errors that we considered for this study. We noted 636 other errors but did not categorize these since we were concentrating on students' conceptual and procedural understanding of calculus ideas and these errors were not specific to calculus and involved mathematics content the students were taught in previous courses. We classified the 1,241 errors into the following categories: (a) Procedural, (b) Conceptual, and (c) Indeterminate. We will describe each category and give its characteristic elements. In order for an error to fit in a certain category, it must meet the criteria for at least one characteristic element.

The *Procedural Error* category consists of errors involving procedural knowledge, as defined by Hiebert and Lafevre (1986); procedural knowledge is composed of two parts: (a) the symbols and syntax of mathematics, and (b) the rules, algorithms, and procedures for performing mathematical tasks. The two parts of procedural knowledge are incorporated in the three characteristic elements of the Procedural Error category. The first characteristic element is that the error violates one or more of the syntactic rules for writing mathematical symbols in an acceptable way. The second characteristic element involves writing a symbol incompletely or improperly. Note that this does not include valid mathematical terms or symbols that are *used* improperly but written correctly. An example of this is from a student who wrote "$\lim_{x \to \infty} =$" without using a function, f. The third characteristic element is that there is an error in the statement of or use of a rule, procedure, or algorithm used for completing mathematical tasks in a step-by-step, linear fashion. Note that this does not include errors in selecting an appropriate procedure or in evaluating the outcome of a procedure. An example illustrating this element is a student who used a distorted version of the quotient rule in calculating a derivative.

The *Conceptual Error* category consists of errors involving conceptual knowledge, as defined by Hiebert and Lefevre (1986); conceptual knowledge is that which is part of a network comprised of individual pieces of information together with the relationships between these pieces of information. We have determined eight characteristic elements for this category.

The first characteristic element of this type of error is that a procedure that is inappropriate for the problem at hand has been selected. For example, part of the solution for a problem involved finding the derivative of a function but a student found the limit of the function instead. The second characteristic element is a failure to reject an answer that is unreasonable or whose incorrectness could have been discovered by checking. An example of this is the student who determined that a particular circle had a radius of -4. We developed these two characteristic elements based on Hiebert and Lefevre's (1986) discussion of errors that involve conceptual knowledge that is associated with a procedure: "Conceptual knowledge, if linked with a procedure, can monitor its selection and use and can evaluate the reasonableness of the procedural outcome" (p. 12).

The third characteristic element is "translating an expression from natural language into a mathematical term or equation that represents a relation different from the one described verbally" (Movshovitz-Hadar, Zaslavsky & Inbar, 1987, p. 10) or vice versa. The fourth characteristic element is "designating a mathematical concept by a symbol traditionally designating another concept" (Movshovitz-Hadar, Zaslavsky & Inbar, 1987, p. 10) or referring to a mathematical concept using language traditionally used in reference to a different concept; for example, using $f'(2x-4)$ to mean $\frac{d}{dx}(2x-4)$. The fifth characteristic element is "incorrectly interpreting graphical symbols as mathematical terms or vice versa" (Movshovitz-Hadar, Zaslavsky & Inbar, 1987, p. 10) or incorrectly interpreting mathematical symbols. An example illustrating this element is a student who interpreted $f'(3) = -8$ as the point (3, -8). The third, fourth and fifth characteristic elements are based on Movshovitz-Hadar, Zaslavsky and Inbar's (1987) Misinterpreted Language category. They describe their error category in the following way: "This category includes those mathematical errors that deal with an incorrect translation of mathematical facts described in one (possibly symbolic) language to another (possibly symbolic)" (p. 10).

The sixth characteristic element of this type of error is making a logically or conceptually invalid inference. That is, invalidly drawing new information from information that was previously given or inferred (Movshovitz-Hadar, Zaslavsky & Inbar, 1987). For example, a student who was given the statement of the Intermediate Value Theorem—"if f is continuous on the closed interval [a, b], then f takes on all f(x) values between f(a) and f(b)"—was then asked "Can the theorem be used to show that f is continuous?" The student responded that this statement of the theorem could be used to show that f is continuous. Another example of this type of error occurred in a problem that required the student to find the absolute maximum value of a function. The student found a critical value (x) for the function and claimed that the function reached a maximum at this x-value without actually determining where f increased and decreased (or any other evidence). The seventh characteristic element is making a statement without providing sufficient motivation for it or explanation of the reasons why the statement is true. Our sixth and seventh characteristic elements are based on Movshovitz-Hadar, Zaslavsky and Inbar's (1987) Logically Invalid Inference category: "In general, this category includes those errors that deal with fallacious reasoning and not with specific content" (p. 10).

The eighth characteristic element of this type of error is making a statement or giving an answer that contradicts or neglects a nonprocedural (in the sense of Hiebert & Lefevre, 1986) principle, definition, or theorem. For example, a student did not list certain x-values as points of discontinuity even though they were points of discontinuity. Another example of this type of error is a student who stated that the limit of a function existed even though the right-hand limit did not equal the left-hand limit. Our eighth characteristic element is related to the Distorted Theorem or Definition error category of Movshovitz-Hadar, Zaslavsky and Inbar (1987) that contains errors concerning "the distortion of a specific and identifiable principle, rule, theorem, or definition" (p. 11).

The *Indeterminate Error* category consists of errors that involved (a) both procedural and conceptual knowledge and where it was not possible for us to categorize the error as predominantly procedural or predominantly conceptual, or (b) neither procedural nor conceptual knowledge. This error category, and examples of error we classified as indeterminate, will be discussed more fully during the presentation.

Connection Between WTLM and the Error Categorization

Rose (1989, 1990) has identified a variety of perceived benefits of writing in mathematics. The benefits she categorized as beneficial to students as writers included, among others, that writing can (a) promote understanding, (b) facilitate reasoning and problem solving, (c) help generate meaning, (d) reveal what was misunderstood, (e) stimulate the posing of questions, (f) promote independent learning, and (g) help retention of content. It was our intent in this study to examine whether some of these perceived benefits are actual benefits. Thus, we explored whether students who were engaged regularly in WTLM over the course of a semester would make fewer and/or a different type of conceptual and procedural errors.

At the present, we have categorized the 1,241 errors into the three categorized that emerged from the data and were supported by the frameworks guiding our study. Up until now, all the data has been anonymous. We are now beginning to examine the connection between WTLM and the type and frequency of errors. This will be discussed in detail during the presentation.

Significance

This study adds to the growing body of research on WTLM in an important way by addressing the lack of comparative research on the proposed benefits of WTLM. This study also yields information that is valuable to educators who seek ways to improve students' conceptual understanding.

References

Gopen, G. D., & Smith, D. A. (1990). What's an assignment like you doing in a course like this?: Writing to learn mathematics. *The College Mathematics Journal, 21*, 2-19.

Guckin, A. M. (1992). The role of mathemátics informal writing in college mathematics instruction. (Doctoral dissertation, University of Minnesota, 1992). *Dissertation Abstracts International, 53*, 1435A.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kolata, G. B. (1988). Calculus reform: Is it needed? Is it possible? In L. A. Steen (Ed.), *Calculus for a new century: A pump, not a filter*. MAA Notes No. 8 (pp. 162-168). Washington, D.C.: Mathematical Association of America.

Movshovitz-Hadar, N., Zaslavsky, O., & Inbar, S. (1987). An empirical classification model for errors in high school mathematics. *Journal for Research in Mathematics Education, 18*, 3-14.

Oaks, A. B. (1988). The effects of the interaction of conception of mathematics and affective constructs on college students in remedial mathematics. (Doctoral dissertation, University of Rochester, 1987). *Dissertation Abstracts International, 49*, 54A.

Oaks, A. B. (1990). *Writing to learn mathematics: Why do we need it and how can it help us?* Paper presented at the Annual Meeting of the Association of Mathematics Teachers of New York State, Ellenville, New York.

Rose, B. J. (1989). Using expressive writing to support the learning of mathematics. (Doctoral dissertation, University of Rochester, 1989). *Dissertation Abstracts International, 50*, 10A.

Rose, B. J. (1990). Using expressive writing to support mathematics instruction: Benefits for the student, teacher, and classroom. In A. Sterrett (Ed.), *Using writing to teach mathematics* (pp. 63-72). Washington, D.C.: Mathematical Association of America.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press.

Youngberg, S. A. T. (1990). The effect of writing assignments on student achievement in a college level elementary algebra class. (Doctoral dissertation, University of Iowa, 1989). *Dissertation Abstracts International, 50*, 2819A.

A CASE STUDY OF SUPPORTING TEACHERS WITH MATHEMATICS REFORM IN LANGUAGE MINORITY CLASSROOMS

Yolanda De La Cruz, Northwestern University

Background of the Study

This study involves six non-Spanish speaking first-year teachers in predominately Spanish-speaking classrooms. These teachers felt unprepared to meet the challenges of teaching mathematical reform methods and especially in doing so with their language minority students. They contacted a former university professor for assistance. The six teachers and the university professor formed a bi-monthly support group. These meetings were used to share ideas and develop mathematical activities that would include the Spanish-speaking students.

Dealing with Language Differences

Math groups were formed by putting a Spanish-speaking student, a bilingual student, and English-speaking students in each group. The bilingual student acted as the language broker whose role was to bridge language barriers. Large charts and class-size manipulatives were used by teachers to facilitate concept understanding. Math centers that reinforced previously taught concepts allowed teachers to work with individual groups that required more assistance. Some of the math centers included activities such as dice games. Parent volunteers and cross-age tutors were available at times to work at the math centers.

Integrated Curriculum

Integrated curriculum units allowed teachers to include math in other content areas. This gave them enough math class time to "play around" with math content and adapt it to their students' needs. One unit connected math and literature. Teachers selected books on Aztec culture and had students write reports on the subject matter. For mathematics they built "Aztec Pyramids" using blocks to make buildings and then calculated the area and perimeter of each model. Science and math were integrated in a body unit where functions of the human body were studied. In mathematics class students measure limbs and recorded all the measurements to find the classroom average and the mean.

Daily surveys created material for graphs. Students worked in small groups to decide how to represent, analyze, and interperet data to the class. Teachers found out more about the personal lives of each student through these daily surveys.

Conclusions

This study clearly indicates that teachers need more support in creating classrooms that will meet the diverse needs of their students as well as use new reform teaching practices. Language does not have to be a barrier so that Spanish-speaking students have only lower-level cognitive experiences if brokers exist for most purposes and if students will work in small groups.

THE SOUND OF SILENCE: REFLECTIONS ON A COOPERATIVE GROUP'S PROBLEM SOLVING INTERACTIONS

Michael D. Hardy, The Florida State University

In "Interaction and Children's Mathematics," a paper presented at the 1993 AERA annual meeting, Steffe relays a tale of two boys working cooperatively on a computer to solve a problem but engaging in very little dialogue. He goes on to argue that despite the scarcity of verbal exchanges, the boys influenced one another via their individual interactions with the computer. This might lead one to wonder, "If dialogue in a larger cooperative group were sparse, would the participants tend to influence one another's learning in a positive manner?" This question is of importance because verbal interaction is often recognized as the source from which cooperative groups draw their power to facilitate learning. Although discourse is a, if not the, major catalyst of learning within a cooperative setting, neither language nor social interaction is limited to the spoken medium. Accordingly, there is no reason to assume that limited verbal interaction prevents the members of a cooperative group from serving as a positive influence on one another's learning. Having been a member of a cooperative group in which verbalizations were often sparse, my and my partners' reflections, on our problem solving interactions, may support broadening the traditional conception of a successful cooperative group.

Silence was commonplace in our group, particularly in the early phases of the problem solving process. There were a myriad of reasons for this, among them pride, competition, mutual respect, a desire to "conquer the problem," the need to internalize problems, and simply a limited need for speech. Neither the silence nor the mild competition prohibited communication which occurred both verbally and nonverbally. Further, our efforts to communicate were facilitated by mutual respect, the diversity of talents within the group, and compatibility of both relevant knowledge and thought processes. Moreover, the frequent periods of silence did not prevent us from influencing one another. In light of this, I conclude, as did Steffe, that limited verbalizations do not preclude the occurrence of meaningful and influential interactions. Accordingly, educators need to take care to construct concepts of successful cooperative groups which are versatile enough to allow some groups which engage in limited verbal interaction to be characterized as successful.

Reference

Steffe, L. P. (1993, April). Interaction and children's mathematics. In Children's construction of the rational numbers of arithmetic. Symposium at the annual meeting of the American Educational Research Association, Atlanta, GA.

THE CONSTRUCTION OF MATHEMATICAL MEANING IN BILINGUAL CONVERSATIONS

Judit Moschkovich, Institute for Research on Learning

While several studies have focused on discourse in monolingual mathematics classrooms (Cobb, Wood and Yackel, 1993; Pimm, 1987), researchers have only recently begun to consider mathematical communication in language minority classrooms (Brenner, 1994; Khisty, McLeod, and Bertilson, 1990). Despite the steadily increasing population of American students, estimated to be 5 million, who are classified as limited in English proficiency (one million of these in California, a large percentage of them Latinos), there has been little research addressing these students' needs in mathematics classrooms.

The poster presents the preliminary analysis of research exploring how Latino students construct mathematical meaning during bilingual (Spanish and English) conversations. The study focuses on the social, linguistic and material resources that support the construction of meaning and the refinement of students' descriptions and explanations. The poster summarizes the preliminary analysis of videotaped conversations between secondary students and examines: 1) how the two languages (Spanish and English) and registers (everyday and mathematical) serve as resources or obstacles for constructing meaning; and 2) how different mathematical activities and representations structure the use of English and Spanish.

In general, the research on language and learning mathematics presents a view of students as facing several discontinuities: from first language to second language, from social talk to academic talk (Cummins, 1981), and from the everyday to the mathematics register (Halliday, 1978; Pimm, 1987). Rather than seeing these as discontinuities, I take the perspective that students' construction of knowledge is socially and materially situated (Lave and Wenger, 1991)-that is, viewing what students are doing as they learn mathematics as constructing meaning while using the social and material resources available to them.

References

Brenner, M. (1994). A communication framework for mathematics: Exemplary instruction for culturally and linguistically diverse students. In B. McLeod (Ed.), *Language and Learning: Educating linguistically diverse students*. Albany: SUNY Press.

Cobb, P., Wood, T., and Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In N. Minick, E. Forman, and C. Stone (Eds.), *Education and Mind: Institutional, social and developmental processes*. New York: Oxford University Press.

Cummins, J. (1981). The role of primary language development in promoting educational success for language minority students. Evaluation, Dissemination and Assessment Center, California State University, Los Angeles.

Halliday, M. A. K. (1978). Sociolinguistics aspects of mathematical education. In M. Halliday, *The social interpretation of language and meaning*. London: University Park Press.

Khisty, L.L., McLeod, D., and Bertilson, K. (1990). Speaking mathematically in bilingual classrooms: An exploratory study of teacher discourse. Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Education, Vol. 3, 105-112. Mexico City: CONACYT.

Lave, J. and Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, England: Cambridge University Press.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Routledge.

Probability and Statistics

CONSTRUCTING STATISTICAL UNDERSTANDING USING MATHEMATICAL STORYTELLING

Susan Prion, University of San Francisco
Mathew Mitchell, University of San Francisco

This paper reports on the effect of a learning method designed to improve students' statistical understanding. The tool is a mathematical "storytelling" process aimed at helping students to construct their own meanings of selected statistical techniques through an innovative and provocative application of the statistical analysis. The storytelling device, called a *Think Paper*, is given to graduate-level statistics students as a weekly homework assignment.

The NCTM and others have encouraged relating new mathematical material to previously acquired student concepts and experiences. One novel way to do that is by asking students to create their own mathematical stories using those concepts. The purpose of the storytelling was to promote both connectedness *and* active learning within students. The results of this storytelling activity was hypothesized to be twofold: (1) students would learn more effectively, and (2) teachers would have a more reliable source for monitoring misconceptions and errors among students. If this storytelling process provides a reasonable and rich measure of student understanding, then storytelling via Think Papers may also provide an important source for improving instruction and remediation in the classroom.

Think Papers were assigned to push students to connect new statistical concepts presented in class with their own knowledge. All *Think Papers* were structured around some sort of controversial issue. Students were provided with some statistical results to help them answer a question concerning the controversy. *Think Papers* were purposely designed so that there was no "right" answer in terms of the social science controversy presented. However, each *Think Paper* contained at least one possible probe for students' misconceptions regarding the interpretation of the numbers presented. Each Think Paper was given after an introductory class about a specific statistical concept over a two semester statistics course. Students responses were limited to 2 written pages.

Since students were required to write a short essay, their responses were much richer and illuminating regarding any "deep understanding of the concept that would be indicated through more traditional measures such as practice calculation problems or multiple-choice questions. This paper illustrates key findings from our initial analysis of these student products.

DEVELOPMENT OF CONCEPTUAL UNDERSTANDING IN DESCRIPTIVE STATISTICS

Mary M. Sullivan, Curry College

Literature that discusses differences between novices and experts describes fields ranging from chess to physics. Researchers have studied novice learners in many contexts including experimental design (Fenker, 1975; Goldsmith, Johnson, & Acton, 1991; Magnello & Spies, 1984), and statistics related to research (Fenker, 1975). Some researchers employed methodology and analysis that resulted in a visual representation of the developing understanding (Fenker, 1975; Goldsmith et al., 1991); however, they did not describe their curriculum or teaching method.

The presenter recently completed a research study in which conceptual understanding in descriptive statistics was analyzed after student-centered, activity-oriented instruction. The four-week curriculum comprised the first part of a semester-long course in undergraduate statistics for non-majors. Its goal was to decrease the amount of lecture instruction and increase the level of active participation by students through hands-on activities, large and small group discussion, and collaborative learning experiences. Faculty who taught the curriculum supported active learning experiences in elementary statistics.

Through use of a word association task comprised of all possible pairs of 13 descriptive statistics concepts, which students rated for relatedness between them, and multidimensional scaling techniques, geometrical representations of student understanding were created before and after instruction and compared to the geometric representation of understanding by faculty experts. Analysis of student configurations reveals that while they develop understanding of central tendency, variability, and data characteristics concepts, their organization of concepts lacks the tight structure apparent in the faculty representation. In addition, the student configurations reveal that they fail to conceptualize broader ideas relative to a distribution of data in the early part of the course. Since descriptive statistics concepts are the foundation upon which concepts related to probability distributions, sampling, and inferential statistics rest, the analysis of the development of basic descriptive statistics concepts points to the need to create oppportunities to revisit elementary concepts throughout the course so that understanding continues to grow.

References

Fenker, R. M. (1975). Ther organization of conceptual materials: A methodology for measuring ideal and actual cognitive structures. *Instructional science, 4*, 33-57.

Goldsmith, T. E., Johnson, P. J., & Acton, W. H. (1991). Assessing structural knowledge. *Journal of Educational Psychology, 83*, 88-96.

Magnello, M. E. & Spies, C. J. (1984). Using organizing concepts to facilitate the teaching of statistics. *Teaching of Psychology, 11*, 220-223.

TELL ME A (STATISTICAL) STORY

Mathew Mitchell, University of San Francisco
Susan Prion, University of San Francisco

This poster presents a selection of student-designed products from an experimental curriculum titled *Statistical Thinking*, and reports on the effect of the curriculum on students' statistical understanding. The primary tool used in the curriculum was the *Teaching Sheet*, a statistical storytelling device used in helping students construct their own meanings and understanding for statistical selected techniques.

The *Statistical Thinking* curriculum was designed for tenth grade students. The curriculum covered the concepts of central tendency, variance, z-scores, effect sizes, correlation, and linear regression. Most of the curriculum was computer-based and used the spreadsheet program Microsoft EXCEL®. The curriculum was implemented over an 11-week span during the Spring of 1995. The pilot high school consisted of students from low to middle income families; 85% of the students were non-white.

Students created two kinds of products with EXCEL®. First, they acquired and practiced statistical concepts by building a statistical playground that would calculate the particular test under study. (Note: students were not allowed to use any of the in-built statistical functions provided by EXCEL® except for SUM, MAX, MIN, and COUNT.) Second, the students were challenged to construct a teaching sheet, an interactive spreadsheet developed for use by a "novice" in order to learn the statistical concept.

Teaching sheets were theorized to be effective because students learn more when they use their new knowledge to teach others (e.g. Benware & Deci, 1984). Students needed to include three key ingredients in their *teaching sheets*: (1) text, sound, or graphics that provide a storyline explaining the purpose and importance of the statistical concept, (2) a well organized number playground where the user could try out various combinations of data and see the effects on the resulting statistical calculation, and (3) a visual representation (in addition to the analytic representation) of the statistical concept.

The student products presented at the poster session provide strong evidence that "regular" students are quite capable of engaging successfully with the process of constructing *teaching sheets*. The poster will highlight the teaching sheets developed by previously low-achieving female students who seemed to do particularily well with this curriculum.

Reference

Benware, C. & Deci, E. (1984). Quality of learning with an active versus passive motivational set. *American Educational Research Journal, 21*, 755-765.

Problem Solving

THE DEVELOPMENT OF PROBLEM-SOLVING PROCESSES IN A HETEROGENEOUS EIGHTH GRADE ALGEBRA CLASS

Sidney L. Rachlin, East Carolina University

The primary purpose of this study is to evaluate the application of a process approach (Rachlin, Matsumoto, and Wada, 1992) for the teaching of algebra with a heterogeneous class of eighth grade students. The assessment is conducted by identifying the processes used by (above average, average, and below average) algebra students in solving standard and nonstandard problems ranging across a content × process × form matrix — (integers, fractions, polynomials) × (generalizations, reversibility, flexibility) × (expression, equation). To give some perspective to the analysis, the processes used by the eighth grade students are compared to processes identified by Wagner, Rachlin, and Jensen (1984) using the same series of interview tasks with ninth grade students in Georgia and Alberta.

Mathematical Form and Content in Algebra

Regardless of what content society ascribes to problem solving and algebra, there is a need for research on the learning and teaching of the curriculum at two levels — that of the students and that of the teachers. The algebra project of the University of Hawaii provides one example of how federal, state, and local funding have combined to support a decade of research on the design of an algebra curriculum.

Much of the content of elementary algebra appears in one of two forms: in *expressions* (combining or simplifying terms, operations on polynomials, operations on rational expressions, etc.) or in *equations* (solving equations and inequalities, graphing of functions, solving systems of equations, etc.). Both of these forms rely upon the use of *variables* (literal symbols: $x, y, z, \ldots$) for their written expression. The algebra tasks used in this study were designed to probe students' conceptual and operational understanding of variables, expressions, and equations.

Content

The mathematical content considered in this study included all of the topics in a typical Algebra I text, with the notable exception of the "standard" algebra word problems. Although all tasks in the study were presented to students in a verbal format, it was felt that the age/coin/mixture/distance problems of elementary algebra involved special translation problems that went beyond the scope of this project. As mentioned earlier, explicit consideration was given to including in the interview tasks problems involving the content areas of integers, rational numbers, and polynomials, as well as the operations of addition/subtraction and multiplication/division.

Psychological Processes in Learning Algebra

A basic premise of this study was that the learning of algebra, beyond the level of rote memorization of formulas and algorithms, can be regarded as a kind of problem-solving process. That is, even the application of formulas to "routine" textbook exercises involves some degree of problem-solving activity on the part of most students, at least initially. Thus, in addition to considerations of mathematical form and content, three well established problem-solving processes were used to guide the development of interview tasks — reversibility, generalization and flexibility (Krutetskii, 1976). Standard problems are used as foundation tasks upon which the process tasks are constructed. For example, the following sample tasks involve polynomials operations and equations:

	Algebraic Expressions	Equations
Standard	Multiply: $(2a + 3)(2a - 3)$.	Solve: $4a^2 - 9 = 0$.
Reversibility	Find the binomial which multiplied by $2a - 3$ equals $4a^2 - 9$.	Find an equation whose solutions are ±f(3,2) .
Ability to Generalize	Find 2 binomials whose product is a binomial ... a trinomial ... has 4 terms ... has 5 terms.	Find a quadratic equation whose solutions are proper fractions.
Flexibility	Can you find the binomial which multiplied by $2a - 3$ equals $4a^2 - 9$, another way?	Solve: $4a^2 - 9 = 0$. Solve: $4(a+1)^2 - 9 = 0$. Solve: $4(2a+1)^2 - 9 = 0$.

Reversibility

Krutetskii (1976) describes reversibility as "an ability to restructure the direction of a mental process from a direct to a reverse train of thought." For example, in the expression $a + b = c$, we might be given values for a and b, and be asked to find a value for c. The reversibility of this addition incorporates three variations: where the values of a and c are given and the value of b is to be found, where the values of b and c are given and the value of a is to be found, and where the value of c is known and both a and b are to be found. To possess complete reversibility of addition of whole numbers, children should be able to solve problems involving all three variations: $5 + b = 7$, and $a + 2 = 7$, and $a + b = 7$.

Correspondingly, a student who possesses complete reversibility of addition of polynomials should be able to solve the following three problems:

1) What polynomial added to $5x^2 + 3xy$ equals $3x^2 + y^2$?

2) The trinomial $2x^2 - 3xy + y^2$ added to what polynomial equals $3x^2 + y^2$?

3) Find two polynomials with at least one non-similar term such that their sum is $3x^2 + y^2$.

Generalization

Krutetskii (1976) considered the ability to generalize mathematical material to be on two levels: first, the ability to subsume a particular case under a known general concept and second, the ability to deduce the general from particular cases, to form a concept. This notion of generalization is commonly reflected in the ordered series of exercises found in most mathematics texts in which increasingly more complicated extensions of a form are made. For example, the following series of polynomials, algebraic fractions, and real numbers provides a generalization for addition:

1) Find three integers whose sum is -2.
2) Find two polynomials whose sum is $5x^2 + 2x + 4$.
3) Find two fractions whose sum is $\frac{3}{8}$.
4) Find two fractions whose sum is $\frac{2x + 7}{8}$.
5) Find two real numbers whose sum is 12.

Krutetskii's (1976) second level of the ability to generalize mathematical material is the ability to deduce the general from particular cases. For example, students' concepts of a difference of two squares are examined through their discussion of open-ended tasks such as: Find two binomials such that their product is a binomial.

Flexibility

Flexibility was identified by Krutetskii (1976) as the ability to switch from one level of thinking about a problem to another. Flexibility can be shown either within or across problems. Within problem flexibility refers to the ease with which a student switches from one method of solving a problem to another method of solving the same problem. How students perceive a problem shapes the approach that they will use to solve the problem. The various solution paths which a student selects establish the structure for the problem. For example, the task "What number divided by 24 equals $\frac{3}{4}$?" has a wide variety of appropriate structures depending on the way in which the task is perceived; e.g., as equivalent fractions, a proportion, a division problem, an equation, etc. A student who is unable to solve this problem as a division problem because of a lack of skill in operations with fractions may still solve it by thinking of the problem as a proportion. Interview tasks such as the following are used to investigate students' alternative ways to solve the same problem:

a. Solve the following equation for x: $7 - 5x = 32$.
b. Solve the equation above another way.
c. Write an equation like the equation above that has a solution of -4.

The ability to switch from one approach to another, more efficient, approach is a question of degree. Across problem flexibility refers to the degree to which a successful solution process on a previous problem *fixes* a student's approach to a subsequent problem. Many students solve the following equations without seeing a connection between them:

Solve each of the following for x:

a. $2x = 12$

b. $2(x + 1) = 12$

c. $2(5x + 1) = 12$

Design Of The Study

This investigation of the development of problem-solving processes in elementary algebra replicates the methodology of an earlier pair of studies conducted in Athens, Georgia with eight ninth-grade algebra students and in Calgary, Alberta, involving four ninth-grade algebra students. The data obtained in these earlier studies is used to represent a norm for *traditional* algebra programs. The present study contrasts the 92 hours of interviews from these studies with over 70 hours of interviews collected from ten high, average, and low achieving eighth grade algebra students in four heterogeneous classes.

Participants

A total of 4 boys and 6 girls (4 with above average, 4 with average, and 2 with below average achievement levels) were selected from four Algebra I classes taught by the same junior high school math teacher in a suburb of the greater Denver area. As an experiment, *all* eighth grade students in the school were enrolled in a concepts of algebra course. This course covered the content of beginning algebra using the text *Algebra I: A Process Approach* (Rachlin, Matsumoto, and Wada, 1992). All teachers using the text, including the special education teacher and a substitute teacher participated in 45 hours of inservice preparation for teaching by a process approach. Since the decision to participate in this experiment was not made until May of the year preceding the project, no effort was made to prepare the seventh grade students for taking algebra in eighth grade. After completing the concepts of algebra course, the students were tested to determine which students would be permitted to use the concepts of algebra course for their algebra credit and which students would follow this course with a year of traditional algebra. At the end of the year students who failed the concepts of algebra course were asked to take the Orleans-Hanna Algebra Readiness Test to determine if they were prepared to take algebra in ninth grade. With the exception of three students who refused to take the test, all students tested were measured as ready for algebra.

The Hawaii Algebra Curriculum

The University of Hawaii Algebra Learning Project designs instructional materials and methods to help students of *all* ability levels develop problem-solving processes as they learn algebra. A basic premise of the project is that if we are to meet the literacy needs of the future, students must do more than memorize formulas and get answers. They must learn to think mathematically and communicate their thinking. Hence this project alters the sequence of algebra content and *instructional methods to foster the development of understanding.*

The Hawaii Algebra Curriculum is:

- is based on research into how students think and learn.
- offers a problem-solving approach to algebra. Concepts are introduced through problem situations.
- allows students to construct their own methods to solve mathematical problems. There is more than one right way to solve a problem.
- offers students non-routine tasks to encourage the development of problem-solving processes such as reversibility, flexibility, and the ability to generalize.
- promotes open-ended inquiry appropriate for individual differences in any classroom.
- allows time for students to grasp concepts, make generalizations, and refine their skills.

Hawaii Algebra has been identified as a promising practice by the Laboratory Networking Program at the U.S. Department of Education's Office of Educational Research and Improvement. The program grew out of research into how students tackle problems. Students were given problems to solve and asked to think aloud as they tried to find different solution paths. The research confirmed the project's belief that students differ in the time they need to grasp a new topic.

The algebra curriculum has been redesigned to include many open-ended questions. As students discuss their approaches to solving homework problems, they gradually internalize the process of algebra. To allow time for this development, students are given one or two nonroutine problems from a topic every day for three to eight days, thus working simultaneously on several concepts each day. From then on, a topic is treated as a skill and reinforced through practice exercises in later problem sets.

Interview Tasks

This study provides a replication of two conducted ten years earlier by in Wagner, Rachlin, and Jensen (1984). The tasks and procedures used in the present study mirror those used earlier. The two populations are very different — the prior studies were conducted with students who took algebra by ninth grade, while the present study includes the total eighth grade student population. To better understand the nature of the role of a process approach in developing algebraic thinking, the processes used by the eighth grade students are compared to processes identified with ninth grade students in Georgia and Alberta.

The interview procedure was adapted from one used by Rachlin (1982) and refined by Wagner, Rachlin, and Jensen (1984). The tasks were given to a student one at a time, with only one problem on a sheet and plenty of room for the student to write. The students were asked to *think aloud* as they attempt to solve each problem. If the students lapsed in their verbalization, they were encouraged to tell what they were thinking. If a student appeared to be having a lot of difficulty, hints were provided. At first the hints were general (*What are you trying to find? What's giving you a problem?*), but if the frustration continued the hints increased in specificity. The hints ranged from pointing out a particular error to directed teaching of a new generalization, concept, or skill.

The interviews were flexible in design. No two interviews were alike. On the one hand, a problem was rarely left incorrect or incomplete. On the other hand, if the interviewer noticed something of interest in a student's response, the interviewer created new questions to follow the direction of the student's thought.

Analysis of data

The analysis of this study is too lengthy to be included in the confines of this abbreviated report. Transcripts of the experimental data have been coded and contrasted with the normed data provided by the earlier studies. What is unusual at this site is the attempt to have *all* eighth grade students (including special education students) taking algebra in un-tracked classes. The results from the study provide qualitative evidence of the strengths and weaknesses of this approach. Copies of the full report will be distributed at the presentation and are available from the author upon request.

References

Krutetskii, V.A. (1976). *The Psychology of Mathematical Abilities in School Children* (J. Kilpatrick & I. Wirszup, Eds.). Chicago: University of Chicago.

Rachlin, S.L. (1982). *The Processes Used by College Students in Understanding Basic Algebra*. Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, and Environmental Education (SE 036 097).

Rachlin, S.L., Matsumoto, A.N., and Wada, L.A. (1992). Algebra I: A Process Approach. *Honolulu, Hawaii: University of Hawaii Curriculum Research & Development Group.*

Wagner, S., Rachlin, S.L., and Jensen, R.J. (1984). Algebra Learning Project Final Report. *Athens, Georgia: University of Georgia Department of Mathematics Education.*

AN EXAMINATION OF THE RELATIONSHIP BETWEEN THE PROBLEM-SOLVING BEHAVIORS AND ACHIEVEMENTS OF STUDENTS IN COOPERATIVE-LEARNING GROUPS

Patrick Mwerinde, University of Delaware
Christine Ebert, University of Delaware

This study examined the problem-solving behaviors, strategies, and achievement of college students enrolled in a one-semester College Algebra and Statistics course, with respect to the content areas of quantitative literacy, connections between algebraic and graphical representations, and mathematical modeling. Four instructional units of this course were chosen - two in which the students were assigned to cooperative-learning groups and two in which the students worked independently. The findings suggest that students who work in cooperative learning groups clearly exhibit important problem-solving behaviors such as persistence and a willingness to explore alternative solutions; however, they still experience difficulty explicating the connections between mathematical actions and/or processes and the mathematical concepts.

Conceptual Framework

Cooperative-learning strategies have been credited with the promotion of critical thinking, higher-level thinking, and improved problem-solving ability of students. Current research that examines behaviors that occur during group problem-solving sessions seem to indicate that groups engage in behaviors that are similar to those exhibited by expert mathematicians when they solve problems (Artz & Newman, 1990; Schoenfeld, 1987); that is, they engage in monitoring their own thoughts, the thoughts of their peers, and the status of the problem-solving process. Researchers who have studied cooperative learning at the college level generally have found that students learn just as well as in more traditional classes and often develop improved attitudes toward each other and toward mathematics (Dees, 1991; Slavin, 1995; Brechting & Hirsh, 1977; Chang, 1977; Davidson, 1971; Olsen, 1973; Shaughnessy, 1977; Treadway, 1983). Although it is not clear which components of cooperative learning are responsible for improvements in higher-level thinking, attempts have been made to identify the components. One conjecture is that dealing with controversy may be such an element. Smith, Johnson, and Johnson (1981) report on a study in which they suggest that higher results on achievement and retention of the students in the "controversy group" may be attributed to the "cognitive rehearsal of their position and the attempts to understand the opponents position" (Smith, Johnson, & Johnson, 1981, p. 652). This work provides important information concerning the efficacy of cooperative-group learning and the key components which contribute to higher-level thinking. However, research which considers both the problem-solving behaviors, strategies, and achievement of college students enrolled in traditional courses also needs to be conducted. In this study, we not only examined the problem-solving behaviors, strategies, and achievement of college students assigned to cooperative learning groups, but designed problem-solving experiences consonant with the course curriculum that focused on the connections between mathematical actions and/or processes and the underlying mathematical concepts.

Methodology

The subjects chosen for this study consist of 108 students enrolled in four instructional units of College Algebra and Statistics at a major university located in the Mid-Atlantic states. Two experimental groups and two control groups were randomly selected. An attitude scale and a pre-test of algebraic ability were administered to both the experimental and control groups on the first day of class. In the experimental sections, students were assigned to cooperative learning groups based on their performance on the pre-test (each group contained four students - 1 high score, two middle scores, and 1 low score). In the control sections, students were told that they could work with fellow students on the various activities/labs, but were not specifically assigned to groups. Throughout the semester, problem-solving behaviors, strategies, and achievement were assessed through four tasks which focused on the connections between the mathematical actions and processes and the mathematical concepts. The first and fourth tasks took place in a regular classroom setting and consisted of problem sets devoted to the topics of quantitative literacy and modeling exponential functions. The second and third tasks took place in the computer laboratory setting and consisted of computer labs devoted to exploring the connections between algebraic and graphical representations of linear functions (set in the context of depreciation) and determining the best mathematical model for a particular set of data. Each of the four tasks were videotaped (some problem-solving activity was recorded for each group) and audiotaped (the problem-solving discourse was recorded for each cooperative-learning group within both experimental sections). In addition, the initial and final problem-solving sessions devoted to quantitative literacy and exponential modeling were also videotaped for the control sections. The written work that accompanied each of these tasks was also analyzed with respect to their ability to explicate the connections between actions and/or processes and mathematical concepts. At the end of the semester, the attitude scale with some additional open ended questions concerning cooperative learning groups was again administered to all of the students.

Results and Conclusions

The results of the students MSAT, the pre-test of algebraic ability, and the questions concerning the number of years of high-school mathematics and their previous university mathematics history were analyzed to determine between-group similarities and differences. Seventy-eight students comprised the experimental group and thirty students made up the control group. The results are summarized in the table at the top of the next page.

These results indicate that the base-line assessments were consistent within each group. The MSAT, pre-test of algebraic ability, and the number of years of high school mathematics all indicate that the experimental sections were more capable and more experienced mathematically than the control sections. In addition,

Table 1. *Baseline Data*

	MSAT	MSAT	Pretest	Yrs. HS	University	Math	History
Group	Mean	St. Dev.			1st Class	Remedial	Other
Ctrl. n=30	470	70.36	4.65	3.62	53%	37%	10%
Exp. n=78	485.14	88	5.30	3.93	64%	28%	8%

more of the students in the control sections had taken and passed the non-credit remedial algebra course prior to enrolling in the current course.

The attitude scale, a 10-item Likert scale administered both prior to instruction and at the end of the semester, assessed the students' views about learning in general, the role of the teacher, and whether they learn mathematics better while working alone or with other students. Prior to instruction, both groups favored working with other students as the better way to learn mathematics. At the end of the semester, students' responses to the question, "I found working in cooperative-learning groups to be (please elaborate)...," ranged along a continuum from "extremely enthusiastic" to" helpful, but..." to "not at all useful." The results of this question are indicated in the following table:

Table 2. *Responses Concerning Cooperative-Group Learning*

Categories of Responses	Extremely helpful	Helpful, but...	No comment	Not useful
Control Group	53%	28%	5%	14%
Experimental Group	60%	13%	7%	20%

Following instruction, both groups still favored working in cooperative-learning groups. However, significant differences emerged with respect to the role of the teacher. The number of experimental group members who strongly agreed that "the role of the teacher is to facilitate learning" increased dramatically. Members of the control group remained ambivalent concerning the role of the teacher. The comments of many of the students with respect to working in cooperative-learning groups are represented by this excerpt from the Attitude and Cooperative Learning Assessment.

> At first I didn't like working with other people because I usually study and work alone in order to memorize and teach myself information (which is hard to do with others). But by the end of the semester I enjoyed working with my group and I studied with 3 others for the final exam.

With respect to whether working in cooperative-learning groups has affected problem-solving strategies, the student writes:

> I have listened to and heard other strategies and learned new solving and thinking patterns.

The course grades indicate how well each group did with respect to the standardized achievement criteria and are recorded in the following table:

Table 3. *Final Course Grades*

Achievement Level	A or A-	B+,B,B-	C+,C,C-	D+,D,D-	F
Control	30%	33 1/3%	26 2/3%	3 1/3%	6 2/3%
Experimental	21%	29%	35%	14%	1%

It is interesting to note that the control group did much better in the course than the experimental group although all of the baseline assessments indicated otherwise. There are several possible explanations. One possible explanation is the differences between the two groups in terms of the number of students who took and passed the remedial algebra course prior to taking the current course. Another possibility centers around the differences between the problem-solving activities and/or labs and the standard exam questions.

Throughout the course, the results of the videotapes, audiotapes, and written work were analyzed to determine the nature of the problem-solving behaviors, strategies, and achievement of both groups. Of particular interest were the videotapes of the cooperative learning groups working on the lab devoted to examining the connections between graphical and algebraic representations of linear depreciation functions. In this laboratory activity, students were asked to estimate graphically, identify the graphical feature they utilized to answer the question, algebraically answer the question, and describe how the graphical and algebraic representations were related. Results indicate that those students in the cooperative-learning groups, like those of the "controversy group" identified by Smith, Johnson, and Johnson engaged in the type of mathematical discourse that would enable them to form connections between graphical and algebraic representations. Results of the written responses on this lab were significantly higher for the students in the cooperative -learning groups than those in the control group. However, students in the cooperative-learning groups still exhibited some difficulty explicating the connections between mathematical actions and/or processes and the mathematical concepts. Furthermore, the standardized assessments (multiple-choice questions with some free-response parts) did not emphasize forming these connections. All of these factors could contribute to the differences in the standardized achievement of the two groups.

These findings provide convergent evidence concerning both the problem-solving behaviors *and* achievements of cooperative learning groups and suggest the kinds of group activities which may facilitate higher-level thinking and improved problem-solving ability.

References

Artzt, A.F. & Newman, C.M. (1990). *How to use cooperative learning in the mathematics classroom.* Reston, VA: National Council of Teachers of Mathematics.

Brechting, M.C., & Hirsh, C.R. (1977). The effects of small-group discovery learning on student achievement and attitudes in calculus. *American Mathematics Association of Two-Year Colleges Journal, 2*, 77-82.

Chang, P.T. (1977). On relationships among academic performance, sex difference, attitude and persistence of small groups in developmental college level mathematics courses (Doctoral dissertation, Georgia State University). *Dissertation Abstracts International, 38*(7), 4002A.

Dees, R.L. (1991). The role of cooperative learning in increasing problem-solving ability in a college remedial course. *Journal for Research in Mathematics Education, 22*(5), 409-421.

Davidson, N. (1971). The small-group discovery method of mathematics instruction as applied in calculus (Doctoral dissertation, University of Wisconsin, 1970). *Dissertation Abstracts International, 31*(11), 5927A (Tech Rep. No. 168). Madison: Wisconsin Research and Development Center for Cognitive Learning.

Olsen, J.C. (1973). A comparison of two methods of teaching a remedial mathematics course at the community college (Doctoral dissertation, Utah State University). *Dissertation Abstracts International, 34*(12), 7522A.

Schoenfeld, A.H. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Ed), *Cognitive Science and Mathematics Education.* Hillsdale, NJ: Lawrence Erlbaum.

Shaughnessy, J.M. (1977). Misconceptions of probability: An experiment with a small-group activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8*(3), 295-316.

Slavin, R.E. (1995). *Cooperative Learning.* Boston, MA: Allyn & Bacon.

Smith, K., Johnson, D.W., & Johnson, R.T. (1981). Can conflict be constructive? Controversy versus concurrence seeking in learning groups. *Journal of Educational Psychology, 73*(5), 651-663.

Treadway, R.T. (1983). An investigation of the real-problem-solving curriculum in the college general education mathematics course (Doctoral dissertation, University of North Carolina at Greensboro). *Dissertation Abstracts International, 45*(1), 108A.

HELP-SEEKING WHILE PROBLEM SOLVING: ADULT CAREGIVERS AND THE ZONE OF PROXIMAL DEVELOPMENT

Vicki Zack, St. George's School and McGill University

This investigation is part of an on-going, larger study which is looking at joint activity and appropriation of new understandings in an inquiry math classroom setting. Instances of help-seeking which occurred while children were endeavoring to solve non-routine problems at home were analyzed. Considered in this study were (1) the kind of help sought by the child, (2) the kind of help offered by the adult, (3) how extensive the help was, and (4) the sense the child made of the help. The data sources included the children's writing in their math logs, and their explanatory presentations subsequently given to peers in class. Findings suggest that (1) the children sought and received help predominantly with respect to problem-solving strategies and mathematical concepts, (2) interactions with adults were evenly distributed among all the students, the adept, moderately adept, and less adept in that some did ask for help while others rarely or never did, and (3) the less adept children were less specific in describing the kind of help sought/received while the more adept children's' requests were more specific and focused.

The overall goal of the on-going study of which this report is a part, is to investigate "how much students appropriate from interactions with others so that they can claim this knowledge as personally meaningful" (Roth, 1995, p. xv). The specific focus of the present investigation is on the child-adult interaction in the context of help-seeking behavior. This is in accordance with Bussi's (1994) recent assertion regarding the need to give attention to the role of the adult vis-a-vis joint activity in problem solving. At the same time, Webb (1989) has pointed out the lack of research on help-seeking. Nelson-Le Gall and her colleagues (Nelson-Le Gall, Gumerman, & Scott-Jones, 1983) have spoken of help-seeking as a problem-solving skill. They suggest that it is vital to look at who seeks help, what type of help is sought, and at what point in the problem-solving process help is sought, and insist that these are all central questions for theories of problem solving (p. 280). Some researchers have concluded that help-seeking denotes dependence, while others have felt that it is a sign of initiative-taking (Nelson-Le Gall et al., 1983). Help-seeking may well be tied up with metacognitive awareness. Do the children know that they do not know? and Do they know what it is they need to know? (Fitzgerald, 1983). Thus for this component of the study, I considered specifically the help-seeking which occurred while the children were solving non-routine mathematics problems at home, and when help was sought, looked at the ensuing written explanations done at home and the oral explanations given by the children in class.

The theoretical framework of the study draws upon Vygotsky's view of the interaction which leads to learning. He maintains that it occurs on two planes, first

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Several aspects discussed in the paper were elaborated in the course of discussions with Barbara Graves. Barbara's contributions have been pivotal to my growth. This research was supported by a Social Studies and Humanities Research Grant from the Government of Canada #410-94-1627.

the social and then the psychological plane; "first it appears between people as an interpsychological category, and then within the child as an intrapsychological category" (Vygotsky, 1981, p. 163). One vital component derived from this basic tenet is that of the *zone of proximal development* . As a theoretical construct the zone of proximal development (ZPD) has been used in a variety of ways to date. Most commonly it has been operationalized in conjunction with an apprenticeship model (see for example, Lave, 1977, Rogoff, 1990, or Cazden, 1981) in which the adult expert is guiding in a step-by-step way, and relinquishing control by degrees when the learner is 'ready'. In contrast to this view, a number of researchers support the premise that any help which leads to conceptual change in the ZPD can be considered to have assisted growth (Tharp & Gallimore, 1988). In the context of this study, and keeping with this latter view, I regard the zone of proximal development as a conceptual space in which new understandings can arise as a result of joint activity.

The school and classroom site of this study is a community of practice of *inquiry math (*Richards, 1991), in which the children are expected to publicly express their thinking and engage in mathematical practice characterized by conjecture, argument and justification (Cobb, Wood, & Yackel, 1993, p. 98). Therefore any analysis of what occurs in the joint activity in this classroom differs in essential ways from much of what has been featured in the literature. Admittedly, the children's work in joint activity with adults and peers may perhaps incorporate some features of the apprenticeship model. However, the activity in a problem-solving environment is more diffuse and in some ways more complex than that in a teacher-centered textbook-based classroom setting. For example, the primary focus is not on the learning of procedural patterns, as was the case in regard to the long-division algorithm which was the focus of learning in the 'construction zone' in the Newman, Griffin & Cole (1989) classroom study. In our setting, the children are encouraged to grapple with non-routine problems, and are seen to deliberate and solve the problems using diverse idiosyncratic approaches (Zack, in press). In our community, there are many potential sources available to the child—peers in class or at home via telephone, the teacher in class, and the caregiver at home. These multiple sources not only enrich the child's environment but also increase its complexity by posing additional challenges as children attempt to understand alternate approaches (Zack, 1993).

Data collection and data analysis

I have been a classroom teacher and researcher in a Grade 5 (10-11 year-olds) classroom for the past 6 years. Problem solving is at the core of the mathematics curriculum. In addition to the in-class problem-solving sessions, the children also work on one challenging mathematics problem at home each week (Problem of the Week), and are expected to write in detail in their math log about what they did as they worked on the problem. The children are told they must work hard alone on the problem at first. They are asked not to seek help, and the parents are asked to refrain from assisting. However, if the children decide they must seek help, they are asked to write about their difficulties, about whom they approached for help, and about how the person helped them. Subsequently in class, each child discusses her/his solution with a partner, and then in a foursome; the problem solutions are then discussed in a group of twelve. The in-school problem-solving sessions are videotaped.

The data analyzed for this paper were drawn from two years worth of Problems of the Week, 1993-1994, and 1994-1995; 8 problems which had been assigned in both years were chosen for the analysis. The class size each year was 25 and 26 children respectively. Considered in this study were (1) the kind of help sought by the child or (2) offered by the adult, (3) how extensive the help was, and (4) the sense the child made of the help (intrapsychological), as seen in the writing in the math logs, and the explanatory presentations subsequently given to peers in class. I looked for instances of the following: (a) the child had a correct answer and had sought help; (b) the child had a correct answer and had not sought help; (c) the child had an incorrect answer and had sought help; (d) the child had an incorrect answer and had not sought help. In the cases where the child had a correct answer and had sought help, and that child subsequently shared her/his steps with a partner or small group, I looked at the videotape and at my focused observation notes to see whether that child was able to explain, and to justify her/his actions. The students know they are expected to go beyond just sharing their answer with each other; each child is expected to tell how she/he arrived at the answer, and to attempt to explain why it works.

I also looked to see at what point in the problem-solving process the child sought help at home. Polya's (1945) stages of clarifying, representing, solving, and checking were considered; instances of help-seeking from adults were almost exclusively related to stages of representing and solving. In order to analyze further the kinds of help the children reported they had been given by the adults, I used categories which had emerged from a previous study which dealt with children's reports of the kinds of help peers gave in class (Zack, 1994). The categories were as follows:

Category #1:	parameters or conditions of the problem
Category #2:	factual, straightforward information
Category #3:	problem-solving strategies (included as well diagnosing errors, getting started)
Category #4:	mathematical concept (e.g., fractions, decimals, percents)
Category #5:	essence or key idea in a problem
Category #6:	alternate solution (i. e. one which is simpler or aesthetically more pleasing).

Findings

Findings suggest that interactions with adults were evenly distributed among all the students, the adept, moderately adept, and less adept; some did ask for help, while others rarely or never did. Overall, the frequency with which children did go to caregivers at home for help was low. This may be due in part to the teacher's request that they do their best to work on the problem diligently on their own; it may be due to the fact that some children feel (as they had reported in reference to another component of the study) that they gain more from working with peers in the classroom than from working with adults, since children of the same age "speak the same language"; it may in part be due to the shying away from challenging problems on the part of some of the caregivers. One finding which emerged was related to cases in which extensive input had been given to less adept children by an adult (parent, babysitter, or tutor). In the 6 cases (out of 13 instances) which I had the opportunity to observe, the children were able to present the solution but they could not adequately defend or explain the specific components of the solution strategy to their peers.

The results indicate that when interacting with adults, the children sought and received help predominantly with respect to problem-solving strategies (Category #3) and mathematical concepts (Category #4). Interestingly, within the child-child interaction the incidence of occurrence of explanations related to mathematical concepts had been very low (Zack, 1994). Other findings suggest that the less adept children are less specific in describing the kind of help sought or received while the more adept children are more specific and focused in their requests and descriptions. In addition, there were two striking instances in which adept children were seen to do much with only minimal input from a parent.

A number of children who did not find the explanations of the caregiver helpful continued to seek to make meaning, and at times were seen to connect the help given by peers in class to the attempts made by the caregiver. In one instance a child did understand his older brother's explanation; however, the child sought and developed another approach (giving me, the teacher, credit for a hint) which he felt would be more accessible and meaningful to his peers when he presented his solution in class the next day. His writing in his math log signaled to me his willingness to pursue alternative ways of solving and presenting, as well as his awareness of the various registers of mathematical discourse, some more 'user-friendly' and more likely to be understood by peers than others. Of interest as well was the finding that the children appeared selective about whom they approached for help at home, and at times spontaneously volunteered the reasons why one candidate was preferred over another.

One aspect worthy of future study is that of the relationship of the gender of the caregiver to the kind of help that is given. Confrey (1995) noted recently that in studies of mother-child versus father-child interactions, researchers have reported that mothers tend to decenter toward the child's activities and goals, while fathers tend to coax the child to accomplish their (the fathers') goals. Due to the small number of instances, no conclusions could be drawn from occurrences in this study; however, it seemed from the few instances that the mode of working in regard to non-routine problem-solving situations might be less related to gender than to the adult's own level of development vis-a-vis mathematics.

This report constitutes a preliminary investigation of how children and caregivers might learn about mathematics through joint activity. The face-to-face interaction between parents (and other caregivers) and children is an important area of investigation which needs to be examined more broadly and in greater detail. The results of such an investigation would contribute both to our general understanding of the social construction of knowledge, and to our more specific understanding of the workings of the zone of proximal development.

References

Bartolini-Bussi, M. (1994). On sociocultural theory and the psychology of mathematics education. *PME News*, May, 1994, p. 7.

Cazden, C. B. (1981). Performance before competence: Assistance to child discourse in the zone of proximal development. *Quarterly Newsletter of the Laboratory of Comparative Human Cognition*, 3 (1), 5-8.

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In E. Forman, N. Minick, & C. A. Stone (Eds.), *Contexts for learning: Sociocultural dynamics in children's development*. New York: Oxford University Press.

Confrey, J. (1995). A theory of intellectual development (Part 3). *For the Learning of Mathematics, 15*, 2, 36-45.

Fitzgerald, J. (1983, December). Helping readers gain self-control over reading comprehension. *The Reading Teacher*, 249-253.

Lave, J. (1977). Tailor-made experiments and evaluating the intellectual consequences of apprenticeship training. *Quarterly Newsletter of the Institute for Comparative Human Development, 1* (2), 1-3.

Nelson-Le Gall, S., Gumerman, R. A., & Scott-Jones, D. (1983). Instrumental help-seeking and everyday problem solving: A developmental perspective. *New Directions in Helping, 2*, 265-282.

Newman, D., Griffin, P. & Cole, M. (1989). *The construction zone: Working for cognitive change in school*. New York: Cambridge University Press.

Polya, G. (1945). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education*. Kluwer Academic.

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. New York: Oxford University Press.

Roth, W. M. (1995). *Authentic school science: Knowing and learning in open-inquiry science laboratories*. Boston: Kluwer Academic.

Tharp, R. G., & Gallimore, R. (1988). *Rousing minds to life: Teaching, learning and schooling in social context*. New York: Cambridge University Press.

Webb, N. (1989). Peer interaction and learning in small groups. *International Journal of Educational Research, 13*, 21-39.

Vygotsky, L. S. (1981). The genesis of higher mental functions. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology*. Armonk, N. Y.: Sharpe.

Zack, V. (1993). Children's perceptions of the usefulness of peer explanations. *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education*, University of Tsukuba, Japan, July 18-23, 1993, 286-292.

Zack, V. (1994). Vygotskian applications in the elementary mathematics classroom: Looking to one's peers for helpful explanations. *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*, University of Lisbon, Portugal, July 29-August 3, 1994, 409-416.

Zack, V. (in press). Algebraic thinking in the upper elementary school: The role of collaboration in making meaning of 'generalization'. *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education* , Recife, Brazil, July 22-27, 1995.

MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF AVERAGE: A PROBLEM-SOLVING APPROACH

Jinfa Cai, Marquette University
John C. Moyer, Marquette University

This study used an open-ended problem-solving approach to teaching and assessing middle school students' understanding of the concept of arithmetic average. Three main results of this study show evidence of positive instructional impact on students' understanding of the concept of average: (1) the number of students who gave correct answers increased from pretest to posttest; (2) on the posttest, more students used appropriate strategies to solve the average problems than on the pretest; (3) more students used multiple representations on the posttest to explain their solutions than on the pretest. The findings of this study indicate that learning the concept of average is cognitively more complex than the computational algorithm suggests. However, with appropriate instruction, students can have an understanding of the concept beyond the computational algorithm.

Arithmetic average is one of the important and basic concepts in data analysis and decision making. It is not only an important concept in statistics, but also an everyday-based concept (National Council of Teachers of Mathematics (NCTM), 1989). The arithmetic average is found by adding the values to be averaged and dividing the sum by the number of values that were summed. Although the computational algorithm suggests that arithmetic average is a simple concept to understand, previous research (e.g., Cai, 1995; Mevarech, 1983; Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988) has indicated that both pre-college and college students have many misconceptions about the average concept. The misconceptions are not due to students' lack of the procedure for calculating an average, rather they are due to their lack of understanding of the concept of average.

The purpose of this study was to examine students' existing understanding of the average concept as well as the impact of open-ended problem solving instruction on their understanding of the concept. This study is an extension of an earlier study in which Cai (1995) used a multiple-choice task and an open-ended task to examine sixth-grade students' knowledge of arithmetic average. He performed a fine-grained cognitive analysis of the students' written responses. He found that the majority of the students knew the "add-them-all-up-and-divide" algorithm for calculating average, but only about half of the students showed evidence of having an understanding of the concept of average. The earlier study (Cai, 1995) also suggests the value of using an open-ended task to assess students' understanding of the average concept and to examine their problem-solving processes. This study extended the earlier study in two ways: (1) this study used two open-ended tasks to examine middle school students' knowledge of arithmetic average; and (2) this study also examined the instructional impact on students' understanding of the arithmetic average through a pretest and posttest design.

Preparation of this paper was supported in part by a grant from the Ford Foundation. Any opinions expressed herein are those of the authors and do not necessarily represent the views of the Ford Foundation.

Method

Subjects

Subjects numbered about 150 middle school students from a public school in a large urban school district. Students in the school are ethnically and culturally diverse, and 75% of the students are on a free or reduced lunch program. In this paper, only those students who took both the pretest and the posttest are used in the analysis, which includes 123 students (46 sixth-graders, 33 seventh-graders, and 44 eighth-graders). It should be indicated that students had been briefly exposed to the average concept in previous years.

Pretests and Posttests

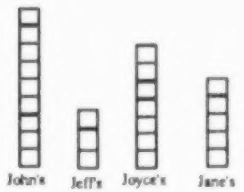
Figure 1 shows the two tasks used as pretests and posttests. In these tasks, students were asked to provide answers and, importantly, they were also asked to explain how they found their answers. In particular, Problem 1 requires students to figure out a simple mean of four numbers, and Problem 2 requires students to find a missing number when the first four numbers and the average of the five numbers (including the missing number) are presented graphically. In order to solve Problem 2, students must have a well-developed understanding of the average concept. Students were allowed about 15 minutes to complete these two problems. The posttest, which consisted of the same two problems as the pretest, was given about six months after the pretest.

Instructional Treatment

In this study, teachers used an open-ended problem-solving approach to teach the average concept with understanding. The instructional materials included those developed by Bennett, Maier, & Nelson (1988), which emphasize "averaging" as

Problem 1

John, Jeff, Joyce, and Jane each has a stack of blocks, which are shown below.

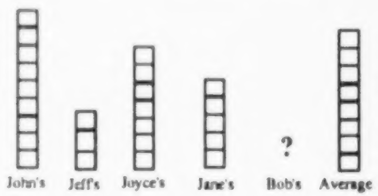


What is the average number of blocks for those four people?

Answer:
Explain how you found your answer.

Problem 2

Later Bob joined them. When Bob came in, the average number of blocks for John, Jeff, Joyce, Jane,



and Bob became 8.
How many blocks did Bob have so that the average for the five people was 8?

Answer:
Explain how you found your answer.

Figure 1. Tasks

an evening-off process. The materials stress that averaging can be used as an effective tool for making sense of a set of data rather than as a simple computation process. In addition to using the materials developed by Bennett et al. (1988), teachers also used a variety of average-related problems in their classroom (Meyer, Browning, & Channell, 1995). The teachers met with two university professors (the authors) regularly to discuss instructional materials and approaches. The teachers were encouraged to develop their own instructional materials based on the discussions in the regular meetings. The focus of the discussions was on ways of teaching the average concept with understanding, not just on the computational algorithm.

Data Coding and Analysis

Data coding and analysis were completed using a classification scheme adapted from Cai (1995). In particular, each response was coded with respect to four distinct perspectives: (1) numerical answer, (2) mathematical error, (3) solution strategy, and (4) representation. To ensure the inter-rater reliability, the two authors randomly selected 20% of the student responses and coded them independently. The inter-rater agreement ranged from 87% to 99%.

Results

Since grade level differences were not a focus of this study, the results are reported in an aggregated manner. There are three separate sections.

Numerical Answer and Mathematical Error

The numerical answer was what the student provided on the answer space on each task, and was judged correct or incorrect. With respect to the correctness of numerical answers, students improved significantly from the pretest to the posttest. Specifically, on the pretest, only 51 and 19 students respectively answered Problems 1 and 2 correctly. On the posttest, however, 104 and 84 students respectively gave the correct answers for Problems 1 and 2. Examination of the correctness of both problems shows that the percentages of students who gave correct answers for both problems increased significantly from 11% (13 of 123) on the pretest to 64% (79 of 123) on the posttest ($z = 7.57$, $p < .001$). The significant increase in students with correct answers from the pretest to posttest provides evidence of the instructional impact on student understanding of the average concept.

Examination of paired answers on the pretest shows that 80% (41 of 51) of the students who were able to solve Problem 1 failed to correctly solve Problem 2. On the posttest, 24% (25 of 104) of the students who were able to solve Problem 1 were still unable to correctly solve Problem 2, but the percentage is statistically smaller than on the pretest ($z = 6.67$, $p < .001$). This implies that after instruction students had a better understanding of the average concept. Interestingly, a few students correctly solved Problem 2 without also correctly solving Problem 1.

Fewer students made mathematical errors on the posttest than on the pretest. However, error analysis shows that students who did not correctly solve the problems tended to make similar types of errors on both tests. For example, a common error that students made in solving Problem 2 was to incorrectly apply the computational algorithm. For example, some students added the numbers of John's blocks (9), Jeff's (3), Joyce's (7), Jane's (5), and the average (8), got a sum of 32, then divided the sum by 5. The students typically gave the whole number part of the quotient (6) as the answer. These students appeared to know the computational procedure for calculating an average (i.e., "add-them-all-up-and-divide"), but they appeared to not know what should be added, what should be divided, or divided by. Thus, although student performance in solving the average problems improved significantly from pretest to posttest, a small proportion of the students still showed a lack of conceptual understanding of the arithmetic average.

Solution Strategy

Three solution strategies were identified, which are described in Table 1. On the pretest, only 42 and 17 students respectively gave a clear indication of using one of the three identified strategies in solving Problems 1 and 2. On the posttest, 94 and 66 students respectively gave a clear indication of using one of the three identified strategies in solving Problems 1 and 2.

Moreover, on the posttest, nearly 50% of the students gave clear indications of using solution strategies in solving both problems, but only 11% of them did so in the pretest. The difference between use of strategies on the pretest and posttest is statistically significant ($z = 6.26$, $p < .001$). This significant increase in the number of students who gave clear indications of using identified solution strategies from the pretest to posttest provides further evidence that instruction had a positive impact on student understanding of the average concept.

On the pretest, students most frequently used the average formula to solve the problems. On the posttest, the number of students who used average formula increased, but the increase was not as dramatic as that for leveling strategy. In fact, only a few students used the leveling strategy on the pretest, but over 40 students used the leveling strategy on the posttest. It should be noted that for those students who gave clear indications of using identified solution strategies in Problems 1 and 2, the majority of them (77%) tended to use the same solution strategies on both problems, either on the pretest or on the posttest. For example, if a student used the leveling strategy to solve Problem 1, he/she would most likely use the same strategy to solve Problem 2.

Representations

The representations were classified into the following categories: verbal (written words), symbolic (mathematical expressions), pictorial (drawings), and any combination of these three. Table 2 shows the number of students who used various representations.

From pretest to posttest, the number of students who did not provide explanations of their solutions decreased. In particular, on the pretest 14 and 29 students respectively did not provide an explanation in solving Problems 1 and 2; while on the posttest, only 2 and 12 students respectively did not provide explanations for Problems 1 and 2. Not only did more students provide explanations on the posttest

Table 1. *Descriptions of Solution Strategies and Frequency of Students Using Each of Them*

		Number of Students			
Strategy	Description	Pretest		Posttest	
		P1	P2	P1	P2
Strategy 1 (Using Average Formula):	The student used the average formula to solve the problems. For example, in solving the first problem, students added blocks that John, Jeff, Joyce, and Jane have, then divided the sum by four. In solving the second problem, students multiplied the 5 by 8, got 40, then subtracted the number of blocks that John, Jeff, Joyce, and Jane had, so the answer was 16 [i.e., $8 \times 5 - (9 + 3 + 7 + 5) = 16$].	39	15	50	24
Strategy 2 (Leveling):	Students tried to even-off the blocks to get the average number of blocks for John, Jeff, Joyce, and Jane in solving the first problem. In the second problem, students tried to use the average number of blocks as the leveling base, then found the number of blocks Bob had.	3	2	44	40
Strategy 3 (Guess-and-Check):	The student first chose a number for Bob, then checked to see if the average was 8. If the average was not 8, then he/she chose another number for Bob and checked again, until the average was 8.	0	0	0	2
Total[1]		42	17	94	66

than on the pretest, but also the quality of student explanations improved from pretest to posttest. For example, more students on the posttest tended to use multiple representations (i.e., any combination of verbal, pictorial, and symbolic representations) to explain their solution processes. In fact, only about 10% of the students used multiple representations on the pretest; while about 40% of the students used multiple representations on the posttest.

The representations students used appear to be related to the strategies they employed. For example, when students used the average formula to solve the problems, they tended to use symbolic-related representations in their explanations. While when students used leveling strategies, they tended to use pictorial-related representations in their explanations.

Discussion

This study used a problem-solving approach to teaching and assessing middle school students' understanding of the concept of arithmetic average. The results of this study suggest that for the pretest a majority of the students only knew the "add-them-all-up-and-divide" algorithm of calculating average. On the posttest, however, the number of students with conceptual understanding increased dramatically. The findings of this study provide evidence of positive instructional impact on students' understanding of the average concept. This evidence includes: (1) the number of students with correct answers increased from pretest to posttest;

Table 2. *Frequency of Students Using Various Representations in Pretest and Posttest*

	Number of Students			
	Pretest		Posttest	
	P1	P2	P1	P2
Verbal	74	60	46	42
Pictorial	3	6	11	10
Symbolic	19	14	11	10
Combination	13	14	53	49
Without Explanation	14	29	2	12

(2) more students on posttest than on pretest gave a clear indication of using appropriate strategies; (3) not only did more students provide explanations on the posttest than on the pretest, but also more students used multiple representations to explain their solutions.

The results of this study provide further evidence that learning the concept of average is cognitively more complex than the computational algorithm suggests, as was shown in previous studies (e.g., Cai, 1995; Strauss & Bichler, 1988). This study shows that if appropriate instructional approach and materials are used in the classroom, students will have an understanding of the average concept, not just the computational algorithm. This study also shows the appropriateness of using open-ended problems to teach and assess students' conceptual understanding of the arithmetic average.

References

Bennett, A., Maier, E., & Nelson, L. T. (1988). *Visualizing number concepts.* Mathematics Learning Center.

Cai, J. (1995). *Beyond the computational algorithm: Students' understanding of the arithmetic average concept.* Paper to be published in the Proceedings of the 19th conference of the International Group of Psychology of Mathematics Education. Recife, Brazil.

Mevarech, Z. (1983). A deep structure model of students' statistical misconceptions. *Educational Studies in Mathematics, 14*, 415-429.

Meyer, R. A., Browning, C., & Channell, D. (1995). Expanding students' conceptions of the arithmetic mean. *School Science and Mathematics, 95*(3), 114-117.

NCTM (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

Pollatsek, A., Lima, S., & Well, A. D. (1981). Concept or computation: Students' understanding of the mean. *Educational Studies in Mathematics, 12*, 191-204.

Strauss, S., & Bichler, E. (1988). The development of children's concepts of the arithmetic average. *Journal for Research in Mathematics Education, 18*(1), 64-80.

STUDENTS' RECOGNITION OF STRUCTURAL FEATURES IN MATHEMATICAL PROBLEM SOLVING INSTRUCTION

Manuel Santos T, Cinvestav-IPN-México

Research in mathematical problem solving has produced significant results in trying to understand what people do to solve problems. An important part of the solution process is the presence of both cognitive and metacognitive strategies. This paper documents the extent to which students are able to recognize the basic structure of a problem given in three different contexts. In the analysis, it was important to distinguish a set of distinctions that the students coordinate during the process of solution. This set of distinctions involves the use some kind of representation of the problem, the search for connections with other ideas, the flexibility in approaching the solutions, and confidence of the results. These ingredients become essential to evaluate qualities of the students' work.

Problem solving has been identified as an important component of mathematical instruction (NCTM, 1989; Schoenfeld, 1994). As a consequence, teachers encourage their students to engage in problem solving activities during the development of their courses. However, what types of problems and to what extent students should discuss these problems during instruction are issues that teachers need to discuss on a regular basis. It is common to hear that it is difficult to find or design good problems for the class discussion, and teachers often continue working with routine problems that they have been using regularly in their classes. Thus, if we accept that problem solving is a way of thinking that should be present not only in mathematics instruction, but in the process of interacting with problems in other contexts, then it becomes important to explore how other contexts could play an important role in the selection of problem solving activities for the classroom. This paper analyzes the work done by tenth grade students who were asked to work on three problems that share similar structure. Thus, it was important to document what type of strategies and difficulties were shown by the students who noticed connections among the problems. The discussion of the students' approaches play an important role not only in understanding the processes shown while working on the problems but also in evaluating the potential of some activities associated with problem solving instruction.

Background to the Study

Research in mathematical problem solving has suggested that it is important to provide learning experiences for the students in which they have opportunity to get engaged in actual mathematical experiences. Schoenfeld (1992) found that the process of doing mathematics includes the use of resources or basic mathematical knowledge (facts, procedures, algorithms), the use of heuristic strategies, the pres-

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This paper was written while the author was a visiting scholar at the University of California at Berkeley. The author thanks Alan Schoenfeld and the function group for their hospitality. He also wants to acknowledge the financial support received from CONACyT and CINVESTAV during his stay.

ence of metacognitive activities (monitoring and control), and an understanding of the nature of the mathematical practice (conception of the discipline). As a consequence, it is necessary to investigate to what extent the students' problem solving behaviors could be improved when the instruction they receive takes into account learning activities related to those dimensions. Santos (1995) pointed out that in order to develop the students' mathematical disposition to learn mathematics is important to provide a class environment in which students consistently are asked to a) work on tasks that offer diverse challenges; b) discuss the importance of using diverse types of strategies including the metacognitive strategies; c) participate in small and whole group discussions; d) reflect on feedback and challenges that emerge from interactions with the instructor and other students; e) communicate their ideas in written and oral forms; and f) search for connections and extensions of the problems. These learning activities play a crucial role in helping students see mathematics as a dynamic discipline in which they have the opportunity to engage in mathematical discussions and thus value the practice of doing mathematics.

The need to document how the students approach different types of tasks is based on the great influence that problem solving has shown in the learning of mathematics. The number of research studies in this area has been significant in the last 25 years (Schoenfeld, 1994; Charles & Silver, 1988, Lester, 1994). One important direction in problem solving has been to categorize the way students solve problems. Several frames of analysis or theoretical models emerged from that research direction and have contributed to the understanding of the process used by the problem solver. The role of qualitative tasks or nonroutine problems has been important during the process of gathering information of the students' work. As a consequence, some research results in problem solving have challenged or transformed the teaching of mathematics. Here, it becomes important to study the potential of diverse tasks or problems that involve different contexts as a means to use them in mathematical problem instruction. The analysis of the students' approaches while working on problems with similar structure will help us understand what aspects of problem solving appear as important when students actually recognize the structure of the problems during the solution process.

Methods, Procedures, and Frame of Analysis

Thirteen grade nine students, all volunteers, participated in the study. They worked on the problems for about 45 minutes. Each student worked the problems individually and was asked to think aloud while solving the problem. It is important to mention that the teacher of this group of students has been implementing problem solving activities during the last three years of his teaching. An interviewer took notes during the whole process and was available to provide clarification questions when required by the students. Three problems were used as means to gather information.

1. A carpenter makes $800 for the first week of work and then $860 for the next two weeks. What were his total earnings for that period, and what was his average salary?
2. A tank is filled to a depth of 80 centimeters and two identical tanks are filled to a depth of 86 centimeters. What is the average depth of the water in the tanks?
3. Peter travels 80 km per hour for one hour, then at 86 km per hour for two hours. How far did Peter travel, and what was his average speed?

The work shown by the students was analyzed by considering the type of resources and strategies that the students used to solve or make progress while working on the tasks. It is important to mention that during the analysis aspects of the mathematical practice which helped students identify similarities among the problems were explored. During this process, three levels were identified as a means to characterize the students' work. The high level appears when a student shows the important mathematical ideas associated with the task in his or her solution and he or she provides a consistent argument that supports such a solution. A medium level is identified when a student shows significant progress to the solution but misses to consider some cases. Finally, a low approach involves the student showing little understanding of the key issues of the task and addresses only superficial parts of the problem solution.

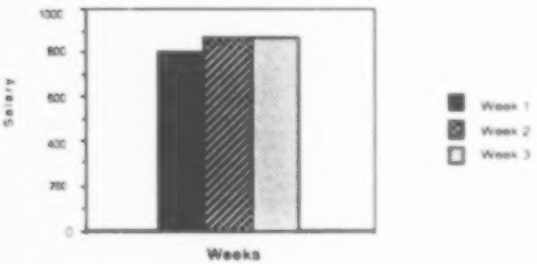
Students' Approaches to the Problems

Eighty percent of the students showed significant progress toward solving the problems. Although the most popular approach was to focus on operations, it was important to observe that various students used graphical representations. For example, seven students relied on a table and figure to solve the first problem, and these students noticed that to solve the second and third problems they were going to use a similar approach. That is, they were able to identify the common structure of the problems. It seems that using a representation helped them make the connections. Some students who relied only on calculations did not make explicit statements about the relationships among the problems. For example, four students were able to solve the first two problems, and they wrote that they did not recall the formula working in the third problem. Only one student graphed the three problems together by presenting the data in accumulative form and explained relationships among the representations. A set of distinctions that students showed during the solution process helped categorize the quality of the responses. These distinctions include: (a) The use of representation as a means to work the data (table, list) and to show the result, (b) Connections in which some students linked the common features among the problems, (c) Flexibility in trying to graph and explain extensions of the problems (accumulative graph), and (d) Confidence shown by some students when they compared the responses to the problems. To illustrate differences among the students' responses, an example taken from the students'

work is used to illustrate the quality of the responses for high, medium and low levels.

Students who decided to represent the data graphically showed tables and, in some cases, bar diagrams. For example, some students utilized the following representations:

For the problem that involves finding the total earnings, seven students arranged the data of the problem on a table and showed a bar diagram. It was interesting to observe that these students also represented the second problem similarly and immediately (while working on the representation) noticed that the three problems could be approached in the same way.

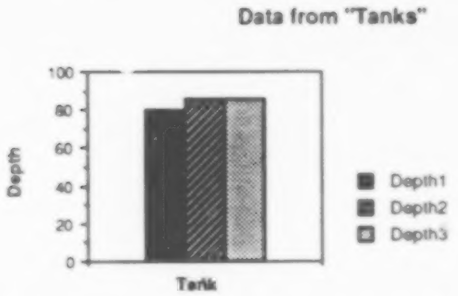

Data from "The Carpenter's Salary"

Week	1	2	3	Average
Salary	800	860	860	840

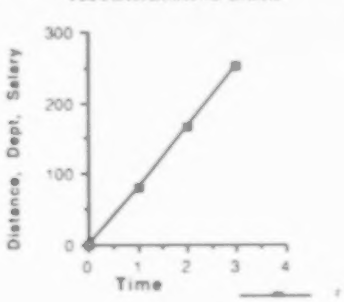
Tank	1	2	3	Average
Depth	80	86	86	84

Hour	1	2	3	Average
Spped	80	86	86	84

It was clear that students who represented the problem graphically were able to identify similar properties among the problems. For example, two students who had used bar graphs to represent the first problems, immediately noticed that the shapes of the graphs were the same. These students mentioned that all three problems could be solved in the same way. They also mentioned that the context of the problem did not influence the form of solution. The responses given by these students were categorized as the high level type. When the students used only a table or paid attention only to the numerical results used to determine the relationships among the problems, then the responses were categorized as the medium level type. For example, three students spotted similarities among the problems



based on a list of what happened in each situation individually. That is, they focused on the average number asked in each problem to support their responses. An interesting contrast with the students who used graphs was that these students worked on the three problems completely before realizing that they shared similar structures; while the students who used graphs did not need to complete the three problems before noticing such similarities.



One student showed the relation between the bar representation and the linear graphs by showing an accumulative representation. He noticed that the information given in the three problems could be easily read from this representation.

The accumulative representation shows exactly how many km had been traveled or how much money had been earned by a given time. The students who failed to solve the problems or make progress toward the solution experienced difficulties in trying to understand the conditions and what they were asked to do. For example, one student asked for the speed formula to approach problem three.

Discussion of Results and Instructional Implications

The results show that it is possible to identify a set of characteristics that distinguishes various approaches in the students' work. On one side there were students who spent significant amount of time analyzing the conditions of the problems and worked on a well structured plan. These students showed the use of different representations as a means to approach the problems. The fact that the students explicitly searched for various representations helped them interpret the information and observe some connections. On the other side, other students tended to approach the problems by using numerical representation, and it was difficult for them to visualize that the problems shared a similar structure. Although the students were asked only to work on the problems, it is interesting to note that those who used more than one representation were able to see the problems in a wider perspective compared with the students who used only one representation. That is, the use of several representations played an important role in the transfer of the students' ideas.

It is also evident that the first group of students (who spent more time understanding the conditions) showed more of a disposition to work on these tasks, and they showed some kind of flexibility in using more than one approach, including graphical representation. It seems that being flexible while representing the information given in the problem allowed students to observe features that were not evident under the numerical representation. An important implication here is that it is important to encourage students to use more than one representation to deal with the information. In addition, it is important that students consistently are asked to identify similarities and differences among methods of solution and structural properties of problems that involve different contexts.

References

Charles, R. I., & Silver, E. A. (1988). (Eds.). The teaching and assessing of mathematical problem solving. Reston, VA: National Council of Teachers of Mathematics.

Lester, F. K. Jr. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education*, 25(6), pp. 660-675.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Santos, T. M. (1995). *On mathematical problem solving instruction: Qualities of some learning activities*. Paper presented to the Function Research Group at the University of California, Berkeley.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.) Handbook of research on mathematics teaching and learning (pp. 334-370). New York: MacMillan.

Schoenfeld, A. (1994). Reflections on doing and teaching mathematics. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving*. Hillsdale, NJ: Erlbaum.

AN EXPERIMENT TO EXAMINE THE EFFECTS OF PROBLEM POSING ACTIVITIES ON STUDENTS' METACOGNITIVE BEHAVIOR

Ted Hodgson, Montana State University

Presumably, students engaged in posing problems develop essential problem-solving skills. The actual effects of problem-posing on students' problem-solving behaviors, however, remain largely unknown. The current study pilots one approach to problem posing, in which students construct and solve problems that are *similar* to designated problems, and offers conjectures regarding the effects of the approach on students' metacognitive behavior. In light of these conjectures, the study reviews students' responses to problem-posing tasks. Moreover, the study offers suggestions for amending the experiment and identifies additional research questions elicited by students' responses.

Subjects participating in the pilot study were eight undergraduate mathematics majors enrolled in a problem-based, history of mathematics course. To isolate the effect of problem posing on students' metacognition, participants were randomly divided into two groups. The members of Group 1 were assigned one additional problem per assignment, whereas Group 2 members posed problems that were *similar* to a designated problem on the problem set, solved the similar problems, and identified the similarities between the original and the newly constructed problem. By looking back at previously solved problems and re-examining their own problem-solving efforts, it is conjectured that students in the problem-posing group will develop reflective patterns of behavior and become more reflective in all problem situations.

Both in terms of the *similar* problem and problem-solving reflections, students' initial responses lacked the depth and richness of their later efforts. In general, subjects (all of whom were mathematics majors) focused on process aspects of the designated problems, a fact that leads to questions concerning less able students' conceptions of "similar" and the effects of these conceptions on students' problem-posing efforts.

PROOF IN THE PROBLEM SOLVING CONTEXT: HOW STUDENTS USE THE RESOURCES AT THEIR DISPOSAL

Barbara R. Smith Reed, University of California, Santa Barbara

This study explores aspects of the uses of various cognitive resources available to pairs of students engaged in joint problem solving. Five pairs of students from a high school geometry class were videotaped while solving a geometry proof problem. The students had been in a geometry course that was organized around cooperative groups, and were using a mathematics reform curriculum developed primarily at U. C. Davis (College Preparatory Mathematics, Kysh et al., 1990). The students in the study were taught to rely more heavily on their own abilities to solve the problems, and to use their peers as a resource, rather than always asking the teacher or referring to the textbook for help. According to Schoenfeld (1985), there are four fundamental aspects of mathematical thinking that are necessary for successful problem-solving: cognitive resources, heuristics, control, and belief systems. Schoenfeld revises this framework (1992) to include the additional category of practices. In this study, the definition of resources has been expanded to include those that were readily available to the students in the CPM class, and places the social construction of knowledge within the resource category, instead of in a separate category (practices) as in Schoenfeld's theories. The expanded resource list includes: Prior Knowledge (spontaneously recalled facts), Each Other (peer interaction), Tool Kit (an artifact of CPM, including theorems, etc.), Diagram (one representation within the problem statement), and Logical Statements (another representation in the problem statement, enclosed in ovals). These resources for solving problems in this class were different from those available to students in a more traditional curriculum, where the teacher and the text are the sources of knowledge. The goal of the study was to explain and describe the ways in which the students used these resources at their disposal, and to discover how the resources helped or hindered their problem-solving ability. In other research of students' problem-solving ability, the problem presented to the students has not been directly tied to a particular curriculum. This study was done to analyze problem solving that was most common within a reform-type class, and therefore the problem presented was directly related to those that the students had previous experience in solving.

References

Kysh, J., Kasimatis, E., & Sallee, T. (1990). *College preparatory mathematics: A change from within.* (v.3.0) Davis, CA: CRESS Center.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

WHY IS PIZZA ROUND? STUDENT RESPONSES TO A NEW PRE-ALGEBRA UNIT

Mary E. Brenner, Theresa Brar, Richard Durán, Richard Mayer,
Bryan Moseley, Barbara R. Smith and David Webb
University of California, Santa Barbara

Classroom instruction which follows the guidelines of the NCTM National Standards for Curriculum and Instruction places many new demands upon students. Students must learn to work within a constructivist framework in which they function more as creators of knowledge rather than recipients of knowledge as in traditional classrooms. Although it will take time for students and teachers to become fully comfortable with constructivist approaches, we feel that students will gain more understanding of mathematical principles and more appreciation for the utility of mathematics in the long run. We have developed a unit to introduce basic algebra to junior high school students using many of the principles of the mathematics reform movement: (a) Instead of emphasizing symbol manipulation, we emphasize problem representation skills. (b) Instead of teaching problem-solving skills in isolation we anchor them within a meaningful thematic situation. (c) Instead of focusing solely on the product of problem solving, we emphasize the process of problem solving, in cooperative groups and through modeling by teachers.

In another report included in these proceedings, we present statistical data that show the effectiveness of our approach (Brenner, Brar, Durán, Mayer, Moseley, Smith & Webb, 1995). This poster is intended to present more information about the activities included in the unit and what kind of work students produced in class. The 20 lesson unit asked students to help make a decision about which of three pizza companies should be contracted to provide pizza in the school cafeteria. Day 1 involves a taste test in which students sample several pizzas, collect data to characterize student preferences, and construct graphs. Days 2 through 5 involve a computer malfunction in which the students look for and describe patterns of errors in order forms and invoice sheets, using tables, graphs, variable expressions and words. Day 6 involves a pizza delivery game in which students must use variable expressions to determine the correct destination. In Days 7 through 10 students learn about formulas for area within the context of an advertising problem. Days 11 through 14 focus on nutrition as students generate equations expressing the fat content of various pizzas. In Days 15 through 18 students use tables and graphs to solve problems about pizza businesses. In the last two days, students write a final report advocating which pizza company should be selected.

Reference

Brenner, M.E., Brar, T., Durán, R., Mayer, R., Moseley, B. Smith, B.R. & Webb, D. (1995). The role of multiple representations in learning algebra. *Proceedings of the International Study Group for the Psychology of Mathematics Education, North American Chapter,* Columbus, OH.

Rational Number Concepts

FACILITATING CHILDREN'S DEVELOPMENT OF RATIONAL NUMBER KNOWLEDGE

Kathleen Cramer, University of Wisconsin-River Falls
Thomas Post, University of Minnesota

At issue in this study was the extent to which large numbers of classroom teachers were able to implement research-based materials with a minimum of inservice education and whether students of these teachers were able to develop the rich mental images for fractions similar to the ones students from previous Rational Number Project (RNP) studies developed in smaller experimental settings. An analysis of student interviews demonstrated that RNP students did in fact develop rich mental images for fractions similar to students in previous studies. As expected the nature of RNP students' thinking about rational number was far richer than students who used textbook curriculum and indicated a more conceptually oriented framework.

Since 1980, the RNP has reported on many investigations into the teaching and learning of fractions among fourth and fifth graders (Bezuk & Cramer, 1989; Post, Wachsmuth, Lesh & Behr, 1985). The curriculum used in the study reported here emanated from this earlier research. The RNP curriculum used in earlier studies reflected the following beliefs: (a) Children learn best through active involvement with multiple concrete models, (b) physical aids are just one component in the acquisition of concepts-verbal, pictorial, symbolic and realistic representations also are important, (c) children should have opportunities to talk together and with their teacher about mathematical ideas, and (d) curriculum must focus on the development of conceptual knowledge prior to formal work with symbols and algorithms.

The curriculum developed the following topics: (a) part-whole model for fractions, (b) concept of unit, (c) order ideas, (d) equivalence concepts and (e) addition and subtraction of fractions at the connect level. It de-emphasized standard paper-pencil procedures for ordering fractions, finding fraction equivalencies and symbolic procedures for operating on fractions. Instead it emphasized the development of a quantitative sense of fraction. To think quantitatively about fractions, students should know something about the relative size of fractions and be able to estimate reasonable answers when fractions are operated on.

The fraction curriculum used in earlier investigations was revised and extended. The goal for this revision was to reorganize lessons from the 30-week teaching experiment into two levels of teaching materials that could be used easily by classroom teachers with fourth and fifth grade students. This study used Level 1 materials (23 lessons) with all students regardless of grade level. Some lessons lasted more than one day.

Treatments

RNP Curriculum

The curriculum was written to reflect cognitive psychological principles as suggested by Piaget (1960), Bruner (1966), and Dienes, (1967). Lesh (1979) elaborated on their ideas and produced a model which suggests that learning is enhanced when children have opportunities to explore mathematical ideas from multiple perspectives - manipulatives, pictures, written symbols, verbal symbols and real life contexts. The model also suggests that it is the translations within and between modes of representation that make ideas meaningful for children. The RNP curriculum reflects this theoretical model. The manipulatives used in the lessons included fraction circles, chips, and paper folding.

An important part of each lesson is the "Notes to the Teacher" section. Here insights into student thinking captured from the initial RNP teaching experiments are communicated to teachers. The notes share examples of student misunderstandings and anecdotes of student thinking from earlier RNP projects. These notes to the teachers also clarify methods for using manipulative materials to model fraction ideas.

Textbook Curriculum

The majority of 33 textbook classrooms used the 1989 edition of *Addison-Wesley Mathematics* (AW). Six of the textbook classrooms piloted the Harcourt Brace Jovanovich (HBJ) 1992 series, *Mathematics Plus*.

The textbook teachers were encouraged to use the resources suggested in the teacher's guides. Fraction bars and pictures of fraction bars were the models suggested by Addison-Wesley textbook series. The HBJ series suggested a larger variety of manipulative materials. These included counters, paper folding, fraction circles and fraction bars made from paper strips. In each case, though, concrete models played only a cursory role in the development of fraction ideas; the primary goal was to develop student competence at the symbolic level. In the RNP lessons translations including extensive use of manipulative materials were the central focus. Symbols were used to record students' observations, discussions and actions with manipulatives.

Procedures

Treatment Assignments

In a suburban school district south of the Twin Cities all 200 fourth and fifth grade teachers were contacted in the fall of 1993 to assess their interest in participating in this study. Sixty-six teachers from 17 schools chose to participate. Teachers were assigned to treatment conditions (RNP or Textbook) by grade level. There were 38 fourth grade classrooms; 19 RNP and 19 Textbook classrooms. There were 28 fifth grade classrooms; 14 RNP and14 Textbook classrooms.

Timeline

The study began with the first of two, two-hour teacher inservices. This inservice was divided into two parts. For the first hour, all teachers heard a presentation that dealt with the following topics: (a) history of the RNP, (b) structure of the study, and (c) research on student learning of fractions. The second hour, teachers broke up into two groups by treatment and reviewed the goals and objectives of their respective treatments conditions.

Instruction was to last a minimum of 28 days and a maximum of 30 days; each class period was for 50 minutes. A second, two-hour inservice session was held half way through instruction. In the first hour all teachers participated in a discussion on assessing fraction learning. During the second hour the RNP teachers worked through activities with manipulatives modeling fraction addition and subtraction. The textbook group considered several fraction enrichment activities.

Interviews

Twenty 4th graders, each from a different classroom, were interviewed by project staff. Ten were selected from the RNP group and 10 from the textbook group. Each student was interviewed three times. Interview topics included concepts, order, equivalence, concept of unit and fraction operations. Each classroom teacher also randomly selected three students from his/her class and interviewed them once at the end of the study. This interview included eight questions in those same five areas.

Questions:

The following questions were of interest:

1. Is the RNP curriculum written and organized so teachers can use it effectively with limited inservice opportunities?
2. Do students taught by classroom teachers using the RNP curriculum develop similar understandings for fractions as compared to students in original RNP teaching experiments taught by project investigators?
3. What differences occur in student achievement and student thinking between students using a conceptually-oriented RNP curriculum and students using district-adopted textbooks?

To investigate these three questions, the RNP personnel relied on several different data sources. Post and retention written test data from some 1600 fourth and fifth grade students provided the foundation from which answers to each of the above questions were generated. Interviews with RNP students were to determine whether student thinking documented during the teaching experiments could be replicated on a large scale with students taught by classroom teachers using

RNP curriculum and to provide anecdotal data depicting treatment differences in student thinking.

The remainder of this paper addresses interview results related to differences in student thinking between RNP and Textbook students. Table 1 reflects data from the final interview given to 17 fourth graders by project staff. [Note: Of the 10 RNP students originally selected to be interviewed, two were tracked into a low math group and did not finish the lessons. These two students were not given the final interview. One Text student did not take the final interview. Data reflecting numbers less than eight RNP students and nine Text students interviewed represent missing data]. Table 2 reflects data from interviews with fourth graders given by classroom teachers. Each table reports student responses to an estimation question. A student response which relied on mental images for the fractions to determine their relative size was categorized as a conceptual response. This was in contrast to a student response that solely relied on symbolic procedures to estimate. Here little thought as to the relative sizes of the fractions was considered. Students determined the exact answer and then estimated from that exact answer. Student responses to the problem in Table 1 are organized below that table to exemplify their correct and incorrect answers. Student responses similar to those reported here also were found in interviews given by classroom teachers. Limited space prohibits a detailed list of examples.

Table 1. *Final interview given by project staff*

Marty was making two types of cookies. He used 3/12 cup of flour for one recipe and 2/3 cup for the other. How much flour did he use altogether? *Without working out the exact answer, give me an estimate that is reasonable. (If needed ask: Is it >1/2 or <1/2? >1 or <1?)*

RNP (8 students)					TEXT (9 students)				
correct concept	correct procedure	correct no explan	incorrect data	missing concept	correct procedure	correct no explan	correct	incorrect data	missing
6	1	1			2		2	4	1

RNP Correct Responses:

KE: It would be more than 1/2. It would be less than a whole. If you had 2-thirds, that's more than half and then you put 3-twelfths to add to it, it would not be a whole.

When asked how she knew it wasn't going to be a whole she said: 3-twelfths isn't very big so you'd add a little more.

JS: About 1 whole. The 3-twelfths - I think 3 of these could fit in the missing spot.

MG: Greater than 1/2, less than one. 2/3 is almost a whole. 4-twelfths plus 2-thirds equals one; so 3/12 plus 2-thirds is not quite one. [All done mentally].

BS: About one. 1/3 is bigger than 1/12. Then 3/12 wouldn't equal 1/3. And you need 2 more thirds to equal a whole.

AR: **Greater than 1/2. It takes 3 reds to cover one blue [fourths] so it probably takes 4 reds to cover a brown [thirds]. So there's only 2 of 3 [browns]. There's a gap when you fill with 3 reds.**

KB: I know it's greater than 1/2 because 2/3 is greater than 1/2. Close to one, a little less. Because I just think that.

RNP Incorrect Responses

LC: About one. 3-twelfths equals 1-third; 2-thirds plus 1-third equals one.

TEXT Correct Responses

AB: Greater than 1/2 and less than one. I know how many times this could go into 12 is four and you go four to get the denominator. And it was four times three, you take three times four equal twelve and then two times four equals eight and then you get 8-twelfths. Then you go 8-twelfths plus 3-twelfths equals 11-twelfths and then its more than 1/2 and less than one.

LB: Greater than 1/2; less than one. Couldn't explain why.

ES: Greater than 1/2; less than one. I am just guessing.

MC: About 1 and a little over. You round this off to twelfths [points to 2/3]. Quadruple that [2] to 8 , (add to 3/12), that's approximately one whole. [He estimates after mentally arriving at exact answer]

TEXT Incorrect Responses

KH: Less than 1/2. Unable to explain reasoning

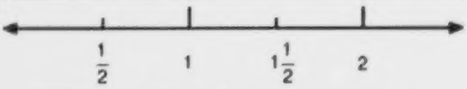
ND: I don't know. Can't guess. [Wanted to use paper and pencil]

KA: More than one. I don't know. It just seems high.

BA: He ate about 1/2. I subtract it. I can't do it in my head.

Table 2. *Interviews given by classroom teachers*

Tell me about where 11/12 - 4/6 would be on this number line:



<table>
<tr><td>correct concept</td><td>correct proced unclear</td><td>correct expl</td><td>correct no expl used</td><td>correct pict/manip</td><td>incor concept</td><td>incor proced</td><td>incor expl unclear</td><td>incor no expl</td><td>miss data</td></tr>
<tr><td colspan="10">RNP Group (53 students)</td></tr>
<tr><td>20(41%)*</td><td>5(10%)</td><td>2(4%)</td><td>2(4%)</td><td>8(16%)</td><td>7(14%)</td><td>2(4%)</td><td></td><td>3(6%)</td><td>4**</td></tr>
<tr><td colspan="10">Text Group (57 students)</td></tr>
<tr><td>2(6%)</td><td>14(41%)</td><td></td><td></td><td></td><td>3(9%)</td><td>6(18%)</td><td>3(9%)</td><td>6(18%)</td><td>23**</td></tr>
</table>

*Percentages based on available data: RNP 49 students; Text 34 students.

** Missing data for the RNP group represents students who did not complete the 23 lessons. Teachers chosenot to ask students this question. Missing data for the textbook group represents teacher error. Severalteachers asked the wrong question. They asked students to place the two fractions on the number line instead of the difference between the two numbers.

Discussion

Almost all the RNP students were able to provide a reasonable estimate to the addition problem given and a large percentage could estimate a subtraction problem. It should be noted that of the 23 lessons only five dealt with the arithmetic operations. These lessons developed addition and subtraction concretely and within context. Paper and pencil procedures for finding fraction equivalencies and common denominators were not taught.

Differences in students' thinking about fractions are evident. RNP students' responses relied on their mental images for fractions considered. Images described relate directly to fraction circles, the manipulative used most frequently in the lessons. Students used images to determine a fraction's relative size (2/3 > 1/2; 2/3 is almost a whole) as well as to determine simple equivalencies (3/12 = 1/4; 3 reds equals 1 blue). Textbook students' responses show most students did not use mental imagery to reason through an addition or subtraction problem. Textbook students most often relied on symbolic procedures (find exact answer and then estimate) or were unable to verbalize reasons for their estimate.

Differences in students' ability to verbalize was evident. RNP lessons emphasized student discussion of ideas and translations to and from the verbal mode of representation. The manipulatives themselves became a focal point for student discussion - students talked about their actions with manipulatives.

The initial studies conducted by the RNP have provided much information and insight into issues involving the teaching and learning of fractions. Our goal for this study was to organize a large scale implementation of a curriculum based on this previous research. Results here provide evidence that large numbers of classroom teachers can effectively implement well structured, conceptual-based curriculum which in this study resulted in student learning that was rich in conceptual understandings as contrasted to the procedural-based learning characterized by students using the textbook curriculum.

References

Bezuk, N. & Cramer, K. (1989). Teaching about fractions: What, when and how? In P. Trafton (Ed.), *National Council of Teachers of Mathematics 1989 yearbook New directions for elementary school mathematics* (pp. 156-167). Reston, VA:National Council of Teachers of Mathematics.

Bruner, J. (1966). *Toward a theory of learning*. New York: Norton.

Dienes, Z. (1967). *Building up mathematics* (revised). London: Hutchinson Educational.

Lesh, R. (1979). Mathematical learning disabilities: Considerations for identification, diagnosis and remediation. In R. Lesh, D. Mierkiewicz & M. G. Kantowski (Eds.), *Applied mathematical problem solving*,. ERIC/SMEAC.

Piaget, J. (1960). *The psychology of intelligence*. Patterson, NJ: Littlefield, Adams.

Post, T., Wachsmuth, I., Lesh, R., & Behr, M. (1985). Order and equivalence of rational numbers: A cognitive analysis. *Journal for Research in Mathematics Education, 16* (1), 18 - 37.

A FIFTH GRADER'S ATTEMPT TO EXPAND HER RATIO AND PROPORTION CONCEPTS

Jane-Jane Lo, Cornell University
Tad Watanabe, Towson State University

One fifth grade student, Martha, was encouraged to develop her informal ratio and proportion strategies during a six-month teaching experiment. In this paper, we describe the challenges Martha faced during the teaching experiment. The current study supports the claim made by Kaput and West (1994) that initial instruction on ratio and proportion which is based on children's informal strategies should be introduced as early as the third grade.

Ratio and proportion are important concepts in current mathematics curricula. Very often multiplication and division tasks in lower grades are presented in unit-rate form, which is a special form of ratio and proportion. In the middle grades, word problems involving equivalent fractions and fraction comparison can also be thought of as ratio and proportion situations. For example, to solve the task "Group A has 4 pizzas and 6 girls. Group B has 6 pizzas and 8 boys. Who gets more pizza, the boys or the girls?" (Adapted from Lamon, 1993), some students may draw pictures to figure out that in group A, each member gets 2/3 of a pizza, while in group B, each member gets 3/4 of a pizza. They can then compare these two fractions with the pictures. Other students may use ratio and proportion reasoning: "If I add 2 pizzas to group A, I would need to add 3 more people. So group A is like having 6 pizzas and 9 members. So, each member in Group B gets more pizza." The ability to recognize structural similarity, and the sense of co-variation and multiplicative comparisons illustrated in such a reasoning process are at the core of algebra and more advanced mathematics (Confrey & Smith, 1995).

Because of the importance of this topic in school mathematics, children's concepts of ratio and proportion have long been a focus of mathematics education research, and much has been learned about students' errors and difficulties in solving ratio and proportion tasks (Hart, 1984; Karplus, Pulos, & Stage, 1983), as well as different task variables which affect students' choices of strategies and performance (Harel, Behr, Post, & Lesh; 1991; Kaput & West, 1994).

But what are the roots of these difficulties? What arithmetic knowledge may be useful in developing the concepts of ratio and proportion? Vergnaud (1988) used the term "multiplicative conceptual field" to refer to "all situations that can be analyzed as simple or multiple proportion problems" (p.141). Mathematical concepts that are tied to those situations include, as pointed out by Vergnaud, multiplication, division, fraction, ratio, proportion, and linear functions. He suggested that students develop these concepts not in isolation, but in concert with each other over long periods of time through experience with a large number of situations. Therefore, research studies on children's ratio and proportion concepts need to consider also the other concepts that are a part of children's developing multiplicative conceptual field.

In this article, we report the findings from such an attempt. The analysis is based on data gathered from fifteen 70-minute teaching sessions with one fifth grade student, Martha, over a period of six months. The goals of this teaching experiment included a) encouraging Martha to develop her informal ratio and proportion strategies, b) documenting the nature of this developmental process, and c) analyzing how the development of her ratio and proportion knowledge might influence or be influenced by other constructs of the multiplicative conceptual field. Specifically, what challenges would Martha face and how would she overcome those challenges?

We were aware that a longitudinal teaching experiment was needed to fully study these questions. We hoped this case study of Martha would provide information that could be used in a much larger research program. Because of the space limitation, we will focus our discussion on one particular type of task—the missing-value proportion task.

Martha and Her Informal Strategy

Martha was a bright and confident fifth grader. She had quite sophisticated methods for solving missing value proportion tasks at the beginning of the teaching experiment. The following are two examples:

Episode 1: 6 quarters can buy 9 candies, how many candies can you buy with 14 quarters?

Martha's strategy and reasoning:

> Through doubling, Martha recognized that 12 quarters could buy 18 candies. Then she figured out that 2 more quarters would get her three more candies, because 6 quarters was like "3 sections of 2" and 9 was like "3 sections of 3." Then she added 3 to 18 and got 21.

Episode 2: 15 quarters can buy 40 candies, how many candies can you buy with 21 quarters.

Martha's strategy and reasoning:

> When the relationship between 15 quarters and the 6 more quarters was not apparent, Martha attempted to find the unit price - "how many candies can one quarter get?" for this particular problem. She first arranged these 15 quarters into five rows of three quarters. Through one-to-one distributing, she then assigned two candies to each quarter. With 10 candies left to be distributed among 15 quarters, she calculated 15 divided by 10 with paper and pencil and did not find the result 1 r 5 useful. Then she started to point in the air with her two right fingers. She pointed 8 times first and did not like the result (We interpreted that she was trying to put one candy by each two quarters). She started over and this time she pointed five times (She was trying

> to put one candy by each three quarters) and was pleased with the result. Then she started to place 5 cubes one at a time. She counted what she had left, thought a while, then repeated the same action with the remaining five candies. By identifying the relationship, "3 quarters for 8 candies," she added 16 to 40 and got 56 as her answer.

These strategies showed that Martha had a concept most researchers consider an important element of proportion concepts, "homogeneity." There was an implicit notion that a relationship existed between the number of quarters and the number of candies in the given condition and this relationship needed to be preserved between certain subsets of the quarters and certain subsets of the candies, thus 2 more quarters required 3 more candies. This approach was different from the one used by another fifth grader, Bruce, we worked with. To find the correct unit-price, he would try different unit price like "one quarter for one and one-half candy," or "one quarter for two and one-third candy" and iterate the amount over the number of quarters they matched the given condition.

One major objective of the teaching experiment with Martha was to help her extend her informal strategies to a variety of problem settings, larger numbers and difficult ratios. In order to achieve these goals, Martha needed to become more reflective to the mathematical nature of her informal strategies. That is, Martha needed to (a) articulate mathematically the goal of her trial-and-error based actions, (b) to give mathematical meaning of these actions, thus making the whole process more systematic, (c) to interiorize her physical actions so that they could be executed mentally without the sensory-motor actions, and (d) to generalize her actions across similar ratio and proportion situations. The following is a brief summary of the major challenges Martha faced when attempting to accomplish these tasks.

Articulating the Mathematical Meaning Behind the Operations

When the numbers involved in a problem were small, and/or a useful common factor between numbers could be identified, Martha used strategies similar to those described above to solve a wide range of ratio and proportion tasks. Neither the problem setting nor the semantic structure seemed to have much influence on her (The only exception was the tasks with enlarging or shrinking objects, which will be discussed later). However, when the numbers became large and/or the common factor could not be identified easily, paper-and-pencil computation became necessary. Martha was efficient with the procedural aspect of the computation, but frequently lost the direction of her solution method in the process of carrying out the computation procedure.

One source of difficulty came from her inability to articulate the mathematical meaning behind the operations within a problem context. For example, to solve the problem "12 quarters can buy 220 candies, how many candies can 3 quarters buy?" Martha quickly carried out the procedure of 220 divided by 12 and got 18 remainder 4. But was not sure what to do next. The interviewer asked Martha to give a meaning to her operation. The following conversation occurred:

M: You can get 18 candies with 4 remainder for 12 quarters.

I: No, you can get 220 candies with 12 quarters.

M: Oh, Yeah, you can get this many for... that's how many times 12 goes into 220. So, do I need that?

I: Yes, you do. But you also need to know what that means.

M: Okay, that is, that is how many times that goes into this, um, um, so, ... if 18 is how many, wait, there is 18 candies for one quarter.

It took some more probing before Martha identified that the remainder 4 meant there were four more candies needed to be distributed among the 12 quarters. She then had no difficulty in reasoning this situation with smaller numbers: Since 3 quarters was one fourth of 12 quarters. So, 3 quarters would get one extra candy in addition to the 54 candies (18 candies x 3) they got first. Martha's explanation showed that she had started making connection between the numerical operation and the physically activity of her strategies as described in Episode 2.

Even though we strongly believe the importance of explaining one's mathematics actions verbally, we recognize certain "problematic situations" which may occur in this process. For example, phrases like "2 quarters for 3 candies" or "2 candies per quarter" are commonly used in daily life which, we believed, facilitated the connection described above. Other ratio and proportion situations were harder to describe. For example, Martha was quick to identify the "7 for 2" information from the initial statement, "Fish A is 56 cm long and needs 16 pieces of food each day." But she needed assistance to verbalize the meaning, "7 cm long got 2 pieces of food." Furthermore, it was less natural to say "3 minute per mile" or "3 minute for each mile" than to say "3 miles per minute" which might have an effect on the choice of operation. The operation which involves norming "There are 3 sets of 2 pairs of socks" (Lamon, 1994) was the hardest to describe. But we also found that Martha learned quickly from her experience to verbalize a variety of ratio and proportion situations. Also with the verbalization, she was less likely to confuse the measure spaces in her computation.

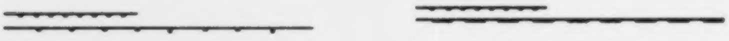
Tasks Involving Similar Figures

Similar to the existing literature, Martha found the tasks involving enlarging and shrinking to be the most difficult ones. "Words" alone simply was not enough to communicate the ideas of shrinking/enlarging or similar figures. With the help of Anno's beautiful illustrations of "Magic Liquids" in *Anno's Math Game III* (Anno, 1991), as well as drawing and cutting of different geometric shapes, Martha was able to solve the following task in her unique way:

Episode 3: An object was 45 cm long and 15 cm wide. It becomes 105 cm long after applying the magic liquid. How wide will it become?
Summary of Martha's strategy and reasoning:

> Martha first drew 9 sets of 5 dots, and explained that 45 cm was like 9 segments of 5 cm. With the 60 cm difference in mind, she then tried to distribute the 60 cm differences among these 9 groups of 5 cm. She knew it would be at least 6 cm for each 5 cm "cause 9 times 6 is 54." Then she attempted to distribute the remaining 6 among 9 (groups of 5 cm). With strategy similar to that described in Episode 2, she figured out each group would get another 2/3 cm. So the growth for each 5 cm was 6 2/3 cm. "That's how much each 5 cm will grow," Martha explained. Since 15 cm equaled to 3 segments of 5 cm, there would be a total growth of 20 cm. So she knew the object would become 35 cm wide.

Even though Martha's strategy helped her identify the correct answer, it was hard to tell whether she had an image of stretching (Figure 1a) where the growth occurred uniformly at "each of the infinitely subdivisible parts of the smaller figure" (Kaput and West, 1994, p. 284), or her concept of change was more additive in nature as her language suggested (Figure 1b). We were also amazed by the observation that Martha treated this so called "continuous" situation as a "discrete" one. To help clarify the nature of stretching activity, the interviewer introduced the phrase, "each of the 5 cm grew into 11 2/3 cm" and used the rubber band to simulate the stretching. Both of these seemed to have some effects on Martha's thinking. Later on, Martha would figure out the amount of change by dividing the new length by the old length. and multiplying the result by the old width to solve similar types of tasks.



(a) (b)

Figure 1

Larger Numbers or Difficult Ratios

The most difficult tasks for Martha were the ones with the second and/or the third quantity smaller than the first quantity, and the numbers involved were less familiar. Furthermore, this difficulty was apparent across a wide range of ratio and proportion situations. Because this phenomenon was identified toward the end of the teaching experiment, we were not able to study it as fully as we would like to. Nevertheless, we would like to offer our tentative findings for further discussion.

First of all, Martha was not comfortable working with three-digit (or larger) numbers. She had no difficulty carrying out the written algorithm quickly and correctly but was unable to verbalize the meaning of the operation even when the problem setting involved buying and selling. Second, Martha did not have enough experience dealing with non-unit fractions directly. Her favorite distributing strategy (as seen in Episode 2) only created unit-fractions and this strategy was less effective when the numbers involved were large. Even though sharing, dividing, and folding are common daily experiences which may help interpret the fractional situations, more carefully developed activities are needed to provide experience with non-unit fractions. One potential activity we used in the study was the weighed sharing, for example, "We bought 150 little toys with 60 dollars. I paid 24 dollars, you paid 36 dollars. How many toys should I get and how many toys should you get?"

Discussion

Our interaction with Martha and the other fifth grader, Bruce, (Lo and Watanabe, 1993, 1994) helped us see the conceptual bases of the formal or informal ratio and proportion strategies. For example, in order to use the unit-price approach meaningfully, a student has to have a solid understanding for division, rational numbers, homogeneity relationship, etc.

However, this does not mean that the instruction on ratio and proportion should wait until children have mastered the four basic operations with both the whole numbers and fractions as it is currently done in school. Our study indicates that children can develop sophisticated ratio and proportion strategy as long as they have a good understanding of numbers and operations which are frequently used in their daily lives. The attempt to generalize such strategies to larger or fractional numbers gives rise to the need to develop a more sophisticated understanding of numbers and operations. The current study supports the claim made by Kaput and West (1994) that initial instruction on ratio and proportion based on children's informal strategies should be introduced as early as the third grade.

Furthermore, our analysis indicates that the process of extending the meanings of four operations from single-digit to multiple-digit numbers should not be taken lightly as "applying analogy." Frequently, students lose their number sense while focusing all their attention on carrying out the computational procedures. Martha's difficulty with the meaning of whole number operations may help to explain why many students do not conserve operations when rational numbers are involved. The problem is really much deeper rooted.

References

Anno, M. (1991). *Anno's math games III*. New York: Philomel Books.

Confrey, J. & Smith, E. (1995). Splitting, covariation and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*, 66-86.

Harel, G., Behr, M., Post, T., & Lesh, R. (1991). Variables affecting proportionality. In F. Furinghetti (Ed.), *Proceedings of the fifteenth Annual Meeting of the International Conference on the Psychology of Mathematics Education*, (vol. 2, pp. 125-132). Assisi, Italy.

Hart, K. M. (1984). *Ratio: Children's strategies and errors*. Windsor, England: NFER-NELSON Publishing Company.

Karplus, R, Pulos, S., & Stage, E. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 45-90). Orlando, FL: Academic Press.

Kaput, J., & West, M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning patterns. In G Harel and J. Confrey (Eds.), *The Development of multiplicative reasoning in the learning of mathematics (237-292).* State University of New York Press, Albany, NY.

Lamon, S. (1993). Ratio and Proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education*, 24 (1), 41-61.

Lamon, S. (1994). Ratio and proportion. In G Harel and J. Confrey (Eds.), *The Development of multiplicative reasoning in the learning of mathematics (89-120).* State University of New York Press, Albany, NY.

Lo, J., & Watanabe, T. (1993, April). *Children's image-based schemes for solving proportion-type tasks.* Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, GA.

Lo, J., & Watanabe, T. (1994, April). Developing proportion schemes: A story of a fifth grader. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades.* National Council of Teachers of Mathematics and Lawrence Erlbaum Associates.

INCONGRUITY AND COMPLEXITY OF YOUNG CHILDREN'S UNDERSTANDING OF SIMPLE FRACTIONS

Tad Watanabe, Towson State University

Fifth grade children's understandings of simple fractions such as 1/2, 1/3 and 1/4 were investigated. The study identified three alternative conceptions of fractions: 1/*N* is one of *N* equal parts, 1/N is one of *N equal* parts, and parts must fit together to form the whole. In addition, many participants believed that the perimeter measured the area. This conception of area measurement also influenced the participants' problems solving activities significantly.

In an earlier work (Watanabe, in press, 1991), it was reported that second grade children held and used different, and often inconsistent, meanings of the fraction one-half in different contexts. The findings from the research raised a number of questions concerning children's understanding of fractions. Do older children hold inconsistent meanings for simple fractions such as 1/2, 1/3, and 1/4? If so, what are some of the meanings that are commonly held by the older students? How do they develop these meanings? What are the factors that influence children's construction of a variety of fraction meanings? Why is it possible for some children to hold inconsistent meanings of simple fractions without perturbation? And, finally, what are the influences of formal instruction and how can teachers cause perturbation in children so that they may construct more consistent meanings of fractions? To answer these and other questions, a series of investigations has been conducted. In this report, I will report findings from a study with fifth grade students on their understanding of simple fractions. The primary focus of this report is to identify and describe the alternative conceptions of these participants.

Theoretical Framework

Dykstra, Boyle & Monarch (1992) pointed out that the phrase, "alternative conception" has been used to describe a variety of meanings. I will use the phrase to mean, "the fundamental beliefs students have about how the world works, which they apply to a variety of different situations" (Dykstra et al., 1992, p.621). In this case, the "world" really refers to the participants' mathematical world. I agree with Dykstra et al. that calling students' alternative conceptions as "misconceptions" is inappropriate. Children's conceptions are not at random, but they are results of rational processes. We must understand how children formulate these conceptions so that we can provide appropriate learning opportunities for the students.

This research was conducted within a framework consistent with the constructivist epistemology. The aim of investigation was, therefore, not to evaluate children's fraction understanding measured against some pre-set standards. Rather, the goal of the study was to understand how children were making sense of mathematical ideas - in this case, simple fractions. Furthermore, this study was conducted with the belief that, if we do take constructivism seriously, the first step in teaching children mathematics must be understanding *children's* understanding, i.e., we must pay close attention to children's prior concepts (Steffe, 1988). In addition, it was also assumed that children's imagery (Wheatley & Reynolds, 1993) and their metaphors (Lakoff & Johnson, 1980) play central roles in their construction of mathematical meanings. Therefore, the analysis of children's concepts will include their imagery and metaphors.

Methodology

Sixteen fifth graders, 7 boys and 9 girls, from a single classroom participated in the research. These children were interviewed individually. The interviews were semi-structured in that a set of common tasks was prepared in advance, and each interview started with the same question. However, the interviewer made changes based on his on-the-spot analysis.

A variety of tasks were used during the interviews; however, the analysis reported in this paper is based on the following four tasks:

One-Half Task: In this task, students were shown 16 partially shaded figures (see Figure 1), and they were asked to identify those that were half-shaded. After the participant had selected those figures s/he believed were half-shaded, the interviewer asked her/him to justify how s/he knew those figures were half-shaded.

Cookie Question: For this question, three figures shown in Figure 2 were used. These figures were obtained from congruent squares by partitioning them into two congruent parts in different ways. The interviewer first showed two copies of each shape and demonstrated that they were identical by placing one on top of the other. He then arranged the two into the square and placed it in front of the participant. This was done with all three figures. After this demonstration, one copy of each shape was given to the participant, and s/he was asked to pretend they were their favorite kind of cookies. Then, the question was posed: You are really hungry, but you can have only one piece. Which one would you choose? After the participant selected one, s/he was asked to justify their selection. If the participant picked one shape as the largest, s/he was reminded of the initial demonstration and asked if that would help them make her/his decision.

Tangram Task: The seven tangram pieces were placed in front of the participant, arranged into a square. An identical square with outlines of each piece drawn inside was also presented on a separate sheet of paper. The participant was then asked to identify what part of the square each tangram piece was.

Identification: Different partially shaded figures were presented on grid papers. The participants were asked if the shape was 1/2 (or 1/3, or 1/4) shaded.

Findings

One of the alternative conceptions identified may be summarized in the phrase, "1/N is one of N equal parts." It is true that one of N equal parts is 1/N of the whole, and this is probably the most common way we approach fractions in lower grades. However, for many of the participants, this conception of fractions limited

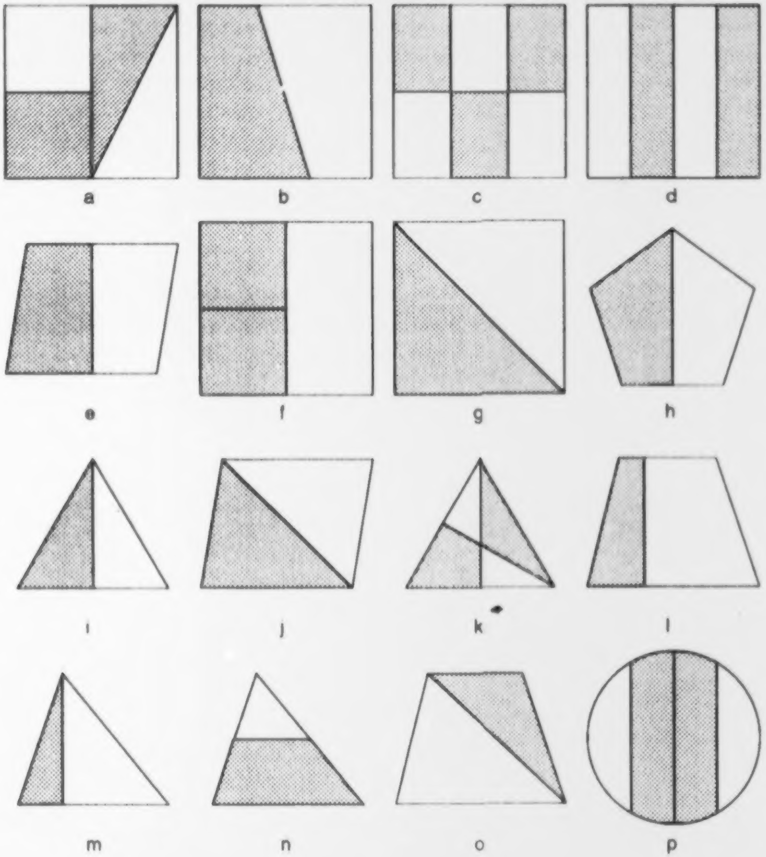

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Figure 1. 16 designs used in the One-Half *task.*

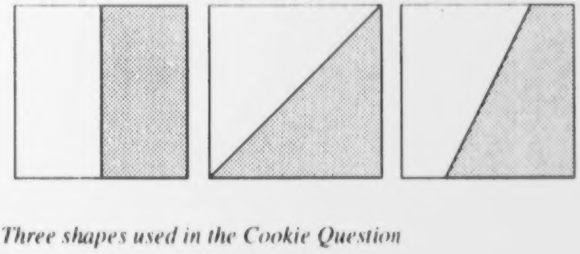


Figure 2. Three shapes used in the Cookie Question

their ability to deal with a number of tasks. There were two versions of this alternative conception.

1/*N* is one of *N* equal parts: SW selected all figures divided into two parts with one part shaded as one-half shaded, including figures (l) - (o) in Figure 1 that were not half-shaded. He had also constructed some form of understanding that 2 out of 4, 3 out of 6, etc., were equivalent to one-half. Therefore, the only shape he did not select as half-shaded was figure (f). Obviously he paid no attention to the *size relationship among the parts, nor the relationship between the shaded part and* the whole. Rather, for him, the important factor is the *number* of the parts and the relationship between the number of shaded parts and the number of unshaded parts. Because this alternative conception does not consider size relationship among parts of the whole or between the part and the whole, the resulting "fractions" have no quantitative significance. Therefore, SW decided that the triangular shape in the Cookie Question was the largest of the three even though, according to his conception, all three figures are one-half of the same square.

1/N is one of *N equal* parts: With this alternative conception, besides having to have N parts, all parts must be equal in size. For example, in response to an Identification question both KR and LD decided that the following figure shown was not 1/3 shaded because the three parts were not equal in size. With this alternative conception, the children were paying attention to the size relationship, but it was the size relationship among the parts, not between the part and the whole, that occupied their attention.

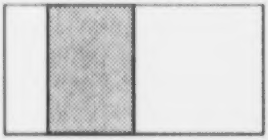


Figure 3. KR and LD said this figure was not 1/3-shaded

Another alternative conception identified was: parts must fit together to make the whole. This conception caused problems for three of the participants as they tried to decide what part of the large square the parallelogram tangram piece was. Many of them tried to cover the square using the parallelogram piece, and one even used two small triangles to make the parallelogram to assist her effort. For most of them, the fact that the parallelogram will not cover the square evenly was the major problem. KR explained why she could not find the answer by saying, "not all the sides are straight, so it won't fit evenly in the box."

Another major alternative conception influenced these participants' problem solving processes involving fractions, although it was not exactly a conception of fractions. This alternative conception was, "perimeter measures the area." In the Cookie Question, eight of the 16 participants selected one of the three figures as the largest even after they were reminded of the initial demonstration. Only two participants were able to decide that the three shapes were the same size when the problem was posed initially. The most common strategy used by the participants to justify their selections, usually the triangular piece, was to compare shapes by placing them next to each other and compare the lengths of the sides. Even when one shape was placed on top of another, many participants simply compared the lengths of "corresponding" sides, not the area. For them, comparing the lengths of sides was a legitimate way of comparing the "largeness" of the pieces. Although this alternative conception may not be directly related to children's conception of fractions, it may be the case that their work with fractions may have facilitated this conception. For example, during the Identification task, KR often counted the number of squares along each side of the both shaded and unshaded figures. This was a valid way for him to test that the parts were equal, and it works fine with most typical fraction exercises where the parts are congruent. If the measures of corresponding sides are equal, then, the parts are equal in size. Such an experience may have encouraged the formation of this alternative conception.

Discussion

Even among fifth grade students who participated in this study, the understanding of fractions was very much context bound. Thus, they were very capable of responding correctly to the One-Half tasks, yet half of the participants were unable to reason that the three pieces of the Cookie Question were the same size, a half of the square, even after they were reminded of the initial demonstration. These participants had received at least three years of formal instruction on fractions. They had studied fraction arithmetic in fourth grade and they were studying decimals. Yet, many participants have little number quantitative sense with fractions.

Furthermore, many of the participants' alternative conceptions identified in this study appeared to have grown out of the formal instruction, unlike many alternative conceptions in science such as, "motion implies force." This is both discouraging and hopeful. It is discouraging to learn that the formal instruction is contributing to the formation of these alternative conceptions. But, it is hopeful that if mathematics educators become aware of the possible problems with some of common ideas about fractions, they would be able to make appropriate adjustments so that they can keep these conceptions from being developed. It appears that the common part-of-a-whole approach to fractions must be complemented with much more emphasis on size relationships, especially the relationship between the part and the whole.

References

Dykstra, D.I., Boyle, C.F., & Monarch, I.A. (1992). Studying conceptual change in learning physics. *Science Education, 76* (6), 615-652.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by.* Chicago: University of Chicago Press.

Steffe, L.P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (eds.) *Number concepts and operations in the middle grades.* (pp. 119-140). Reston, VA: Lawrence Erlbaum Associates and National Council of Teachers of Mathematics.

Watanabe, T. (in press). Ben's understanding of one-half. *Teaching Children Mathematics.*

Watanabe, T. (1991). *Coordination of units: An investigation of second grade children's pre-rational number concepts.* Unpublished doctoral dissertation. Department of Curriculum and Instruction, Florida State University.

Wheatley, G.H. & Reynolds, A. (1993). *The construction and coordination of mathematics units.* Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA, April 12-16, 1993.

INTRODUCING FIFTH GRADERS TO DECIMAL NOTATION THROUGH RATIO AND PROPORTION

Andrea Lachance, Cornell University
Jere Confrey, Cornell University

This paper reports on a teaching experiment involving decimal instruction. After doing extensive work with multiplication, division, ratio and fractions through an innovative mathematics curriculum, fifth grade students were introduced to decimal numbers. To develop their understanding of decimal notation, students worked through three open-ended, contextual problems which encouraged them to make connections between decimals and previously encountered mathematical constructs such as ratio and fraction. After instruction, students' performance on decimal tasks indicate that students developed a robust understanding of decimal concepts. Based on these positive results, the authors assert that building decimal instruction upon students' ratio reasoning and fraction sense is a key component to helping students develop meaningful strategies for understanding and working with decimal numbers.

The various difficulties elementary school students have as they begin to work with decimal fractions have been well documented by mathematics education researchers (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989; Wearne & Hiebert, 1988, 1989). Hiebert and Wearne (1985) have hypothesized that children's struggles with decimals stem from the fact that what students learn about decimals is largely syntactic. In many schools in the United States, students are taught the rules governing decimal operations but are not given sufficient time or opportunity to develop a deep understanding of the notation itself. Without developing a meaning for the symbols from which decimals are constructed, students struggle to conceptually understand and successfully compute with decimals.

In the past, several teaching experiments involving decimal instruction have been conducted in an attempt to understand what types of classroom activities might help students construct a meaningful understanding of decimal numbers (Hiebert, Wearne, & Taber, 1991; Wearne & Hiebert, 1988, 1989). However, these teaching experiments have typically treated the teaching of decimals as a distinct and separate instructional unit (Ibid.). Little attention is given to the elementary school mathematics curriculum in which the decimal instruction is embedded and where in that curriculum such instruction belongs. Consequently, minimal effort in the research of decimal instruction has been given to how the understanding of decimals is connected to the understanding of other mathematical constructs presented earlier in the elementary school mathematics curriculum.

In this paper, we will report on the results from a teaching experiment involving instruction of decimal fractions embedded in a unique curriculum. Students who participated in this curriculum appeared to use their prior mathematical experiences with ratio and fractions to develop a strong conceptual understanding of decimal notation.

Research Context

The novel curriculum in which this decimal instruction took place is built around the construct of splitting (Confrey, 1994). Splitting actions, which include sharing, folding, and magnifying, are believed to stem from primitive notions which occur intuitively in children. Since these notions can lead directly to multiplication, and concurrently division and ratio, Confrey (1994) has argued that in order to support the intuitive splitting actions of children, students should be introduced to the constructs of multiplication, division, and ratio as a trio, early in their schooling. Consequently, Confrey developed a curriculum in which students are first introduced to all these multiplicative constructs simultaneously in the third grade.

Over the past three years, Confrey has piloted her curriculum with a class of elementary school students (n = 20) in a public school of a small city. When these students were in third grade, they were introduced to multiplication, division, and ratio as a trio of mathematical ideas. When this same class was in fourth grade, instruction focused on strengthening the construct of ratio and introducing fractions as a subset of ratio (Confrey & Scarano, 1995). As fifth graders, having developed a rich network of mathematical ideas, students were then introduced to decimal fractions. This paper focuses only on the aspects of this curriculum related to decimal instruction.

Decimal Instruction

Students' introduction to decimals was done over a six-week period and was built around three open-ended, contextual problems which students worked on in small groups. The first of these problems gave students the opportunity to review and further develop the ratio concepts they encountered earlier in the curriculum and to begin to connect these concepts to decimal notation. In the second and third contextual problems, students worked directly with decimal notation and computation involving decimals.

In addition to the contextual problems, numerous whole class discussions were held. The whole class discussions were used to help students with decimal concepts and operations which they would need to work on the contextual problems and on the homework. Students were typically assigned homework four nights per week and worked on these assignments individually. The homework gave students the opportunity to further practice and develop problem-solving and computational skills with decimals.

Assessment Tools

Students' understanding of decimal concepts was assessed through a series of written and interview tasks. All twenty students were tested prior to the start of decimal instruction and were given a similar test at the end of decimal instruction. The items on the pre and posttests were taken largely from previous research studies on decimal instruction (Hiebert & Wearne, 1985; Hiebert, Wearne, & Taber, 1991; Wearne & Hiebert, 1988, 1989; Resnick et al., 1989) and included a diversity of decimal tasks. In addition to the written test, four students worked on decimal tasks in an interview setting two weeks after decimal instruction ended. Tasks given in interviews were similar to the tasks on the written tests.

Results

In general, the students in this study performed exceptionally well on all written and interview tasks. Individually, only three students out of twenty did not show significant improvement (p-value < .05) between pre and posttests. As a group, the pretest average of 15.5% correct responses rose to 80.8% correct responses on the posttest (Table 1). A paired t-test conducted on this data revealed the group's improvement was highly significant (p < .001).

The items on the written tests were grouped into four different content scales: tasks relating to the meaning of decimal notation, ordering tasks, fraction to decimal tasks, and computation with decimal tasks. Group performance on each scale was computed to get pre and posttest averages (see Table 1 for scale averages). On all four scales, student improvement from pre to posttest was highly significant (p < .001).

Table 1
Class pre and posttest averages on the items in each content scale and on overall written test.

Scale	Pretest	Postest:
Meaning of Decimal Notation	41.7	80.6
Ordering Tasks	43.5	84.0
Fraction to Decimal Tasks	38.8	88.8
Computation with Decimals	18.3	67.8
Overall	**15.5**	**80.8**

Because many of the items on the written tests were taken from other studies done on decimal instruction, it was possible to compare the performance of students in this study to the performance of students in other studies. In their investigation of the invented rules students use to order decimals, both Resnick et al.(1989) and Sackur-Grisvard and Leonard (1989) classify students according to the type of rule they used to complete an ordering task. They also report on what percentage of their sample consistently ordered decimal numbers correctly, and thus could be classified as experts.

For the purposes of comparison, we reviewed the individual student's performances on decimal ordering tasks in our study. Fourteen students out of twenty got all ten of the ordering tasks on the written tasks correct and could thus be classified as experts. As the data in Table 2 illustrates, the percentage of students in our study who could be classified as experts is substantially higher than the percentage of experts found in other groups of students reported in other studies.

In another study, Hiebert & Wearne (1985) collected data on students' performances on different computation tasks. Table 3 presents both the addition and

Table 2

Percent of students classified as experts in completing decimal ordering tasks across three studies

Study Group	#students	%experts
THIS STUDY (1995)		
U.S. Fifth Graders	20	70.0
Resnick et. al. (1989)		
U.S. Fifth Graders	17	18.0
Israeli Sixth Graders	21	19.0
French Fifth Graders	38	53.0
Sackur-Grisvard & Leonard (1985)		
French Fifth Graders	49	53.1
French Sixth Graders	57	52.6

Table 3

Percent of correct student responses on decimal computation items found in the Hiebert & Wearne (1985) study and in this study.

H&W	Grade 5	Grade 6	Grade 7	Grade 9	This Studys	Grade 5
Items:	n=99	n=55	n=272	n=196	Items:	n=20
5.1+.46	20	62	70	79	4.5+8.6	80
6+.32	6	25	39	77	6.24+8+.873	70
.86-.3	12	35	51	81	5.42-.3	75
4.7-.24	5	42	53	69	7.6-.34	85

subtraction items and the student performance data on these items from Hiebert & Wearne's study along with similar information from our study. In comparison to students tested in Hiebert & Wearne's study, the students in our study performed substantially better than their peers in the same grade, and at least as well or better than older students.

Discussion

From the results presented above, it seems evident that students in our study developed a strong and robust conceptual understanding of decimal notation which allowed them to successfully complete a variety of decimal tasks. From our test data and our observations and interviews with these students, we believe that students' prior work with ratio and fraction was crucial to students' ability to develop a deep, conceptual understanding of decimals numbers.

For instance, because of their work with ratio and fractions, students viewed decimals as simply another form of fractions. In response to the question, "What are decimals?", three students (of the four who responded) said:

Carrie: A way to write fractions in base 10.

Max: Well, I just think it's sort of like a fancy way of writing fractions. Like if you can't use fractions in everything like the computer or calculator, you need to adapt it.

Kai: I think it's just a shorter way of writing fractions. Instead of writing 97/100, you just do point 97.

Such a vision of decimals gave students great flexibility in dealing with these numbers. When they had difficulties, they simply converted the decimal numbers to fractional form. In fact, whenever students were initially introduced to an operation (addition, subtraction, multiplication, or division) with decimals, their first reaction was usually to use fractions to complete the given computation.

In completing ordering tasks, students often used ratio reasoning. For example, in comparing .8 to .34, one student said, ".8 goes to 10 and .34 goes to 100. 8 is a lot closer to 10 than 34 is to 100, so .8 is bigger." In addition, students who used the "add a zero to the end trick" to compare two decimals of different lengths (e.g. .8 and .08) usually understood why the "trick" worked. "Since .8 is 8/10 and 8/10 is equal to 80/100, I can just write .8 as .80. It's the same thing," explained one student.

Conclusion

As the above examples illustrate, students frequently and easily connected and applied their knowledge of ratio and fraction to their work with decimals. It is our contention that the ratio curriculum and its approach to decimals supported and encouraged these connections and applications. Thus, given the meaningful, conceptual understanding of decimals that the students in this study developed, we recommend that ratio and fraction concepts be developed earlier and more broadly in the elementary school mathematics curriculum. Effective decimal instruction can then be grounded in students' understanding of these multiplicative constructs.

References

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. E. (1981). Decimals: Results and implications from the second NAEP mathematics assessment. *Arithmetic Teacher, 28*(8), 34-37.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel and J. Confrey (eds.), *The Development of Multiplicative Reasoning In Learning of Mathematics*. Albany: State University of New York Press.

Confrey, J. & Scarano, G. H. (1995). Splitting reexamined: Results from a three-year longitudinal study of children in grades three to five. Proceedings of the Seventeenth Annual Meeting of PME-NA. Columbus, OH: The Ohio State University. October 17-24.

Hiebert J. & Wearne, D. (1985). A model of students' decimal computation procedures. *Cognition and Instruction, 2*(3&4), 175-205.

Hiebert, J., Wearne, D., & Taber, S. (1991). Fourth graders' gradual construction of decimal fractions during instruction using different physical representations. *Elementary School Journal* 91(4), 321-341.

Resnick, L., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20*(1), 8-27.

Sackur-Grisvard, C. & Leonard, F. (1985). Intermediate cognitive organizations in the process of learning a mathematical concept: The order of positive decimal numbers. *Cognition and Instruction, 2*(2), 157-174.

Wearne, D. & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for Research in Mathematics Education, 19*(5), 371-384.

Wearne, D., & Hiebert, J. (1989). Cognitive changes during conceptually based instruction on decimal fractions. *Journal of Educational Psychology, 81*(4), 507-513.

LINKING INFORMAL KNOWLEDGE AND FORMAL SKILLS: THE CASE OF PERCENTS

Lynda Ginsburg, University of Pennsylvania
Iddo Gal, University of Pennsylvania

This study examined adult students' informal knowledge of percent and its relationship to their computational skills. Sixty adults studying in adult education programs were interviewed to ascertain their ideas of the meanings of benchmark percents, 100%, 50%, and 25%, as they appear in advertising and media contexts; ability to use these percents in everyday mental math tasks; and visual representations of these percents. Students also completed written computational percent exercises. Students' responses were examined to determine the nature of their informal knowledge and skills and a number of patterns were identified. The range and fragility of student responses and the diversity of knowledge gaps suggest the acquisition of isolated ideas, but the absence of elaborated frameworks.

Many studies have been published in the last few decades exploring the informal knowledge of mathematics that students develop. Much of the research has examined the mathematical knowledge young children bring with them to their early schooling experiences (e.g., Carpenter, Moser, and Romberg, 1982). Other studies have focused on the mathematical knowledge older children or adults, usually with little or no prior schooling, develop in out-of-school, functional contexts (e.g., Nunes, Schliemann, and Carraher, 1993). This research demonstrated that individuals can and do acquire informal mathematical knowledge as it is needed without the benefit of school learning and that this knowledge has important similarities to and differences from school-based knowledge.

Daily functioning in numerous real-world situations (e.g., dealing with work-related tasks, shopping, and understanding messages in the media) necessitates frequent encounters with percents. Therefore, it was postulated that almost all adults, even those with limited school-based knowledge, will have formed some ideas about the meaning of the percents they encounter and developed strategies to support percent related activities. Many everyday tasks do not require extensive computations but rather interpretive skills based on an understanding of the ideas underlying the percent system, "number sense," and mental math skills relating to percents.

The goals of the study were to examine some aspects of the informal knowledge of percent displayed by adult literacy students, identify its limitations and gaps, and examine the relationship between this knowledge and computational skills.

Design of the Study

Semi-structured interviews were conducted with sixty adults studying in 7 urban and suburban adult education programs. The 57 women and 3 men ranged in age from 18 to 53 years (mean=27.5) and had completed a mean of 10.6 years of schooling. While all interviewees were studying mathematics, none had begun working with percents in their present programs.

The adults were presented with explanatory, shopping, visual, and computation tasks involving the benchmark percents 100%, 50%, and 25%. They were shown everyday percent-laden stimuli such as newspaper articles and advertising flyers to elicit their ideas about five separate but related facets of the role of 100% as the basis of the percent system. Questioning about 50% and 25% centered around the adults' interpretations of the meaning and use of the percents in shopping contexts and the mental math strategies they use in those situations and in visualization tasks. For the computation task, the students completed a series of decontextualized percent exercises.

Findings And Discussion

Knowledge about 100% as the basis of the percent system

The 5 facets of the meaning of 100% explored in the study and the percentages of students' appropriate responses are shown in Table 1. Most of the errors in the visual task involved confusion between 15% and one-fifteenth, with students dividing the circle into 15 parts and identifying one part as "15%."

The question that caused the most difficulty required students to justify their use of 100% and clarify its meaning. Of those whose responses were considered appropriate, some at first seemed to be unsure or tentative about their ideas, but then appeared to be crystallizing and thinking through their ideas during the response process, suggesting that their ideas about 100% as the basis for percent may be fragile and still evolving.

Interviewer: Why did you use 100%?

Dorothy: It all depends on how you're breaking it down. You can use any number for a whole: fifty fiftieths, four fourths.

Interviewer: And when you are dealing with percent?

Dorothy: It would have to be over 100, 200% could be a whole, 250% couldn't be a whole because that breaks the rhythm.

Interviewer: So which numbers can be a whole?

Dorothy: Zeros: 100, 200, 300.

Interviewer: As high as you want?

Dorothy: All depends on what type of money you're dealing with. Got 10 million dollars (pause). No keep it at 100%, forget the 200%, etc. 100% is a whole.

In their justifications, 48% of the adult students seemed to be unsure that 100% represents a whole and is the reference point for other percents. Some students were unable to separate percent ideas from the contexts in which they were en-

Table 1

Percentages of Appropriate Responses for Each Facet of 100% as the Basis of the Percent System

	Grade level				
Facet	6th and below n=15	7th-8th n=24	9th and above n=15	Unclass. n=6	Total n=60
Percents lie on a 0-100 scale.	87%	96%	93%	100%	93%
Percentages of the of a whole sum to 100%	80%	79%	100%	67%	83%
Visual representation of % as proportional part of whole.	67%	71%	67%	100%	72%
100% mean whole of all.	80%	83%	87%	83%	83%
Justify use of 100% as the reference point for percents.	40%	38%	80%	67%	52%

countered while others ignored the proportional nature of percent and treated percents as absolute numbers.

> Interviewer: Would you always use 100% [to evaluate 90% in "This new test detects cancer correctly in 90% of the cases"]?
>
> Theresa: Yes. In a way, you don't know. It all depends on how many cases they used. 90% is good out of 100% of the people. If you have 250, 90% is not good. It's not half of 250 people. 125 would be half.

The number of appropriate responses by students for all 5 tasks involving 100% are shown in Table 2. A majority of the students, including a sizable group from the most advanced cohort, appeared to grasp some facets of the meaning of 100% but were unable to grasp others, demonstrating gaps, limitations, or inconsistencies in knowledge. There were no patterns of errors; knowledge gaps varied across students within all grade level groups.

Response patterns across tasks involving 50%

When asked about the meaning of 50% as it appeared in department store sales flyers, all students responded that 50% means one-half . However, when asked to explain their statement that 50% is the same as one half, 40% of the students did not relate 50% to 100% but rather explained the meaning of 50% as an artifact of our monetary system: "because 50 cents is one half of a dollar" or "$50 is one half of $100."

Table 2

Percentage of Students Within Grade Levels by Number of Appropriate Responses to Questions About 100% as the Basis of the Percent System*

Number of Appropriate responses	Grade level				
	6th and below n=15	7th-8th n=24	9th and above n=15	Unclass. n=6	Total n=60
0	7%	0%	0%	0%	2%
1	0%	4%	0%	0%	2%
2	20%	4%	7%	0%	8%
3	13%	29%	13%	17%	20%
4	27%	46%	27%	50%	37%
5	33%	17%	53%	33%	32%

Percentages are calculated within each grade level. Columns may not sum to 100% due to rounding.

One third of the students were able to solve all tasks involving 50%. Thirty students (50%) were able to solve tasks in shopping and visualization contexts, yet failed to solve at least one of the two mathematically equivalent written computational problems. Perhaps the test-like environment requiring written responses created an expectation that problems had to be solved using school-based computational algorithms and prevented students with limited knowledge of percent algorithms from assuming that they could create mental (or visual) models of test items to support meaning. The remaining students (17%) displayed various patterns of responses to questions involving 50%. Included in this group were two students who were able to solve one written computational task (50% x 10=?) but were unable to solve either an arithmetically equivalent shopping task or the visual task.

Response patterns across tasks involving 25%

Reliance on the monetary system was also found in students' explanations of the meaning of 25%. The responses of 77% of the students referred to fractions (one-fourth or one-quarter), money (25 cents off a dollar), or a combination of fractions and money. ("One quarter" was a difficult response to classify since the students could not always decide if they meant a fractional part, the name of a coin, or both.) The remaining 23% of the students were unable to explain the meaning of 25% (in the context of "25% off sale") although they did know that 50% was one half.

Of the 60 interviewees, 12 students (20%) responded appropriately to all tasks involving 25% and seven (12%) were unable to respond correctly to any task. The remaining students exhibited a variety of patterns of responses with the two most common patterns being success with only the visual task (23% of all students), and success with only the written computational task (15% of all students).

General trends across tasks involving 100%, 50%, and 25% and implications

Using a criterion of a maximum of one incorrect response to the tasks within a category (i.e., 4 correct out of the 5 tasks involving 100%, 4 out of 5 tasks involving 50%, or 3 out of 4 tasks involving 25%), success rates in the categories of "100%" and "50% " were quite similar. Of the 60 students, 41 (68%) were successful with "100% tasks" and 43 students (72%) were successful with "50% tasks." When individual performance across these two categories of tasks was considered, 55% of all students were successful in both categories and 15% were unsuccessful in both categories. The remaining students (30%) were successful in only one of the categories, with about half successful in each category.

The finding that 30% of the students were successful in one category but not the other suggests that the ideas targeted by the two categories of questions may not inform each other. A demonstrated knowledge of one set of ideas or skills does not necessarily lead to knowledge of the other; each body of information or skills is attainable in isolation for these students. Apparently, some students have some knowledge of the different facets of 100%, yet this knowledge does not help them sufficiently to make sense of situations in which 50% appears. Other students realize that 50% is equivalent to one half and are able to apply that knowledge in a useful way, yet do not have an elaborated conceptualization of a system based on 100% within which 50% has meaning. Perhaps the knowledge of the meaning and application of 50% is not mathematically based but was developed through personal experiences and encountering percent words in everyday usage in which the term "50%" is treated as a word synonymous with "half" rather than as part of a mathematical system.

The tasks using 25% were more difficult for students in all grade level groups than were the 50% tasks. Yet, 24 students (40%) were successful with at least three of the four 25% tasks, including 2 students from the group with the lowest scores on the standardized tests. The successful students were found to be those who also demonstrated proficiency on the "knowledge of 100%" questions (only 1 of the 24 students responded appropriately to *less* than four of the five questions) *and* on the tasks using 50% (only 2 of the 24 students were not successful here and all of their missed questions were written computations). These data suggest that those who were competent in comprehending and using 25% also demonstrated *both* a knowledge of the role of 100% within the percent system and the ability to use at least one other percent (50%) in a meaningful way.

On the other hand, demonstrated knowledge of the facets of 100% did not necessarily imply an ability to activate that knowledge across a variety of tasks using 25% (for 18 students), nor did an ability to work with 50% necessarily transfer to an ability to work with 25% (for 21 students). Knowledge of 100% and the ability to use 50% appropriately, to the extent these constructs were measured, was apparently not always sufficient for students to be in a position to generalize their knowledge to apply to 25%.

As expected, the highest grade-level group was the most successful with the various percent tasks. However, even within this group, there was evidence of gaps in understanding as well as some limitations on how and when knowledge was applied. Less expected was the ability of many in the lowest grade-level group to respond successfully to many of the questions. Apparently, many of the adult students who are classified as needing much remedial mathematics education (based on existing testing practices), do have some knowledge of the percent system and/or some familiarity with 50%; this knowledge, however, often seems to be limited to isolated informal ideas that do not inform activities involving 25%.

Many of the adult students in this study have acquired bodies of informal knowledge of percent and are able to apply that knowledge in some contexts but not others. Often this knowledge includes misinformation or gaps, but this does not seem apparent to the individual. Much of the students' knowledge of percent consists of isolated pieces of information tied to those contexts in which it was developed, either everyday contexts or school contexts, but is not integrated into an elaborated mathematical structure.

References

Carpenter, T. P., Moser, J. M., & Romberg, T. A. (1982). *Addition and subtraction: A cognitive perspective*. Hillsdale, NJ: Erlbaum.

Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics*. New York: Cambridge University Press.

PRESERVATION OF THE COMMON REFERENT IN THE ADDITION OF FRACTIONS: A CASE STUDY

Marta Elena Valdemoros, Research and Advanced Studies of the IPN, Mexico

This case study was carried out with a student who was 9 years old and in the fourth grade when the the study began. She was selected after the application of an exploratory questionnaire to 66 pupils in primary school. The main features of the case were the absence of a common referent and a corresponding unit to the free generation of references related to the addition of fractions. The girl was interviewed twice with the same instrument. The interviews consisted of 10 tasks. We confirmed that the link generated by this student between different classes of objects (referents) through a sum affected only fraction references (not natural number references). Likewise, the child was unable to establish a unit when facing the requirement to construct the additive situation by her own means while she could assign sense, recognize and select an adequate unit in simple tasks, and reconstruction of whole processes.

The problem considered herein centers on the difficulties experienced by the child in the elementary school, when he or she must assign a common referent to the addition of fractions. Are such difficulties of a general nature and do they arise in the context of addition of natural numbers? Or, rather, do they come out in the field of fractional numbers, connected with cognitive processes of greater complexity?

Evidence obtained in some previous studies (Valdemoros, 1993a, 1994a) offers direct support to the outline of this problem, since they allow one to recognize that most of the students included in that research (attending third and fourth grades of primary school, whose teaching of such numbers is undertaken in these grades in Mexico) relate different concrete referents with certain sums of fractions; that is, they refer a specific additive situation to various kinds of objects. Likewise, the aforesaid study established that the difficulty to construct compatible references with the addition of fractions has always been accompanied by the absence of a unit of measure to which each fraction involved in the sum is referred, in the field of "problem invention" by those students.

Supported by the weight that linguists and semiologists (Ducrot & Todorov, 1981; Eco, 1991, among others) assign to *references*, at the level of meaning formation, we grant here great attention to those, in the concrete framework of addition of fractions. The correlative concept of elaborations built around such references is constituted by the *unit*, which has been widely recognized as a fundamental cognitive component, both for the construction of fractional number ideas (Piaget et al., 1966; Kieren, 1983, 1984, 1988; Bergeron & Herscovics, 1987; Hiebert & Behr, 1988) as well as for the resultant integration of their relations and operations.

Method

In order to make an in-depth inquiry, we designed a case study for third and fourth grades of primary school. The institution chosen was a public school with a good performance within the local educational system.

So as to select the subjects that would be interviewed, we administered to 66 students a questionnaire comprised of 13 adaptations of tasks previously pilot-tested and submitted to the analysis of several specialists in the area. We assigned the questionnaire an eminently selective character, since it was the starting point for the case studies. The questionnaire included 5 problems involving the identification of fractions (with the meanings of the part-whole relation, measure, indicated quotient, and ratio); 4 tasks centered on the equivalence relation between fractions, 2 problems referred to the pictorial resolution of a sum (in the presence of a given figure), and 2 tasks requiring the "invention of a problem" by the pupils (each one related to a sum and a subtraction of fractions and without any given figure).

Seven children were chosen to carry out individual, videotaped interviews. The design thereof was specific for each case taking into account its particular profile (in spite that they all showed difficulties with the construction of references, the intrinsic details thereof differed). All cases were controlled by means of a triangulation scheme consisting of the comparison of results obtained during the interviews with responses to the questionnaire and with the notes of an observer. The results were submitted to a qualitative analysis (Valdemoros, 1993b, Valdemoros & Orendain, 1994, Valdemoros & Campa, 1994).

The case presented herein is that of Belen (a 9-year-old fourth-grade student) who exhibited good performance on the questionnaire, where (at the "problem inventing level") the corresponding task required her to "Invent a problem which contains 1/5 + 1/10," and she wrote:

> Two ladies are going to make a cake and they need 1/5 of mixture and 1/10 of lard. How much did the two of them get together? 1/5 + 1/10 = 15/50 (a text without a common referent for addition and lacking of a unit of measure—a common occurrence among these children).

This obstacle was observed for 23 children of the described group (that is, 23/66).

Belen was interviewed twice. The first interview took place some weeks after the application of the questionnaire. The second interview was developed eleven months later.

Belen's Interviews

For Belen's case study we designed ten different tasks (Valdemoros & Campa, 1994):

- The "re-invention" of an additive problem with fractions from the questionnaire (see Fig. 1).

TASK 1
Invent a problem which contains 1/5 + 1/10

Figure 1

- Two tasks referred to the "invention of problems" with natural number addition and subtraction (see Fig. 2).

TASK 2	TASK 3
Invent a problem which contains: 12+6	Invent a problem which contains: 19-9

Figure 2

- Three identification tasks of certain fractions with the aid of concrete materials and in the presence of two continuous wholes and a discrete whole (see Fig. 3).

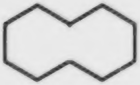
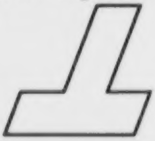
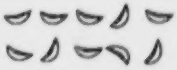
TASK 4	TASK 5	TASK 10
Use the blocks to represent 3/4 of this figure 	Use the blocks to represent 2/5 of this figure 	Identify 1/5 in the following set: 

Figure 3

- Two tasks for the reconstruction of the continuous whole from the part (see Fig. 4).

TASK 6	TASK 7
If △ is 1/3, draw 1	If ◇ is 1/4, draw 1

Figure 4

- Finally, two addition and subtraction tasks with fractions, in the presence of a certain figure and by means of the manipulation of concrete materials (Zullie's geometrical blocks, 1975). See Fig. 5.

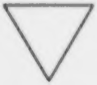
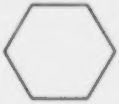
TASK 4	TASK 5
In this figure use the blocks to represent: 2/4 + 1/4 	In this figure use the blocks to represent: 1 - 2/6 

Figure 5. This was the common design for both interviews maintained with Belen.

The Main Results and Their Interpretation

During the first interview, when faced again with the task of fraction addition from the questionnaire, Belen "invented" a text of similar in nature to the above, with identical difficulties and semantic distortions (that is, she generated a text without a common referent for the sum and lacking of a unit of measure). She "invented" and adequately solved the natural number addition and subtraction problems. Belen expressed correct solutions with respect to the elementary activities of fraction recognition and reconstruction of the whole from the part. The girl, with more caution and effort, did also adequately solve the fraction addition and subtraction tasks, in the presence of an already identified figure and using concrete materials (geometrical blocks of different shape and size).

In the second interview, Belen exhibited results similar to those on the previous instrument. When we asked her what part of Task 1 could be changed, Belen wrote that the text was adequate and she wouldn't change anything in it. The girl easily solved the other tasks included in the second interview (specifically, the elementary activities of fraction recognition, the reconstruction of the whole from the part, and the natural number addition and subtraction problems). With more difficulty, she also correctly solved the fraction addition and subtraction tasks in the presence of an already identified figure and using concrete materials (Zullie's blocks).

Confronting both interviews, we confirmed that the main progress evidenced by Belen was the final development of a more efficient algorithm (during the second interview she wrote: 1/5 + 1/10 = 3/10). However, the central feature of this case—the generation of a text without a common referent for the sum and lacking of a unit of measure—didn't change.

In general, Belen did not exhibit difficulties in recognizing the unit in simple fractional contexts (identification of the fraction tasks). The girl could also configure the unit from the part. She also did not produce errors when she added and subtracted fractions in more complex situations and using concrete material (Tasks 8 and 9), because the respective unit of each task was already established. But, the most complex requirement—the selection of a unit for the sum of Task 1—couldn't be adequately solved by Belen when she produced the corresponding text. Due to this outcome, we infer that the unit couldn't be "re-signified" by the girl (that is, endowed with new meanings) at the last elaboration level where there were some evidences of incomplete semantic processes.

With regard to the loss of the common referent for addition and the construction of unsuitable additive referents when Belen chose a referent without taking into consideration the need of its preservation in the frame of addition, we confirmed that it had its origin in the terrain of fractions. The second interview showed us that the problem we detected during the first interview didn't disappear with posterior teaching. Maybe it was not considered an important cognitive obstacle for the student.

Conclusions

We confirmed that the link generated by Belen between different classes of objects (referents) through a sum only affected fraction references (not natural number references). She was able to assign sense, recognize and select an adequate unit in simple tasks, and complete reconstruction of whole processes.

The "re-signification" of unit in additive contexts (that is, the production of new meanings for unit in more complex frames) was possible when we presented her a certain figure related to the respective task. Facing the requirement to construct the additive situation by her own means, Belen was unable to establish an unit.

Perspective of this Case

We are now carrying out other studies of the difficulties related to the preservation of a common referent for addition of fractions among students in diverse public schools. The conclusions stated in the Belen's case allow us to establish the hypotheses of the new studies. The design and results will be communicated in future reports.

References

Bergeron, J. & Herscovics, N. (1987). Unit fractions of a continuous whole. *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education*. 357- 365.

Ducrot, O. & Todorov, T. (1981). *Diccionario enciclopédico de las ciencias del lenguaje*. Mexico: Siglo XXI.

Eco, U. (1991). *Tratado de semiótica general*. Barcelona: Lumen.

Hiebert, J. & Behr, M. (1988). Capturing the major themes. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades*. Reston, VA: National Council of Teachers of Mathematics. 1-18.

Kieren, T. E. (1983). Partitioning, equivalence and the construction of rational number ideas. *Proceedings of the Fourth International Congress on Mathematical Education*. 506-108.

Kieren, T. E. (1984). Mathematical knowledge building: The mathematics teacher as consulting architect. *Proceedings of the Fifth International Congress on Mathematical Education*. 186-194.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades*, 2. 162- 181.

Piaget, J., Inhelder, B. & Szeminska, A. (1966). *The child's conception of geometry*. London: Routledge and Keagan Paul.

Valdemoros, M. (1993a). *La construcción del lenguaje de las fracciones y de los conceptos involucrados en él*. Doctoral Dissertation. Matemática Educativa-CINVESTAV, Mexico.

Valdemoros, M. (1993b). *Fracciones y referencias: El caso de los cambios arbitrarios de unidad en situaciones aditivas*. Matemática Educativa-CINVESTAV, Mexico.

Valdemoros, M. (1994a). Various representations of the fraction through a case study. *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*, II. 16-23.

Valdemoros, M (1994b). Fracciones, referentes concretos y vínculos referenciales. *Memorias de la Octava Reunión Centroamericana y del Caribe sobre Formación de Profesores e Investigación en Matemática Educativa*. 21-30.

Valdemoros, M. & Orendain, M. (1994c). *Fracciones y referencias: El caso de la ausencia inicial de referentes concretos*. Matemática Educativa-CINVESTAV, Mexico.

Valdemoros, M. & Campa, A. L. (1994d). *Fracciones y referencias: El caso de la relación entre distintas clases de objetos a través de la suma*. Matemática Educativa-CINVESTAV, Mexico.

Zullie, M. E. (1975). *Fractions with pattern blocks*. Palo Alto: Creative Publications.

A PROPOSED CONSTRUCTIVE ITINERARY FROM ITERATING COMPOSITE UNITS TO RATIO AND PROPORTION CONCEPTS

Michael T. Battista, Kent State University
Caroline Van Auken Borrow, Kent State University

In this article, we attempt to describe the meanings students construct and the conceptual advances they make as they deal with ratio and proportion problems. We argue that a critical factor in students' comprehension of and solution to these problems is their explicit recognition of the action that links composite units. We highlight critical transitions in students' constructive itineraries, arguing that an essential component of these transitions is students' development of related concepts and their integration of that conceptual knowledge with ratio and proportion reasoning.

Conceptual Milestones

Iterating Composite Units

Multiplicative thinking is the foundation on which students construct notions of ratio and proportion. Steffe (1988) has argued that the key to students' meaningful dealings with multiplication is the ability to iterate abstract composite units. This involves taking a set as a countable unit while maintaining the unit nature of its elements. For example, suppose a student is asked "If there are 9 groups of 3 blocks, how many blocks are there?" If the student can solve this problem by coordinating two number sequences, he or she has established an iterable composite unit. That is, the student counts: 1 group is 3, 2 groups is 6, 3 is 9, 4 is 12, 5 is 15, ..., 9 is 27.

Extending the Thinking

Once students are able to iterate composite units, they can extend their multiplicative thinking to ratio situations. Episode 1 describes how a second grader who regularly iterated composite units in solving multiplication problems extended his multiplicative schemes to ratio situations.

Episode 1. The interviewer made a bundle of 5 white and 3 red sticks and asked how many of the same kind of bundles would be behind his back if he had 10 white sticks. JB recited "5, 10," then answered 2. With the same bundle, the interviewer asked how many whites there would be if there were 12 reds. JB figured "You need four bundles to get 12 reds. Then 5, 10, 15, 20." Hence, JB coordinated the iteration of composite units of 5 and 3.

Iterating "Linked Composites"

In Episode 1, JB extended his coordination of a counting-by-1 scheme with another counting scheme to coordinating counting-by-1 with two other counting schemes. That is, before these examples, JB had constructed counting sequences in which he coordinated a counting-by-1 scheme with, for example, a counting-by-3 scheme.

1 group	2 groups	3 groups	4 groups
3 objects	6 objects	9 objeects	12 objects

But in Episode 1, he extended this counting scheme to construct a "linked composite" counting sequence:

1 group	2 groups	3 groups	4 groups
3 of object A	6 of object A	9 of object A	12 of object A
5 of object B	10 of object B	15 of object B	20 of object B

JB was able to iterate a composite consisting of a composite of 3 linked together with a composite of 5. He solved the ratio problem by analyzing the iteration of linked composites. In Episode 1, he first iterated the 3 until he got 12 to determine how many of the 3-to-5 composites there were. He then iterated 5 that same number of times.

It is our contention that the type of thinking that JB exhibited in Episode 1 can serve as a foundation for future meaningful dealings with ratio and proportion. In fact, the strategy of iterating linked composites used by JB (often called a build-up strategy) is used widely and successfully by older students (Hart, 1984; Kaput & West, 1994; Lamon, 1994). In Episode 2, we see a seventh grader making this iterated linked-composites thinking more sophisticated. JR used the same reasoning as JB in Episode 1, except that JR used division to find the number of linked composites, and multiplication rather than skip-counting to find the answer. Kaput and West (1994) call this an abbreviated build-up strategy.

Episode 2. At a dining room table, there are 3 serving utensils for every 2 plates. If there are 10 plates, how many serving utensils are there?

JR: I got 15.

Int: What were you thinking?

JR: Well, I used the 2 and 10. I divided the 10 by 2 and got 5.

Int: Why?

JR: Well for every 10 plates, I got 15 utensils. There are two plates in a set and there are 5 sets. For every set, there are 3 utensils. So for 5 sets, 2 plates and 3 utensils, so 3 times 5 is the number of utensils in the number of sets.

The Transition from Iteration to Multiplication and Division

It is essential to determine what enables students to make the transition from solving ratio problems by iterating linked composites to using multiplication and division. The work of the two students below, who just completed fifth grade, suggests some elements of this transition.

EB dealt with linked composites in ratio problems by iterating, making drawings, and using the operations of multiplication and division. When she used the former two methods, the interviewer often asked her if there were other ways she could solve the problems, hoping EB would see how the use of operations could shorten the iteration process. But EB often struggled with her use of the arithmetic operations.

Episode 3. Mitch paid $4.50 for 5 computer disks. How much did he pay for a dozen?

> EB iterated linked composites, reasoning that 10 disks cost $4.50 + $4.50 = $9.00. But to find the cost of the additional 2 disks, she simply divided $4.50 by 2. (We saw this same mistake by a ninth grader.) However, EB noticed that she made a mistake, so she divided $4.50 by 5 to get $.90: "One disk would be 90 cents. Then another one, plus them together and it would be $1.80. So it would be $10.80." Significantly, even though EB found the price per disk, she used it only to find the cost of the left over 2 disks; she was still iterating linked composites. When asked if there was another way to solve the problem, "now that you've done this division," EB did not know until the interviewer asked how much each computer disk cost. EB: "Do $4.50 + 5, you'd get $.90; then do $.90 ¥ 12."

As the next episode illustrates, EB also needed to explicitly conceptualize linking the two composites to make sense of ratio problems. In particular, she had difficulty conceptualizing the linking action in unfamiliar contexts.

Episode 4. If you can exchange $3 for 2 pounds, how many pounds can you exchange for $21?

> EB initially said that you'd get 5 pounds for $6: "You always get 1 less." The interviewer then asked questions to make explicit the pairing of $3 for every 2 pounds: How many pounds for the first $3? How many pounds for the next $3? So how many pounds for $6?... Although EB was able to answer the questions correctly, she focused on patterns in the separate sequences, first noting the differences between successive values in the linked composites, "It's minus 2, minus 3, minus 4, ..." then differences between the differences in the two sequences "It's always 3 then 2." Indeed, EB seemed to see two separate sequences; the unfamiliarity of the context prevented EB from seeing the problem in terms of the action of exchanging $3 for every 2 pounds.

EB's ability to use operations with linked composites seemed to involve three essential components. First, she needed to explicitly conceptualize the repeated action of linking the two composites to make sense of ratio problems. Second, she needed to have sufficient understanding of the meaning of multiplication and division so that she could see their relevance in the iteration process. Third, and finally, EB needed to have sufficiently abstracted the iteration process so that she could reflect on it, then reconceptualize it in terms of her knowledge of the multiplication and division operations.

CR also dealt with linked composites in ratio problems by iterating, but attempted to shorten the iteration process in the episode below.

Episode 5. (Problem 1) There are 3 boys for every 4 girls in Mrs. Smith's class. If there are 28 students in the class, how many girls and how many boys are there?

> CR: Well 4 plus 3, so 7 altogether. There are 28 students. 7 times 3 is 21. And you need 28. So another 7 on to 21 would equal 28. That's 4 different groups of 4 girls and 3 boys. For every group there's 3 boys and 4 girls. So you have 3 times 4 which equals 12 boys, and 4 times 4 which equals 16. So 12 boys and 16 girls.

(Problem 2) Suppose in a large class, there are 4 girls for every 6 boys. There are 250 students altogether. How many boys and how many girls are in this class?

> CR: I know that there are 4 girls and 6 boys, and that equals 10. There are 250 students in the class. And so to make 100, that's 10 of them. So double that to make 200, that's 20 of them. And then to make 50, that's 5 of them. So that would be 25. Int: 25 what?

> CR: 25 groups of 10, groups of 4 girls and 6 boys. You take 6, and times that by 25 (she does the computation 25 times 6). So there's 150 boys in the class.

In these problems, CR curtailed the iteration process by using known multiplication facts to aid her in determining the total number of iterations. She then correctly multiplied the relevant composite unit by that total. This curtailment required CR to sufficiently abstract the iteration action so that she could reflect on it and anticipate that the result of several iterations could be captured by a known multiplication fact. After CR completed Problem 2, the interviewer queried her about other ways to solve the problem. CR mentioned guess and check, then division; but it wasn't immediately obvious to her how division could be used to solve the problem. CR also admitted to getting confused by division. However, she solved several subsequent problems by dividing with a calculator.

Extending Linked-Composite Sequences beyond Whole Numbers

One of the major accommodations that students have to make to the multiplicative scheme employed by JB and JR occurs when the numbers do not "divide evenly." For instance, Lesh, Post, and Behr (1988) report a seventh grader enlarging a 2x3 rectangle by doubling the lengths of the sides to produce a 4x6 rectangle. However, when asked to enlarge this rectangle so that the base would be 9, the student responded that doubling would make the base 12. So he added 3 to 4 because 3 had to be added to 6 to get 9. This student was unable to find a way to appropriately alter his linked composite scheme to deal with this new situation, so he switched to an additive scheme. Two other seventh graders, TM and JR, however, were able to make proper accommodations to their linked pair iteration schemes, although their strategies were quite different.

Episode 6. In hot chocolate, for every 2 cups of milk, one needs 4 teaspoons of cocoa. If a person has 5 cups of milk to make hot chocolate with, how many teaspoons of cocoa are needed?

TM: There are 2 cups for every 4 teaspoons, so this is a ratio, proportion kind of thingy. Two goes into 4, 2 times. Four goes into 8, 2 times, and there is a fifth cup left, so divide 2 by 2 to get 1. So 10 is the answer.
Int: How did you get that?

TM: O.K. There are 2 cups of milk, and 2 cups of milk. That's 4 cups of milk. We need 5 cups, so one left over. The first 2 cups of milk is 4 teaspoons. The second 2 cups is 4 more. That's 8 teaspoons so far. There is 1 cup left over. 1 cup is 1/2 of 2 cups. What's 1/2 of 4 teaspoons? That's 2 teaspoons. 8+2 is 10.

JR: I divided 2 cups of milk and 4 teaspoons by 1/2, and got 2 teaspoons. I divided by 2 to come up with 1/2 of it.

Int: Half of what?

JR: I needed to know 1/2 of it because 2 doesn't go into 5. I divided it so I could find out what 1 cup was. Half of 2 is 1 cup; half of the cocoa is 2 teaspoons. So, 5 cups is 10 teaspoons.

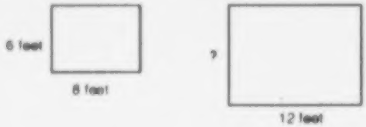
Because 2 does not divide 5, TM returned, in essence, to a skip-counting approach: 2 cups of hot chocolate for 4 tablespoons of cocoa, 2 more cups for another 4 tablespoons. TM then altered his strategy; instead of adding a full 2-to-4 linked composite, he added half of such a composite, getting 1 cup with 2 tablespoons. In essence, he started to extend his iteration scheme beyond whole-number increments, so that instead of making whole-unit increments of a 2-to-4 linked composite, he made a half-unit increment of this composite.

JR's method, on the other hand, addressed the problem by recalibrating the 2-to-4 linked composite to make it a 1-to-2 composite. He divided the 2-to-4 composite by 2 so that the 2-cup component of that composite evenly divided 5 cups. JR seemed to anticipate what unit he needed by envisioning the whole iteration sequence (i.e., by embedding his linked composite in a whole sequence). In the episode below, JR did a similar thing, but gave it a new interpretation. He kept dividing by 2 until he again obtained an increment unit of 1. But this time he interpreted the final ratio as a unit ratio.

Episode 7. Mr. Short is 4 large buttons in height. Mr. Tall is similar to Mr. Short but is 6 large buttons in height. Measure Mr. Short's height in paper clips and predict the height of Mr. Tall if you could measure him in paper clips. Explain your prediction.

JR: I took half this [6 paper clips] which is 3, then he [Mr. Short] is half of 4 buttons, which is 2. The 3 and the 2 are the same thing. Then I divided the 3 by 2 to get 1.5. I needed to figure out the number of paper clips in a button; 1.5 times 6 = 9.

That JR's method included creating new increment units seems to be verified by his strategy use in Episode 8.



Episode 8. These rectangles are the same shape, but one is larger than the other. Explain how you would find the height of the larger rectangle.

JR: I got 9. These two are the same proportions. Everything is the same other than the size.

Obs: Why do you say that?

JR: If you blew this up and made it bigger, or shrink it, it would be the same size.

Obs: How can it be the same size?

JR: It would be the same size as the bigger one, if you blew up the smaller one. So I took this rectangle [the 6 by 8] and divided all the sides by 2, also I multiplied this [the sides] by 3 and that's the same size as this [the larger rectangle].

Obs: What sides did you get?

JR: I took the smaller box, I got 3 and 4, then I multiplied this by 3, so you get 9 and 12.

Obs: Why did you divide by 2?

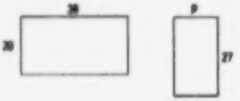
JR: Because I knew that if I divided by 2, I could find the missing side. The smaller rectangle [6 by 8] I could find the missing side, then of the larger rectangle [9 by 12] if I multiplied by something, and I knew I could do this.

In this episode, JR saw that 8 did not evenly divide 12. So he divided the 8 by 2 to get 4, which divides 12; so the iteration sequence included the target 12. He then saw that it takes three 4s to get 12, and concluded that it takes 3 of the 3s to get the desired side length. Also, he seemed to be able to use his thinking about stretching and shrinking to help him reason through this problem, especially with the difficult interpretation of what he got when he first divided by 2 (a "smaller box"). Essential to JR's last step seem to be numerical transformations that stretch and shrink rectangles, while preserving their shape and the ratio of the lengths of their sides.

Proportional Thinking

Students have achieved proportional thinking when they see how to numerically transform the terms in one ratio to the corresponding terms in an equivalent ratio, when they see that the same transformation applies to corresponding terms of equivalent ratios.

Episode 9. Find the value of p in these similar rectangles.



JB (sixth grade) writes the following:

$$\frac{20}{36} = \frac{p}{27}$$

JB then solved the problem by figuring that you get 27 from 36 by dividing by 4 then multiplying by 3, so you must do the same to 20: 20 ÷ 4 = 5, × 3 = 15.

Obs: How did you know to do this?

JB: They're equivalent fractions.

In Episode 9, JB has made the equivalence of ratios explicit. He performs a more complex numeric transformation on the elements of the first ratio to get the second, a natural evolution of the type utilized by JR.

Episode 10. Sue can walk 15 miles in 5 hours. How far can she walk in 3 hours?

JB (seventh grade) writes: $\frac{15}{5} = \frac{x}{3}$ $\qquad \frac{3}{1}$

JB: You have to multiply 5 by 3/5 to get 3, so x is 3/5 times 15; so it's 9.

So JB has extended his thinking from the year before to be even more sophisticated. He combined the two operations of multiplying and dividing into the single multiplication by a fraction. He could even extend this thinking to irrational numbers.

Obs: How far can she walk in $\sqrt{2}$ hours?

JB: Because it's 3 over 1, you multiply 1 by $\sqrt{2}$ to get $\sqrt{2}$, so you multiply 3 by $\sqrt{2}$.

JB, who was in algebra when this interview occurred, also used cross multiplying to find answers to some proportional problems.

Cross Multiplying

Solving proportional equations by using cross multiplying requires the use of structural operations from algebra, which is a difficult step for most students to make. Thus, students are likely to make sense of this strategy only when they understand such operations in algebra.

Conclusion

In our proposed constructive itinerary for ratio and proportion, students move from iterating single composites to iterating linked composites to solve ratio problems. They progress to using operations with linked composites when they have sufficiently abstracted the iterative process so that it can be connected to already firm conceptualizations of multiplication and division. They also extend the iterative process from whole number to fractional increments. Students make the transition to proportional reasoning as their focus shifts from implementing the iterative process to reflecting on the numerical operations that transform one ratio to equivalent ratios. (This shift may be strongly connected with their emerging knowledge of fractions and equivalent fractions.) The final step occurs as students apply structural operations from algebra to classical proportional equations. In all cases, transitions to more sophisticated thinking occur as students reflectively abstract their current ratio schemes, taking them to a higher level in which they can be integrated with knowledge of other relevant mathematical concepts.

References

Hart, K. M. (1984). *Ratio: Children's strategies and errors.* Windsor: NFER-Nelson.

Kaput, J. J., & West, M. M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Ed.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 235-287). Albany: SUNY Press.

Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Ed.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 89-120). Albany: SUNY Press.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Ed.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Ed.), *Number concepts and operations in the middle grades* (pp. 119-140). Reston, VA: National Council of Teachers of Mathematics.

SPLITTING REEXAMINED: RESULTS FROM A THREE-YEAR LONGITUDINAL STUDY OF CHILDREN IN GRADES THREE TO FIVE

Jere Confrey, Cornell University
Grace Hotchkiss Scarano, Cornell University

A report is made on the results of a three-year teaching experiment introducing students to the concepts of multiplication, division and ratio as a trio, and to ratio and proportion in a project-based curriculum with heterogeneous grouping. Fractions were introduced as a subset of ratio and proportion. The paper outlines curricular changes in the third through fifth grades and focuses on the major representational forms used by the students including: Venn diagrams, daisy chains, contingency tables, tables of values, dot drawings, two-dimensional graphs and ratio boxes, and discusses the role these tools played in the development of students' understandings of the multiplicative world. Results of the study are presented showing that these 10 and 11 year olds exceeded the comparative performance of 14 and 15 year olds on ratio and proportion test items.

Ratio and proportion is arguably the most critical concept to learn in the elementary curriculum in order to make a successful transition into advanced mathematics. Its centrality is secured by both its conceptual and practical characteristics. Proportional thinking represents increased cognitive complexity in comparison to other arithmetic procedures of the elementary curriculum and demands considerable mental flexibility. It underlies such notions as scale, rate of change, acceleration, algebraic fractions, etc. Proportional thinking is involved in all kinds of applications of mathematics, from gears to weights, from motion to conversion tables. The learning of ratio and proportion has garnered significant attention from researchers around the world (Hart, 1988; Lamon, 1994). Its relationship to fractions has been hotly debated (Behr, Harel, Post & Lesh, 1992), its placement in multiplicative conceptual fields explored (Harel & Confrey, 1994; Vergnaud, 1994), and its developmental sequences articulated multiple times (Karplus, Pulos & Stage, 1983; Noelting, 1980; Piaget, Berthoud-Papandropoulou & Kilcher, 1987).

One of the most compelling and startling analyses of personal knowledge of rational numbers is offered by Kieren (1988). He proposes that this "complex and textured" (Kieren, 1988, p. 162) knowledge is comprised of multiple constructs including partitioning, equivalencing, measure, quotient, ratio number, and others. More recently extending and simplifying these constructs, Confrey proposed the *splitting conjecture* (Confrey, 1988). This conjecture posits that counting and splitting are two of the primitives that spawn our number system. Confrey argued that just as the act of partitioning is a primitive that cannot be reduced to repeated subtraction, a complementary construct, the inverse of partitioning, exists that is the precursor to multiplication and cannot be reduced to repeated addition. These partitioning acts which are precursors to multiplication and division evolve from a primitive she called "splitting" that involves the activities of sharing and folding, and geometric constructs which create a fundamental relationship to similarity.

Furthermore, Confrey argues that splitting is a basic cognitive structure that parallels, but differs from, counting.

This conjecture implies profound alterations in the scope and sequence of the typical course, particularly from third through fifth grade (and before and beyond). To examine these changes, a three-year longitudinal study at Belle Sherman Elementary School in Ithaca, NY was undertaken starting with a group in the third grade who would remain together until fifth grade and entry to middle school. The curriculum used by the experimental class incorporated significant changes which are described below.

In third grade, 1) multiplication, division and ratio were introduced as a trio. The order of introduction followed a splitting sequence, starting with twos, fours, fives, tens, eights, threes, sixes, nines and then sevens. 2) Extensive exploration of partitive and quotitive division and their interrelationship through the use of arrays was investigated.

In fourth grade, 1) least common multiple (LCM) and greatest common factor (GCF) were introduced early in the curriculum using prime factoring as students were encouraged to increase their mental flexibility in multiplicative conceptual space. 2) Ratio and proportion were introduced prior to the development of any operations on fractions, except simple recognition and naming of fractional parts. 3) The operations of multiplication and division within rational numbers were developed as extensions of ratio relations. 4) Explorations of ratio involved the two-dimensional plane and similarity relations on geometric figures. 5) Fractions were developed as a subset of ratios which share a common unit, and addition and subtraction of fractions as therefore requiring the identification of a common measurement unit.

In fifth grade, further extensions of ratio thinking, especially as regards multiplication and division, were developed. 1) Transitions to decimals were facilitated using the notion of ratio conversions between smaller and larger units. Mixed systems (such as weight measured in ounces, pounds, and tons) were contrasted with the "pure" system (where one n:1 ratio serves as the conversion factor between adjacent sized units) of decimal notation which utilizes the 10:1 ratio (Lachance, 1995 this volume). 2) Percent was treated in relation to decimal as ratio is in relation to fraction. 3) A transition to the use of algebraic symbolism was undertaken (Luthuli & Confrey, in progress).

Most units were taught using a project-based approach. Students were presented with project challenges, and materials and tools were provided for explorations (Preyer, in progress). For example, during the fourth grade year, the children designed handicap ramps. They were given a child's wheelchair and went outside to find a slope they could go both up and down while remaining in control. Students used a plumb line, measuring tape, and level to figure out how to describe their slope. Each group of students used their slope to create scale drawings and a model ramp for a given height of stairs. They also predicted the cost of materials given a certain set of conditions. Children were heterogeneously grouped with the assumption that all students would complete a performance assessment and an individual open-ended written assessment on all topics.

During the three-year teaching experiment, an exploratory methodology was used. Curriculum units were developed that were aligned with the splitting conjecture. This meant that ratio and proportion were assumed to be intimately connected to multiplication and division, that addition and subtraction of fractions were assumed secondary to multiplication and division, and that connections to geometry were given priority over additive relations. These subconjectures were modified as the experiment evolved in light of student work. All classes were videotaped and when the children worked in small groups, a single group was selected for videotaping for the duration of the project.

We will introduce the major forms of representation used extensively by the students and then report on the quantitative data concerning the students' performance on written ratio and proportion assessments.

1. Venn diagrams for LCM and GCF. The children were taught to prime factor numbers and to find their LCM and GCF using Venn diagrams. For two prime factorizations A and B, $A \cap B$ yields the GCF and $A \cup B$ yields the LCM. LCM was explored in the context of clapping rhythms to predict when two clappers would clap simultaneously. The idea behind this exploration was to explore numbers' "multiplicative biographies."

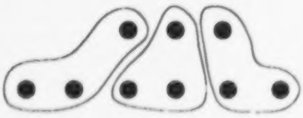
2. Daisy chains. Before beginning the introduction to ratio and proportion, students were asked to explore multiplicative space by creating sequences of operations, only using multiplication and division, to move from one number to another. Thus, to go from 28 to 36, a student might write: $28 \div 7 \Rightarrow 4 \times 3 \Rightarrow 12 \times 3 \Rightarrow 36$. Later this notation would be curtailed to $28 \times \frac{9}{7} = 36$. The students discovered two important methods concerning how to move from a to b (where a and b are rational numbers): 1) divide by a to get 1 and then multiply by b; 2) multiply by b to get ab and then divide by a. We claim that this is the critical meaning of multiplication by a ratio.

3. Contingency tables. The introduction to comparing ratios was undertaken in the context of polling. Students used 2 by 2 contingency tables, often divided into boys' and girls' responses categorized into "yes" and "no." Totals were listed in the margins. This format, in contrast to writing the proportions as $\frac{yes(part)}{no(part)}$, encouraged the students to work flexibly with their data concerning both "numerators" and "denominators." Children described results as part to part or part to total, depending on what they wanted to claim from their data.

	Boys	Girls	Total
Yes	3	5	8
No	7	9	16
Total	10	14	24

4. Tables of values. As the children extended their explorations from the comparison of ratios to the equivalence of ratios, the use of the contingency table was extended to the use of a table of values. Employing the context of a two-ingredient recipe (one ingredient in each column), students easily made larger recipes by doubling or tripling the original recipe. Then they explored halving it and argued for its equivalence. One student recognized the "pace" of the ingredients (water and oranges) claiming, "Just add three (waters) on every time you add an orange" (Confrey, 1995, p. 9). Students used other terms including "basic combination" and "little recipe" (which became the class favorite) to refer to the smallest whole number ratio for a given proportion. We suggest that this be named a ratio unit (Ibid., p. 11), to recognize its importance as a multiplicative unit .

5. Dot drawings. The children used dot drawings in order to find the "littlest recipe" (2 to 1 in the figure below). Rather than finding the littlest recipe numerically using factoring, they used dot drawings and employed a recursive process to check the validity of "little recipes." Validation was determined when the children's search image of a series of circled little recipes left no dots uncircled and each set was identical, or when they regrouped to see the recursiveness in the whole picture. If uncircled dots remained after using the attempted ratio unit, children would recursively select another ratio unit for a further attempt at determining the "little recipe."



6. Graphing on the two dimensional plane. The children were introduced to the idea of a ratio (a:b) as a vector from (0,0) to (a,b). The axes' labels allowed them to distinguish a:b and b:a. Equivalent ratios lie along a vector. The children found this notion extremely generative and connected, and explored its relations to rectangles, stairsteps, triangles and straightness. They were able to make sense of the meaning of steeper and less steep, and learned to interpolate and to extrapolate using data. Later, in the context of falling domino chains, without any formal introduction, the students extended their analyses to include discussions of acceleration and deceleration based on the shapes of curves.

7. Ratio boxes. Ratio boxes encouraged student exploration of the ratio relations both across and down (as an isomorphism of measures and as a functional relation in Vergnaud's terms). Using ratio boxes avoided problems concerning the lack of distinguishing notation between ratio and fraction, and supported a smooth conceptual networking of contingency tables, tables of values, and ratio boxes. Having become a primary tool for the children, the use of ratio boxes led to three significant results: 1) all students in the class believed that for any three values, a fourth could be found; 2) a student recognized that the fourth value could be obtained by multiplying the two numbers in the diagonal positions, and then dividing that product by the number in the third position ($\underline{9} \times \underline{8} = 72$. $72 \div \underline{12} = 6$); and 3) students learned that if they could find a daisy chain to go from one cell to an adjacent cell, that same daisy chain should also work for the other pair of cells ($\underline{12} \div 4 \Rightarrow 3 \times 3 = \underline{9}$, so $\underline{8} \div 4 \Rightarrow 2 \times 3 = \underline{6}$).

12	9
8	

This paper claims that, given appropriate contextual challenges and representational tools with which to approach ratios, an earlier and more robust introduction to ratio can be presented. The table below presents the results of these students' written assessments given at the end of fourth grade and repeated at the end

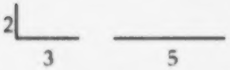
of fifth grade, and the comparative data from the "Concepts in Secondary Mathematics and Science" (CSMS) studies (Hart, 1988).

	CSMS			Belle Sherman	
PROBLEM	Age 13	Age 14	Age 15	Ages 9-10	Ages 10-11
Short/Tall	28.1 (51.4)	29.6 (50.6)	42.0 (39.1)	45 (15)	90 (5)
Ls	7.9 (47.6)	11.0 (39.4)	19.7 (39.7)	35 (30)	65 (10)
Onion Soup:					
Water for 4	94	94	96	90	100
Cubes for 4	95	95	96	90	100
Water for 6	85	84	88	70	70
Cubes for 6	75	75	79	75	75
Cream for 6	24	23	21	35	45

Percentage correct (Percentage using additive strategy)

In Mr. Tall and Mr. Short, students are given, "Mr. Short's height is 6 paper clips or 4 buttons. His friend Mr. Tall's height is 6 buttons" and are asked, "How many paper clips are needed for Mr. Tall's height?" (Karplus R., Karplus, E., & Wollman, W., 1972). Of these fifth graders, 90% answered correctly and only 5% (one student) showed any evidence of using additive strategies. These are striking results in light of the CSMS data for 15 year olds where only 42% answered correctly and nearly as many (39.1%) used additive strategies.

On the Ls problem, students are asked to "work out how long the missing line should be if this diagram is to be the same shape but bigger than the one on the left" (Hart, 1988). Of the fifth graders in this study, 65% solved it correctly and only 10% showed any evidence of additive strategies. This is in contrast to the 15 year olds in CSMS where 19.7% got the item correct and nearly 40% gave evidence of additive approaches.



On the onion soup problem, given a recipe to serve 8 people, 100% of the fifth graders figured out the amount of ingredient needed to serve 4 people. When faced with what seems to be the most difficult problem for students (determining the amount of cream for 6 people given that the amount for 8 people is F(1,2) pint), the data from non-experimental students shows dramatic drops in performance (to 21% among 15 year olds). However, with the children in our study, the drop in performance is considerably less (to 45% among 10 and 11 year olds). In fact, these 10 and 11 year olds more than doubled the accuracy of the 15 year olds in the CSMS studies on this challenging question, demonstrating the robustness of their approaches.

The data presented here suggests that conceptual analyses and developmental studies (Karplus et al., 1983; Noelting, 1980) have tended to underestimate the power of different representational forms in allowing students access to the conceptual understanding of ratio and proportion. In contrast to traditional curricula which have treated ratio and proportion in settings with little (single) or no context, we are embedding the study of ratio and proportion in a rich interactional system with physical and representational tools, and in multiple problem contexts. Our results suggest that higher success may be achieved by many students at an earlier age.

References

Behr, J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In Douglas A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan Publishing Company.

Confrey, J. (1988). Multiplication and splitting: Their role in understanding exponential functions. *Proceedings of the Tenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 250-259). DeKalb: Northern Illinois University Press.

Confrey, J. (1995). Student voice in examining "splitting" as an approach to ratio, proportions and fractions. *Proceedings of the 19th Annual Meeting of the International Group for the Psychology of Mathematics Education*. Recife, Brazil: Universidade Federal de Pernambuco, July 22-27.

Harel, G., & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: State University of New York Press.

Hart, K. M. (1988). Ratio and proportion. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 198-219). Reston, VA: National Council of Teachers of Mathematics, Inc.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 45-89). New York: Academic Press, Inc.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, VA: National Council of Teachers of Mathematics, Inc.

Lachance, A. (1995). Introducing fifth graders to decimal notation through ratio and proportion. *Proceedings of the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: The Ohio State University, October 21-24, 1995.

Lamon, S. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-120). Albany, NY: State University Press.

Luthuli, D., & Confrey, J. (in progress). The introduction of fifth graders to algebraic symbolism through the use of contextual problems and function data tables. Paper to be presented at the American Educational Research Association 1996 Annual Meeting. New York, NY: Columbia University, April 8-12, 1996.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept: Part 1. The differentiation of stages. *Educational Studies in Mathematics 11*, 217-253.

Piaget, J., Berthoud-Papandropoulou, I., & Kilcher, H. (1987). Multiplication and multiplicative associativity. In J. Piaget (Ed.), *Possibility and necessity* (pp. 78-97). Minneapolis: University of Minnesota Press.

Preyer, R. (in progress). *Tool based investigation and the development of experiences that support mathematical understanding*. Unpublished master's thesis, Cornell University, Ithaca, NY.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41-60). Albany, NY: State University of New York Press.

A COMPARISON BETWEEN PROPORTIONAL AND NON-PROPORTIONAL MANIPULATIVES IN DECIMAL ADDITION AND SUBTRACTION INSTRUCTION

Rosemary S. Mitchell, Straley Elementary School
Rose Sinicrope, East Carolina University

The generalization of decimal computational algorithms from whole number computational algorithms appears simple symbolically; however, for many students it is not (Hiebert & Wearne, 1986). Critical to the generalization are the students' prior concrete experiences (Wearne & Hiebert, 1988). What must be the nature of these concrete experiences? Is the use of a non-proportional manipulative adequate as the primary model for the development of addition and subtraction algorithms for decimals?

Subjects. The 26 students in a self-contained, heterogeneously-formed, third-grade classroom were the subjects of this study. Eleven of the students were females; 15 were males. Approximately two-thirds of the children were from low socio-economic families.

Method. Prior to any decimal instruction, the students completed a 15 item test on addition, subtraction, and comparison of decimals and decimal-fraction conversions. After five days of whole-class instruction on decimal numbers, the students were randomly assigned to two equal-sized groups. Students in one group were given Dienes' blocks, with the flat representing a whole, and place value mats (ones, tenths, and hundredths). The students in the other group were given red, blue, and yellow cubes with color-coded place value mats. For the following 14 school days, the students, using the manipulatives assigned to their respective groups, solved problems and discussed their solutions within their groups. Students kept daily journals and submitted written assignments. The teacher provided written and oral feedback. Immediately after instruction and approximately one month later, the students were given the same 15 item test. The test was not timed, and students could choose to use manipulatives in answering the questions.

Results. A comparison of gain scores between the pre-test and post-test for the two groups yielded a significant difference at the 0.10 level in favor of the proportional group. The t statistic for the delayed post-test comparisons was not significant.

Summary. Decimal addition and subtraction instruction is most effective when a proportional manipulative is used. Students need both concrete representations of decimals and of place value. The lack of a difference between the two groups on the delayed post-test raises questions about student interactions between groups and the roles of time and cognitive conflict in making generalizations.

References

Hiebert, J., & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.). *Conceptual and procedural knowledge: The case of mathematics* (pp. 199 - 223). Hillsdale, NJ: Erlbaum.

Wearne, D., & Hiebert, J. (1988). Constructing and using meaning for mathematical symbols: The case of decimal fractions. In J. Hiebert & M. Behr (Eds.). *Number concepts and operations in the middle grades, volume 2* (pp. 220 - 235). Reston, VA: NCTM.

RECONSTRUCTING THE WHOLE: A GAUGE OF FRACTION UNDERSTANDING

Sylvia R. Taube, University of Texas-Pan American

This study analyzed graphical solutions of 260 students (Gr 4-college) who were administered a 40-item assessment (r_{xy}=.83) of basic fraction concepts. Of particular interest are three problems which required completing the whole given a fractional part drawn on a square dot paper. The fractions studied were "4/6" (irregular-shaped), "3/4" and "2/3" (both regular-shaped).

Results indicated the younger students frequently used the "doubling" strategy even if the fractional part showed "2/3". At other times, they represented the whole by drawing a rectangular region of any size. Videotaped interviews clearly revealed these two dominant strategies. Moreover, misconceptions held by both preservice teachers and elementary students include the assumption that the unit is always a regular-shaped geometric region rather than irregular.

The findings indicate the influence of the rectangular region often used in traditional mathematics textbooks and support the use of varied models or contexts (e.g., money, pattern blocks, dot paper) to strengthen understanding of part-whole relationships in fractions. Reforms in the teaching of fractions should include not only unit partitioning but also activities involving completing the whole. Various ways to reconstruct the whole using the area model must be stressed.

TOOL BASED INVESTIGATION: A MATHEMATICAL EXPERIENCE

Richard M. Preyer, Cornell University

This report will discuss the role of physical classroom tools and materials in mathematics education. Through an evaluation of a fifth grade project using balances to investigate ratio and proportion, I will focus on students' use of materials as tools for investigation and how, as a result of these actions, students develop and express mathematical experiences. It will be argued that tool-based investigation, in contrast to the use of mathematical manipulatives for demonstration, provides a medium for student reflection on kinesthetic experience. This gives them a means to express their insights that can be subsequently transformed into the traditional symbolic codes. Just as tools are used in our daily activities to organize, control and construct our world, tools allow students to construct sophisticated mathematical understandings of the world by virtue of their experimental processes. An analysis of their actions and verbalizations while involved in this process reveals six components of such experience: 1) *A sense of accuracy* 2) *Kinesthetic knowledge*, 3) *Investigative methods* 4) *Structural thinking*, 5) *Metaphorical relationships*, and 6) *Perception of concept applications*. Examples of the tools themselves, and videotaped excerpts of students' work will illustrate these six types of mathematical experiences.

A four-week design problem based on international monetary exchange presented students with the challenge to find and describe the relationships amoung weights of various shapes and materials. First, each group of students needed to establish ratio relations (all were rational) among the weights in their own system, so that the information could be given to tourists wanting to purchase goods in this fictitious kingdom. The students were given materials for the construction of a balance that allowed a range of choices regarding its possible organization and calibration. The balance was designed particularly for this problem and encouraged the students to progressively extend their understanding of the balance's mechanics as well as their physical skills in its operation. The implications for future tool design, implementation and assessment will be discussed as they relates to the project findings.

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October 21-24, 1995
The Ohio State University
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PME-NA XVII

October 21-24, 1995
The Ohio State University
Columbus, Ohio, U.S.A.

Editors:
Douglas T. Owens
Michelle K. Reed
Gayle M. Millsaps

Published by:
The ERIC Clearinghouse for Science,
Mathematics, and Environmental Education
Columbus, OH

PRINTED WITH SOY INK

Cite as:

Owens, D. T., Reed, M. K., and Millsaps, G. M. (Eds.). (1995). *Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Document development:

David L. Haury, *Editor in Chief*
Linda A. Milbourne, *Copyeditor*
J. Eric Bush, *Cover Design and Page Layout*

Accession number: SE 057 177

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This publication was funded in part by the Office of Educational Research and Improvement, U. S. Department of Education under contract no. RR93002013. Opinions expressed in this publication do not necessarily reflect the positions or policies of OERI or the Department of Education.

History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

This program began with a meeting of interested volunteers in November 1994 at Baton Rouge during the PME-NA meeting. The results of the suggestions made were taken to a meeting of the local program committee at The Ohio State University where the overarching theme of research on teaching and learning mathematics in diverse settings and the subthemes of diversity, constructivism and algebra were selected. These emphases are achieved in the plenary papers. Constructivism from a social perspective in the paper by Paul Cobb and Erna Yackel takes account of diverse learning idiosyncrasies. Fairness in dealing with diversity in characteristics and background of the learners was addressed in the paper by Suzanne Damarin. Reform in algebra toward making algebra more accessible to all students is addressed in the paper by James Kaput. No reaction paper to the Cobb and Yackel paper was requested because the opening plenary session ended with questions posed by the speaker, Paul Cobb, for roundtable discussions. Reactions to the paper by Suzanne Damarin were prepared by Ruth Cossey and Edward Silver from their perspectives of the work in which they are involved. Reactions to Jim Kaput's paper were written by Gail Burrill and Elizabeth Phillips who are involved in aspects of change in algebra curriculum.

Included in these *Proceedings* are 84 research reports, two discussion groups, 40 oral reports and 43 poster presentation entries. The one-page synopses of discussion groups, oral reports and poster presentations are organized by topic along with the research reports following the pattern begun with the *Proceedings* of the 1994 PME-NA meeting. Proposers expressed first choice: 124 research reports (2 withdrawals), 12 oral presentations, 35 poster presentations, and two discussion groups. Proposals for all categories were blind reviewed by three reviewers with expertise in the topic of the submission. Cases of disagreement among reviewers were refereed by subcommittees of the Program Committee at The Ohio State University. This process resulted in rejection or reassignment of about 31% of the research report proposals and about 25% overall.

For the first time, the submissions for the Proceedings were made on disk. These *Proceedings* were produced by the ERIC/CSMEE staff. The format of the papers were adjusted to make them uniform. As papers were assigned to topic areas for the table of contents, possible secondary or tertiary topic areas were noted. Thus, most papers are included with more than one descriptor in the index in the appendix in Volume 2. An alphabetical list of addresses of authors is included in the appendix in Volume 2 with page numbers of their report or synopsis. For the first time the electronic mail address is included in this address list. In the case of multiple authors, submissions were made with presenting author(s) name(s) underlined.

The editors wish to express thanks to all those who submitted proposals, the reviewers, the 1995 Program Committee, the PME-NA Steering Committee for making the program an excellent contribution to ongoing research and discussions of psychology and mathematics education; Dean Nancy Zimpher, College of Education, and the administration of the Department of Educational Theory and Practice at The Ohio State University, for their support; The Mathematics Education faculty and graduate students for their endless committee work; and the ERIC/CSMEE staff, especially Director David Haury, Linda Milbourne, and J. Eric Bush for the production of these *Proceedings*.

Douglas T. Owens
Michelle K. Reed
Gayle M. Millsaps
October, 1995

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Research Methods

HOW SHALL WE DEFINE AND RESEARCH CULTURAL DIVERSITY?

Todd M. Johnson, Illinois State University

The overall theme of PME-NA XVII is *Mathematics Learning in Culturally Diverse Settings*. Concern about how cultural diversity has been defined and researched has been raised by numerous authors.

Nickerson (1992) identifies *culture* as one of the popular catchwords often used by educational researchers without clarifying what the term means or considering the relevance. For researchers concerned with the classroom as their unit of study, Nickerson recommends coming to some understanding of culture at that level before considering how wider aspects of culture impinge on that of the classroom.

Secada (1992) questions the categories used to describe student diversity. Secada suggests that the categories used are often treated as unquestioned givens and unless our ways of viewing the world are examined, our research may grant legitimacy to the social arrangements that originally lead to the disparity under study.

Like Secada, Fennema (1981) is concerned that research may perpetuate myths that affect students. As an example, Fennema includes the following justification a middle school boy gave for believing a boy was more likely than a girl to answer an arithmetic problem correctly.

> Okay, cuz I read somewhere ... that, um boys are, it's some kind of scientific thing that boys are better in math than girls are.

All individuals interested in exploring the categories that we use when we consider cultural diversity and how these categories are maintained in our research are invited to take part in this discussion group.

References

Fennema, E. (1981). Women and mathematics: Does research matter? *Journal for Research in Mathematics Education*. 12, 380-385.

Nickerson, M. (1992). The culture of the mathematics classroom: An unknown quantity? In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 101-114). New York: Macmillian Publishing Company.

Secada, W. (1992). Race, ethnicity, social class, language, and achievement in mathematics. *Handbook of research on mathematics teaching and learning* (pp. 623-660). New York: Macmillan Publishing Company.

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Michelle K. Reed, The Ohio State University

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HEIDEGGER AND HALL DUTY: USING VIGNETTES OF TEACHER'S DAILY PRACTICE TO TRIANGULATE OBSERVATIONAL DATA

Sharon B. Walen, Montana State University
Steven R. Williams, Brigham Young University

Little is known about teachers' reflections on current reform efforts in real classrooms (Cuban, 1993). This study describes methodology and data on teachers' efforts to implement a four-year secondary mathematics reform curriculum at a statewide level. The methodology uses vignettes in an attempt to retain the best part of careful observational case-study techniques while bowing to practical pressures of dealing with hundreds of teachers in as many sites. As such, it allows us to triangulate observational data, to affirm working hypotheses based on that data, and to provide teachers with a rich opportunity for reflection on classroom practices.

Objectives

The importance of reflection to changing teaching practice is widely recognized. Indeed, the model of teachers as *reflective practitioners* (Schon, 1987) makes it clear that reflection on the practice of teaching is vital to professional growth. Also important is the use of reflection as a data source in the study of teaching. This is true especially as we attempt to understand the everyday world of the teacher and how that world transforms. The coupling of teachers' reflections with observational data is a valuable means of triangulation.

Recognizing the importance of reflection does not obviate the theoretical and practical problems connected with its use as a data source. The act of reflection is necessarily distanced from the acts reflected on and is thus affected by memory, emotions, and various *subjective* variables. An important question, then, is how best to stimulate reflection on practice to obtain maximally useful results. With this in mind, the purposes of this paper are fourfold: 1) Introduce a method of stimulating reflection on practice that we find particularly compelling, 2) Outline its advantages, 3) Report particularly insightful results obtained by the method, and 4) Analyze the reasons for the method's success by way of an interpretive theoretical model.

Theoretical Framework

We base our analysis of reflection in general on Heidegger's work and more recent writings in psychology and in education (for a complete discussion see Williams, 1993). We suggest that the primary mode of human functioning is a kind of unreflective being-in-the-world, a mode of existence that Heidegger calls the ready-to-hand. To use his favorite example, when we are hammering, we are not reflecting on the hammer; indeed, we are hardly aware of it. Rather, we are immersed in hammering. From this fundamental mode, we can move to a more reflective mode if, for example, the hammer breaks. We then focus on the hammer, seeing it as broken, and attempt to replace it or work around it. This mode is called the unready-to-hand and suggests a kind of reflection still embedded in the context of the larger program of which hammering was part. A more abstract reflection can be performed when, for example, one wishes to design a hammer or to choose a particular kind of hammer for a job. Here, more than in the other two modes, the hammer emerges as an object. Then, abstract knowledge about the hammer (weight, balance, etc.) gains salience. This most abstract mode is called the present-at-hand and represents the mode in which much of what we would call reflection occurs.

Reflection is most often provoked by a breakdown in the normal, ready-to-hand flow of everyday coping. To reflect, we must step out of that deeply contextual mode, but as we do so, the original actions on which we reflect begin to recede. As we enter the unready-to-hand mode, salient features of our actions and environment emerge for reflection. The context of the original actions remains important as background; the reflection remains oriented toward ongoing activity. If the breakdown is more fundamental, salient features emerge and are dealt with as separated from their original contexts–as abstract objects of reflection. The original acts then become separated from their context. This characterizes the present-at-hand mode.

Our problem, then, is to encourage reflection that stays oriented toward practice, as in the unready-to-hand mode, away from the abstract knowledge of the present-at-hand. We would ideally deal with individual teachers contemplating their daily practice. However, when more systemic reform is being studied, involving many teachers across multiple sites, the burden of collecting data in this way proves insupportable. We describe below a methodology that, although taking steps away from typical case study techniques, remains sensitive to the underlying ontology of reflection we have described.

Methods and Data Sources

Although our philosophical framework mandated the gathering and analysis of rich ethnographic data, the size of our project and the multiple sites involved presented a particularly intriguing challenge. We met this challenge by modifying and merging current research methods. The following section presents the story of how we used observation to identify issues, vignettes to set the stage for a discussion of important themes, and the resulting discussions to validate the nature of the issues.

Previous data from 16 participant observers' visits provided us with richly detailed snapshots of 35 individual classrooms. Although rich, the data did not result in common threads necessary to weave a picture across reform classrooms. In an attempt to find these threads, the observers designed and employed a semi-structured observation form. Observers indicated this allowed them the individual freedom to make original insightful observations yet assisted them to focus on curricular objectives essential for implementation of the reform curriculum.

As analysis of the data gathered using the semi-structured form began to outline common themes across this subset of reform classrooms (assessment, communication, and time management), we felt the need to triangulate the themes within the larger group of participating teachers. We began thinking about how classroom vignettes enhance communication and, with this in mind, we wrote 4 vignettes with the themes found in the observational data as story-lines. Several revision cycles occurred before the participant observers agreed that each vignette told a valid classroom story. During the write and revise cycle, we took care not to present the stories as *moral lessons*. We wanted **all** teachers in **all** stages of reform to willingly talk about issues and be comfortable with their currently held beliefs. We also remained sensitive to the fact that many reform teachers previously taught using traditional methods and currently worked with and respected teachers preferring to use traditional methods. The vignettes did not portray mathematics education reform as *us against them* but rather as an opportunity to debate issues.

During a two-day grant sponsored teacher meeting, 115 teachers from across the state read and discussed the 4 vignettes. The 16 original classroom observers facilitated each teacher discussion group and recorded individual teacher comments. Other group members also provided written comments and documentation of consensus on primary issues. Triangulation of these data validated the importance of the themes presented in the vignettes.

Vignettes and Teachers' Comments on Key Reform Issues

The following discussion revolves around 1 of the 4 vignettes. Although each vignette allowed us to validate a common theme, this particular discussion illustrates the richness of the data. During these discussions, teachers interpreted issues both individually from their personal context of teaching and generally from their common knowledge of what it is to teach. The meaning and significance of the themes became clear as each group assisted the *vignette-teachers*.

Vignette #1: Assessment and Communication Issues in Reform Classrooms

The vignette in Figure 1 focuses on 2 of the 3 central reform themes–assessment and communication. We had heard teachers across the state say that new forms of assessment were more difficult to implement than the reform curriculum, itself. Many teachers indicated they had used the integrated curriculum for 2 years without changing classroom assessment. They stated that they were beginning to recognize the mismatch between traditional assessment and the reform curriculum. Comments from these teachers included, "I know that I need to change how I assess, but I can only do one thing at a time. I'm working on the curriculum and technology. I'll get to the assessment next." One teacher commented, "OK, I have the kids writing journals. Now I don't know what to do with them [the journals]." Grading was a genuine concern. Generally, teachers using alternate assessment did not recognize the value of their efforts or express confidence with their results but repeatedly requested assistance. The vignette-teacher modeled these teachers' views.

Beth Davis looked up, startled for a second at the loud crackling voice on the intercom. "Yes, this is Ms. Davis."

"Sorry to interrupt, Ms. Davis, but could you please stop in the main office before you leave tonight?"

"Yes." At that note, Beth turned toward the clock surprised to see that an hour had gone by since the last of the Level 1 students had left her room for the day and she had begun to read the student portfolios. With an inward smile, Beth thought; The projects are really well done. Even though there was a wide range of skills in her class this year, Beth had decided to require the students to choose research projects. They seemed to like their first projects, but their second projects were fantastic, especially Faith's. She seemed to have real talent.

Beth gathered a few of the student projects to finish reading, slipped on her coat, and headed toward the office.

"Ms. Davis, the principal, Ms. Waters, wishes to make an appointment with you tomorrow during your free period. Faith Old Elk's father will be here. He has called and expressed concern about Faith's math work. Will 10:15 be OK?" Beth finished discussing the details of tomorrow's meeting and walked to her car.

"Good morning, Ms. Davis. Please sit down." Ms. Waters' voice was low and friendly, but Beth still felt uneasy. "So, Mr. Old Elk, you said over the phone that you would like to talk to us about Faith's math work?"

Mr. Old Elk looked uncomfortable. "Yes, I did. You know that Faith has always done well in math. This is hard for me to say. Faith says she doesn't know what math she is learning. Her cousin spent the weekend with us and his book looks like the one I had in high school. We want Faith to be able to go to college. She has to do well on the tests for college so she can get a scholarship. Faith says you are giving her an A for this quarter in math, but I'm afraid that you're not preparing her for college. Faith spent the last month writing about the traditional ways to tan hides and comparing that to chemical tans for her math project. But she couldn't help her cousin with his algebra homework until she read the whole chapter in his book. Faith has always been better than him at math but now he's ahead of her. What does her A mean, Ms. Davis? Will she be able to do well in college math?"

Later that evening as Beth *replayed* that morning's conversation in her mind, she started to wonder about the changes she had made in her mathematics classroom. How can I be sure I'm really right about Faith's A? Will Faith really do OK? What do I really know about Faith's math skills and concepts?

What is the problem? Whose problem is it? What should Beth do next?

Figure 1. Vignette #1: What's an A?

It did not surprise us that every teacher-discussion group validated student assessment as a primary issue. Assessment and grading were genuine concerns across the state. During the discussions, teacher groups validated another key issue related to classroom assessment; they were afraid that students' scores on standardized tests would not reflect what their students really knew. If this were to happen, colleges would not be able to assess the depth of their students' understanding. Teachers expressed frustration with *the way things were* and offered solution strategies ranging from supplementing with more drill-and-practice (so the curriculum resembled traditional texts) to *teaching to* standardized tests. "We ought to just take 2 days out and teach them the standardized test–teach them to take the test," commented one teacher.

However, these solutions did not satisfy the groups. They were comfortable with what they currently saw their students learning and doing. The teachers valued the improvements they saw in their students' problem solving skills, abilities to use technology, and attitudes towards mathematics. After a brief discussion, teachers stopped talking about returning to traditional teaching and started talking about challenging the standardized testing structure. The following 4 comments represents their suggestions: 1. "Will letters from teachers help on scholarships and college entrance?"; 2. "Can we start sending portfolios?"; 3. "Are these tests ACT SAT and placement tests valid anyway?"; and 4. "Our system is in a transition and integrated math is strong on teaching kids *how* to learn. We can show them that." As teachers continued to talk about the values they saw in using the integrated mathematics curriculum, discussions turned to communication as critical to *fixing* problems.

Teachers identified two areas of communication needing improvement–between high schools and universities and between parents and teachers. In the high-school-university discussions, teachers talked about how they had previously followed colleges and universities lead. University calculus course content previously drove many high school curriculum choices. Now they were not so certain that was going to continue to work. One teacher expressed her frustration, "I don't know what I'm preparing my students for with all the changes in calculus. Students can't drive those changes. We need to help people at the university. We need to start talking to them–telling them what we are doing." These teachers wanted to *align* the two curriculums rather than to continue to passively feed their students into the university. In the parent-high-school discussions, teachers stated that the vignette-teacher needed to improve communication with parents. This teacher's comment summarized the discussion, "Resistance to change–parents are just resistant to change." Time changes attitudes. Communication with community is a must. That's when we all get in trouble–failing to communicate." Within this segment of the discussion, several teacher groups validated proactive information sessions with parents, community, and other teachers. They discovered that all the teachers whose overall interaction with parents had been positive had all had parent nights. Therefore, parent night was a critical way to avoid problems. Teachers shared their plans and experiences. Several teachers volunteered to drive hundreds of miles to help others plan and present parent night.

In summary, the teacher discussions began in a passive mode with acceptance of the current situations and ideas for fitting into the existing system then quickly progressed to challenging the system. As individuals became aware of similarities in their ideas, teachers expressed confidence in their ability to find ways to change the system. Teachers genuinely enjoyed the opportunity that vignettes provided for discussion. One teacher summed it up by saying, "This let us have lots of good discussion about situations similar to the vignette and how we handled it." Another teacher added, "Yes, and it let us find out that we weren't alone in our concerns." This discussion confirmed, explained, and triangulated 2 of the themes key to reform teaching–assessment and communication.

Implications of the Method

Teachers' discussions and solutions reflected a personalized, contextual richness. We believe the methodology facilitated this in two ways. First, the vignettes focused on familiar situations of practice for the teachers. These were not the same as engaging in practice, but oriented the teachers toward a ready-to-hand mode. The vignettes also provided a safe forum for discussion of problems they recognized, yet were previously unwilling or unable to offer for discussion. One individual commented after the conference that she had not previously talked about her problems because she thought of them as "specific to my own classroom and not important enough to take up the valuable conference time." However, she now recognized the problems as valid and a valuable forum for discussion.

Second, the vignettes dealt with breakdowns in practice calling for solutions that make sense within the context of that practice. This oriented teachers toward an unready-to-hand mode and enabled more context-sensitive reflection. We might say that the vignettes supported reflection on action that was largely oriented toward reflection in action. One teacher commented during a break in the group discussions, "I've seen myself in each of the discussions. These are problems that I've had. [Laughs] I wonder what will be in the next 2 vignettes that we are going to read after break." She identified with the reform efforts in the vignettes. One facilitator who attempted to direct the discussion said, "These teachers saw and discussed what was of concern to them in each vignette and wouldn't be moved." This data indicated the vignette-teacher's voice deeply resonated with the individuals composing the discussion groups. After the group discussions, 7 individual teachers said that although we had not visited them, they still felt the stories were from their classrooms. This confirmatory data validated the practice-oriented nature of the vignette topics.

Although the problems described in the vignettes were not the teachers' own, they showed considerable engagement with the issues and felt compelled to find solutions. Again, we feel that this was supported by the methodology. Teachers in the study were sensitive to the plight of the vignette-teachers and seemed to be called out by some sense of responsibility or camaraderie to deal with their problems. Specific data indicated that these teachers valued the vignette-teacher's situations and identified with the teachers in the stories. Although we informed the teachers that only the issues were real, they referred to the vignette-teachers as actual teachers needing advice. As they assisted the vignette-teachers, they found themselves reflecting on their classrooms. To return to Heidegger's example, the teachers seemed to engage willingly in dealing with another teacher's broken hammer–even though that teacher was not real.

As efforts increase nationwide to reform the way mathematics is taught and learned, issues clearly center around individual teacher's practice. This methodology allowed us to triangulate observational data and affirm working hypotheses. Most importantly, this methodology provided rich opportunities for teachers to engage in reflective activity and to support each other's reform efforts.

References

Cuban, L. (1993) The lure of curricular reform and its pitiful history. *Phi Delta Kappan*, 75(2), 181-85.

Schon, D.A. (1987) *Educating the reflective practitioner.* San Francisco: Jossey-Bass.

Williams, S.R. (1993) Mathematics and being in the world: Toward an interpretive framework. *For the Learning of Mathematics, 13*(2), 2-7.

MODELS OF NEURAL PLASTICITY AND CLASSROOM PRACTICE

Dawn L. Brown, Tallahassee Community College
Grayson H. Wheatley, Florida State University

Mathematics instruction involves assumptions about the nature of learning. Because of new discoveries about brain activity during cognition, some of these previously held assumptions are now being questioned. Although we acknowledge that learning occurs in the brain, little attention has been paid to the mechanisms by which this occurs. Historically, there are many reasons for this stance.

As psychology undergoes a paradigm shift from a behavioristic to a cognitive science approach new interest is being kindled in brain functioning. A number of new research techniques have become available which permit the study of brain activity which has previously been inaccessible. The result has been an explosion of knowledge and new views about how the human brain functions in complex reasoning tasks. Many of these views are quite consistent with the ideas of constructivism and therefore should be of interest to mathematics educators.

One of the strongest arguments for radical constructivism is the transduction process which occurs at the level of the sensory receptor. This process and its consequences for constructivism has been discussed elsewhere.

At the level of the central nervous system the idea of neural plasticity and many of its consequences is also highly consistent with the constructivist philosophy. Neural plasticity has been defined as the capacity of the central nervous system to modify its own structural organization in response to environmental conditions. This concept allows one to consider changes we normally think of as "learning" to be considered as similar to changes which occur during the normal growth process and recovery of function after neural damage. In a similar fashion, a constructivist perspective groups classroom learning with learning which occurs in other settings.

Neural plasticity research and the ideas about brain function when it has generated also have implications for classroom practice. One of these findings is that the brain is organized to process material along sequential and parallel pathways. The other is the idea of critical periods for experience during development. Both of these concepts can be taken to support constructivist classroom practices for learning mathematics.

This presentation will discuss recent developments in neural functioning during cognition. In particular, we will discuss neural plasticity, the concepts mentioned above and their implications for classroom practice. We feel that neuroscientific research has great potential for providing insights into mathematics learning.

MATHEMATICS TEACHER DEVELOPMENT: LESSONS LEARNED FROM TWO COLLABORATIVE ACTION RESEARCH PARTNERSHIPS

Anne M. Raymond, Indiana State University

Collaborative action research partnerships create rich opportunities for the professional development of teachers and classroom reform. One of the primary goals of collaborative teacher research is to bridge the gap and strengthen the relationship between universities and schools (Miller & Pine, 1990). Theorists claim (Cardelle-Elawar, 1993) and studies show (e.g. Raymond, 1995) that teacher-inspired action research has the potential to result in immediate classroom reform because the results are more context specific and meaningful to the teacher.

In the past year I have had the opportunity to participate in two collaborative action research projects with two middle school mathematics teachers. The first project was one in which a seventh-grade mathematics teacher wanted to change his mathematics teaching in an effort to more effectively address the National Council of Teachers of Mathematics' (1989) call to infuse problem solving into the mathematics classroom. In doing so he hoped to improve his students' attitudes toward and abilities in mathematical problem solving. The second collaborative project stemmed from an eighth-grade teacher's desire to investigate whether or not her efforts in teaching algebra through the use of a "Hands-On" manipulative program were worthwhile. Specifically, she wanted to compare students' attitudes and academic achievement when working solely from a traditional algebra textbook to their attitudes and achievement when working with the "Hands-On" manipulatives.

Both projects were teacher-driven and yielded a number of interesting results. In my poster session I propose to (a) briefly present findings from the two studies, (b) discuss the similarities and differences between my roles in the two studies, (c) provide suggestions for engaging in collaborative action research projects, and (d) offer my thoughts on the role of action research in the professional development of mathematics teachers.

References

Cardelle-Elawar, M. (1993). The teacher as researcher in the classroom. *Action in Teacher Education, 15*(1), 49-57.

Miller, D. M. & Pine, G. J. (1990). Advancing professional inquiry for educational improvement through action research. *Journal of Staff Development*, 2(3), 56-61.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Raymond, A. M., & Hamersley, B. (1995). Collaborative action research in a seventh-grade mathematics classroom. *Paper presented at the Research Presession of the Annual Meeting of the National Council of Teachers of Mathematics*. Boston, MA.

Social and Cultural Factors

DEVELOPMENT OF ALGEBRAIC REASONING IN CHILDREN AND ADOLESCENTS: A CROSS–CULTURAL AND CROSS–CURRICULAR PERSPECTIVE

Anne Morris, University of Delaware
Vladimir Sloutsky, The Ohio State University

This study examined effects of cultural and curricular variables on algebraic reasoning in early and middle adolescence. Four algebra curricula in England and Russia were included in the design. Two age groups were included in the samples: 10 to 14 years, and 14 to 16 years. Algebraic reasoning processes were examined using a written test and interviews. Profound cross–cultural and cross–curricular differences were found in students' algebraic deductive reasoning for both age groups.

Comparative and developmental studies consistently reveal cross–cultural and cross–sectional differences in mathematical reasoning (e.g., Crosswhite et al., 1985; Inhelder & Piaget, 1958). While cross–cultural variability has been established, specific cultural and social variables accountable for the variation remain unclear. Candidate factors include educational variables; family processes; cultural belief systems and practices; semiotic systems; social factors; and interactions between these variables. Sources of within–cultural, cross–sectional variability in mathematical reasoning also remain unclear. Age has been firmly established as an important contributing factor, and cross–sectional differences have been primarily attributed to cognitive developmental stages (e.g., Küchemann, 1981; Piaget, 1983), or to an interaction between developmental and socio–cultural factors (e.g., curriculum, cognitive tools, SES) (e.g., Davydov, 1975).

There are inherent difficulties in identifying explanatory variables, and establishing mechanisms via which these factors affect reasoning. First, sources of variation are usually examined in isolation from one another. Consequently, critical sources of group differences, and interrelationships among contributing variables remain unclear (see, e.g., Reyes & Stanic, 1988). For example, within–cultural cross–sectional studies have attributed variation in algebraic reasoning to cognitive developmental factors without an accompanying analysis of the effects of socio–cultural and curricular variables (see, e.g., Küchemann, 1981). To develop a cohesive model of reasoning, multiple sources of variation must be examined within a single design (Stedman, 1994). Second, to establish relationships between specific socio–cultural variables and specific cognitive outcomes and processes involved in mathematical reasoning, sufficient variability has to be obtained in both sets of variables.

This study attempted to identify explanatory variables affecting mathematical reasoning (to detect and measure effects, and to point to likely candidate factors), and to establish underlying mechanisms by (1) examining multiple sources of variation in a single design; and (2) obtaining sufficient variability in socio–cultural contexts. The latter was viewed as a preliminary move toward establishing links between variability in specific socio–cultural factors and variability in reasoning.

To address the problems, two curricular settings were identified that incorporated profoundly different models for developing algebraic reasoning. The first curriculum, *National Mathematics Project* in England (Harper, Küchemann, et al., 1987), has a strong concrete–to–abstract orientation. The curriculum tends to replicate a natural progression in development—the progression from more concrete to more abstract concepts (e.g., in developing algebraic letter concepts and concepts of mathematical structure) (e.g., Piaget, 1983). Thus, *NMP* emphasizes inductive, case–based reasoning and learning—the investigation of a number of particular cases to formulate and to assess the validity of algebraic generalizations, emphasizing the importance of empirical checks. With respect to developing children's ability to create and operate on abstract algebraic objects, to recognize and use structure, and to perform algebraic transformations, *NMP* uses a procedural–to–structural curricular model (Kieran, 1992). Emphasis on less abstract numerical input–output interpretations ("procedural interpretations") of algebraic constructs precedes emphasis on structural interpretations, and on algebraic transformations.

The second, Davydov's elementary mathematics curriculum in Russia, assumes the natural progression in concept development is not the most efficient. Davydov draws on Vygotsky's distinction between development of spontaneous and scientific concepts. While spontaneous concepts develop as an abstraction of properties of concrete instances, scientific concepts develop in the opposite direction: from formal definitions of properties, to an ability to identify those properties in concrete instances. Formal education is viewed as the environment that can, and must, foster development of scientific concepts. Thus, Davydov's curriculum emphasizes abstract deductive, law–based reasoning—the logical derivation of particular (e.g., numerical) cases from general mathematical principles and relationships where those principles and relationships are first expressed algebraically. Davydov's curriculum can be characterized as abstract–to–concrete and structural–to–procedural. Concepts of "relation or structure" are developed prior to numerical work, and prior to emphasis on algebraic transformations.

Curricular variables across cultures were confounded with other potentially important variables such as language, family and cultural beliefs and practices, etc. To control potential confounds, and to investigate alternative sources of variation, two "non–experimental schools" were selected in the same countries. Schools were in the same geographical area, and had comparable student and teacher populations; however, they did not have curricula that were designed to develop specific kinds of algebraic reasoning.

This paper specifically examines curricular effects on components of algebraic deductive reasoning, including letter interpretation, formulation of equations, and children's understanding of the logical necessity of deductive conclusions derived from algebraic proof.

Method

For purposes of comparison, four groups were included: (1) students and graduates of Davydov's curriculum, implemented in Moscow School #91 in Moscow, Russia (n=120); (2) students in a non–experimental school in Moscow (n=89); (3) students in an upper school in England that had implemented *NMP* for seven years (n=120); and (4) students in an upper school in England with a "non–experimental" curriculum (n=120).

Outcome variables were measured through written open–ended problems and follow–up interviews. Students were tested and interviewed within their schools. The following task from the *CSMS* study (Küchemann, 1981) examined ability to interpret letters as variables:

Which is larger, $2n$ or $n+2$? Explain.

The following task, adapted from Clement, Lochhead, and Monk (1981), measured ability to formulate algebraic equations that represented verbally described quantitative relationships:

> Write an equation using the letters S and T to represent the following statement:
>
> "There are six times as many students as teachers at this school." Use S for the number of students and T for the number of teachers.

The following problem from Lee and Wheeler (1989) examined students' tendency and ability to formulate algebraic deductive arguments:

> A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why *your* answer is right.

Data were aggregated by age and culture–curriculum composites ("Groups"). Thus, four groups were used in the analyses: Russian non–experimental curriculum ("R–NEX"), Russian experimental curriculum ("DV"), English non–experimental curriculum ("E–NEX"), and English experimental curriculum ("NMP"). Subjects within each of the curricular groups were divided into two age groups: 10–14 and 14–16 years. Since the data were categorical, and frequencies of categories of responses were aggregated across groups and ages, log–linear analysis was deemed an appropriate approach to data analysis. Log–linear models were used to discern cultural, curricular, and age–related effects on algebraic reasoning.

Results

For Küchemann's task, Figure 1 compares the percentages of correct conditional responses (e.g., $2n$, when n>2) for the various curricular groups and age groups.

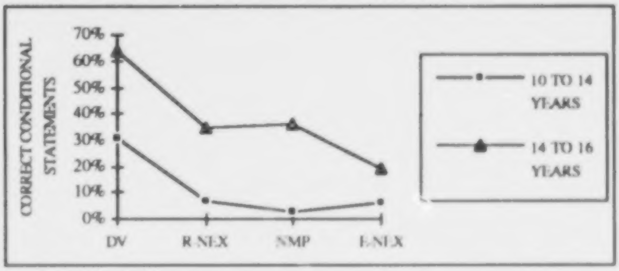

Figure 1. Percentages giving correct conditional responses

For interpretation of letters as variables, Group and Age had independent effects (goodness of fit $c^2(3) = 2.26$; $p=0.52$). The effect size of Group was large (0.53), while the effect size of Age was moderate (0.30). Data analyses suggested curricular effects. Davydov's group gave correct responses more often than other groups ($p<.0001$), and the English non-experimental group gave correct responses less often than other groups ($p<.0001$).

Figure 2 shows percentages formulating a correct algebraic equation in response to Clement et al.'s (1981) task. For formulation of a correct equation, Group and Age had independent effects (goodness of fit $c^2(3) = 7.49$; $p=0.06$). The effect size of Group was moderate (0.42), while the effect size of Age was small (0.14). Analyses revealed curricular effects. Davydov's group wrote correct equations more often than other groups ($p<.0001$). Experimental groups wrote correct equations more often than non-experimental groups ($p<.0001$).

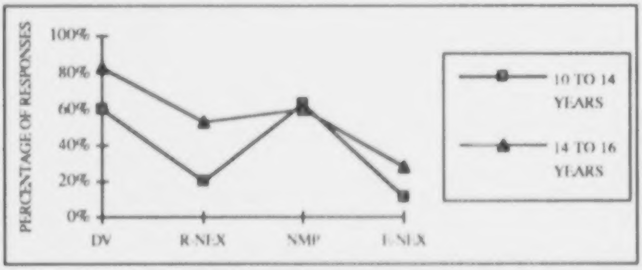

Figure 2. Percentages formulating a correct algebraic equation (S=6T)

Figure 3 compares the percentages of students independently formulating an algebraic deductive argument in response to Lee and Wheeler's task. For use of algebraic deductive reasoning, Group and Age had independent effects (goodness of fit $c^2(3)=0.85$; $p=0.8375$). Effect sizes of Group (0.5) and Age (0.67) were large. Analyses suggested cultural effects, and a combination of cultural and curricular effects. Russian groups formulated proofs more often than English groups

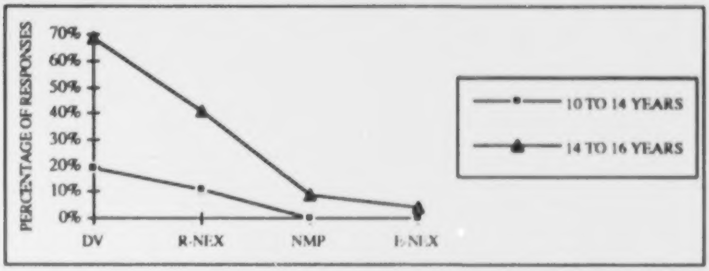

Figure 3. Percentages formulating an algebraic proof during the written test

(p<.0001), and Davydov's group formulated proofs more often than other groups (p<.0001).

English groups were more likely than Russian groups to use purely numerical reasoning on this task (p<.0001); e.g., 76% of *NMP* students used only numerical examples, with no use of algebra. "Empirical proofs" were often formulated in response to this task; i.e., children used inductive numerical arguments, concluding a generalization held for an infinite set after verifying that a generalization held for particular numerical cases. Curricular effects were evident: Davydov's group used "empirical proofs" less often than other groups (p<.0001), whereas *NMP* students used "empirical proofs" more often than other groups (p<.0001). Forty-seven percent of *NMP* students formulated an empirical proof, while 10% of Davydov's group used empirical arguments.

If a student did not use algebra on this item during the written test, he/she was asked to do so during the follow-up interview. Students' ability to use algebra as a tool for reasoning could therefore be examined, as well as their tendency to do so. When prompted to use algebra, 17% of high track ("Red Track") *NMP* students formulated an algebraic proof, while 38% of high track *NMP* students used an algebraic equation/expression only as a template to generate numerical examples.

Discussion

Davydov's group was more likely to interpret letters as variables, to formulate correct equations, and to formulate algebraic proofs. For algebraic deductive reasoning, differences between Davydov's group and other groups tended to increase with children's age—suggesting effects of instruction tend to increase with age. Though age is an important contributing factor in development of algebraic reasoning, comparison of younger and older children's responses across groups suggests socio-cultural factors can amplify development of algebraic reasoning—over-shadowing effects of age (Figure 3).

While the Russian groups and *NMP* group acquired component understandings required in algebraic deductive reasoning (variables, equations), there were profound differences in their use of algebraic deductive arguments. Differences did not appear to be due to across-group differences in children's tendency to use, or apply algebraic concepts and skills. Rather, findings suggested children in the curricular groups had acquired very different kinds of understandings of algebraic reasoning, constructs, and operations.

These results were consistent with other findings from this study (Morris, 1995). In comparison with other curricular groups, Davydov's group was more likely to use algebraic deductive arguments; to believe algebraic proof establishes "universal validity"; to use arithmetical structure; to manipulate algebraic expressions correctly; and to acquire concepts of generalized numbers, variables, and givens. In comparison with other curricular groups, the *NMP* group was more likely to use inductive, numerical arguments on proof tasks; to believe algebraic proof requires empirical support; to compute, rather than use arithmetical structure; and to use only procedural interpretations of algebraic constructs. In comparison with the English non-experimental group, the *NMP* group was more likely to acquire concepts of generalized numbers and variables; to formulate correct equations; and to manipulate algebraic expressions correctly. Thus, *while the approach developed some component understandings*, prolonged emphasis on inductive, case-based reasoning and numerical input-output interpretations seems to promote empirical, rather than theoretical reasoning (see Hatano et al., 1995). Using numerical reasoning, children attempted to establish "whether a generalization worked," rather than "why it worked."

When prompted to use algebra on proof tasks, *NMP* students tended to substitute numbers to make sense of algebraic statements, to test cases, and to generate empirical evidence. Russian groups operated at a different level—operating at the level of *relationship or structure*. This was particularly evident among Davydov's group. Approximately 70% of Davydov's graduates operated at the level of structure—writing proofs, and demonstrating an understanding of the logical necessity of deductive conclusions derived from proofs.

Findings suggest different curricular approaches tend to lead to different conceptual organizations of children's mathematical knowledge. This conceptual organization, in turn, affects how and whether children utilize and apply their algebraic knowledge and skills in the solution of particular problems.

References

Crosswhite, F. J., Dossey, J. A., Swafford, J. O., McKnight, C. C., & Cooney, T. J. (1985). *Second International Mathematics Study summary report for the United States*. Champaign, IL: Stipes.

Davydov, V. V. (1975). The psychological characteristics of the "prenumerical" period of mathematics instruction. In L. P. Steffe (Ed.), *Soviet studies in the psychology of learning and teaching mathematics* (Vol. 7, pp. 109–206). Chicago: The University of Chicago Press.

Harper, E., Küchemann, D., Mahoney, M., Marshall, S., Martin, E., McLeay, H., Reed, P., & Russell, S. (1987). *National Mathematics Project: Mathematics for secondary schools*. London: Longman.

Hatano, G., Morita, E., & Inagaki, K. (1995, March). *From empiric to theoretic: Conceptual change underlying the differentiation between conclusions and justifications*. Paper presented at the Biennial Meeting of the Society for Research in Child Development. Indianapolis, Indiana.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking*. New York: Basic Books.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Küchemann, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11–16* (pp. 102–119). London: John Murray.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics, 20*, 41–54.

Morris, A. K. (1995). *Development of algebraic reasoning in children and adolescents: Cultural, curricular, and age-related effects*. Unpublished doctoral dissertation, The Ohio State University.

Piaget, J. (1983). Piaget's theory. In P. H. Mussen (Ed.), *Handbook of child psychology, Volume I* (pp. 103–128). New York: John Wiley & Sons.

Reyes, L. H., & Stanic, G. M. A. (1988). Race, sex, socioeconomic status, and mathematics. *Journal for Research in Mathematics Education, 19* (1), 26–43.

Stedman, L. C. (1994). Incomplete explanations: The case of U.S. performance in the international assessments of education. *Educational Researcher, 23* (7), 24–32.

EXPLORING GENDER DIFFERENCES IN SOLVING OPEN-ENDED MATHEMATICAL PROBLEMS

Jinfa Cai, Marquette University

Open-ended tasks were used to examine gender differences in complex mathematical problem solving. The results of this study suggest that overall, males perform better than females, but the gender differences vary from task to task. A qualitative analysis of student responses to those tasks with gender differences showed that male and female students had many similarities in their solution processes of solving these problems, such as making similar types of mathematical errors and using similar strategies and representations. This study suggests not only the complexity of the issue of gender differences, but also the feasibility and usefulness of using open-ended tasks to explore the issue.

Gender differences in mathematics performance have been a popular but unresolved issue in educational research (Fennema & Leder, 1990). Since Sells (1973) expressed the concern that early decisions not to study mathematics in schools might be excluding students, and especially female students, from higher paying occupations such as those in science, engineering, and medicine, there has been an increased interest in research about gender and mathematics (Fennema & Leder, 1990). In particular, researchers have focused on investigations of gender-related differences in mathematical performance and have provided different theoretical models to explain the gender differences in mathematics performance.

In the early 1970s, Maccoby and Jacklin (1974) conducted a comprehensive review of research on gender differences and concluded that "boys excel in mathematical ability" (p. 352). Others indicated that there were no gender differences in the earlier years of elementary school, but in upper elementary, junior high, and senior high school, males outperformed females in mathematics. A review of research on gender differences in mathematics by Hyde, Fennema, and Lamon (1990) suggests that gender differences are declining. However, females continue to express less confidence in their mathematical ability and a lower perception of the usefulness of mathematics to them in the future (Lindquist, 1989). Even among the mathematically gifted students, females have lower educational aspirations in mathematics and sciences than do males (Benbow, 1992). As Leder (1990) indicates, "the issue of gender differences in mathematics learning is complex and there are many perspectives from which it can be explored" (p. 21).

Most of the previous studies used multiple-choice tasks to examine the gender-related differences in solving routine mathematical problems (Marshall, 1983). How male and female students differ in solving more complex mathematical problems remains to be investigated. The purpose of this study is to use open-ended tasks to explore the gender-related performance differences in solving complex mathematical problems. The open-ended problems allow students to display their solution processes, so male and female students' thinking and reasoning can be examined beyond the correctness of the numerical answers. Thus, this study is intended to provide more in-depth information about male and female students' thinking and reasoning in solving complex mathematical problems.

Method

Data Source

The data used in this study were from an earlier research project (Cai, in press). In particular, 227 sixth-grade students (96 females and 131 males) from the Pittsburgh area participated in this study. Subjects were asked to complete seven open-ended problems within a regular classroom setting (about 40 minutes). These open-ended problems involve a variety of important mathematical content areas, such as number sense, pattern, number theory, pre-algebra, ratio and proportion, estimation, and statistics. The appendix shows two of the seven tasks. These problems were from the QUASAR project (Silver, 1993).

Data Coding and Analysis

The data were coded and analyzed according to two analysis schemes: quantitative analysis (Lane, 1993) and qualitative analysis (Cai, Magone, Wang, & Lane, in press; Magone, Cai, Silver, & Wang, 1994). In the quantitative analysis, each student response to an open-ended problem was scored using a five-point scale (0 - 4) with 0 = no understanding, 1 = beginning understanding, 2 = some understanding, 3 = nearly complete and correct understanding, and 4 = complete and correct understanding. In the qualitative analysis, each student's response is examined in detail in terms of cognitive aspects of solving the open-ended problems, such as solution strategies, mathematical errors, mathematical justifications, and representations. These cognitive aspects are the focus of the qualitative analysis since they have been identified as important and significant dimensions in cognitive psychology in general and in mathematical problem solving in particular. An elaborate description of the framework for the qualitative analysis can be found in Cai (in press).

Inter-rater Agreement

In order to ensure a high reliability of coding student responses to open-ended problems, two raters independently coded about 50 student responses to three of the open-ended problems. The inter-rater agreements for the quantitative analysis range from 84–89%. Inter-rater agreements for the qualitative analysis range from 86– 98%.

Results

Quantitative Results

Overall, male students have significantly higher aggregated mean scores than female students ($M_{males} = 18.79$, $M_{females} = 16.36$; $\underline{t} = 2.43$, $\underline{p} < .01$). The gender differences were also examined for each open-ended problem. Table 1 shows the mean scores of male and female students on each open-ended problem. Males have significantly higher mean scores than females on four of the problems. This implies that the overall difference between male and female students is mainly due to the differences on the four problems. It is interesting to note that the four problems on which there are statistically significant gender differences require computation, but the others on which there are no significant gender differences require less computation.

Table 1.
Mean Scores of Male and Female Students on Each of the Open-ended Problems

	Division* Problem	Estimation Problem	Average Problem	Number Theory* Problem	Pattern Problem	Ratio &* Proportion Problem	Pre-algebra* Problem
Male(n=131)	3.15	2.15	2.18	2.27	3.05	2.90	3.09
Female(n=96)	2.79	2.19	2.07	1.66	2.96	2.08	2.61

* For this problem, the difference in mean scores between male and female students is statistically significant ($p < .05$).

Qualitative Results

Because of the space limitation, the qualitative results based on the first three of the problems for which there exist gender differences are reported in this paper. Of these problems, two of them (Division Problem and Number Theory Problem) appear in the appendix.

Division Problem. The first open-ended problem which shows gender differences is a division-with-remainder story problem (see appendix). In solving the division problem, one not only needs to apply and execute division computations correctly (computation phase), but also one needs to interpret the computational results with respect to a given story situation (sense-making phase). The qualitative analysis of the Division Problem was conducted from four aspects: (1) solution process, (2) execution of procedures, (3) numerical answer, and (4) interpretation.

Over 90% of the male and female students selected the appropriate procedure (e.g., long division) to solve the problem. However, a significantly larger percentage of male (86%) than female students (75%) executed the procedure flawlessly ($z = 2.01$, $p < .05$). A larger percentage of male (70%) than female students (61%) provided the correct answer of 13, but the difference is not statistically significant. For those who had incorrect answers, both male and female students frequently gave 12 or 12 with a whole number remainder (i.e., 12 R 8) as their answers. Only a few students expressed their numerical answer as 12 with a decimal remainder or 12 with a fractional remainder. Similarly, although a larger percentage of male (50%) than female students (44%) provided appropriate interpretation of their answers, the difference is not statistically significant. Therefore, the qualitative results of the Division Problem suggest that there is a significant gender difference in the computation phase of solving the problem (favoring males), but gender difference does not exist in the sense-making phase.

Number Theory Problem. This problem (see appendix) assesses student number sense and the ability to use basic concepts of number theory to solve a problem. It allows for multiple correct answers. In particular, the number 1 and any multiple of 12 plus 1 are correct answers (i.e., 1 + 12n, for n = 0, 1, 2, ...). Each student response was coded with respect to: (1) numerical answer, (2) solution strategy, (3) mathematical error, and (4) representation.

A significant larger percentage of male (61%) than female students (43%) had correct numerical answers ($z = 2.69$, $p <.01$). For those male and female students who had correct answers, 83% of female and 86% of male students had the correct answer of 13. The remaining 17% of female and 14% of male students had a correct answer other than 13, including 25, 49 etc. This implies that for those who had correct answers, female and male students tend to provide similar types of correct answers.

In solving the Number Theory Problem, eight different solution strategies were identified. Some students used common-multiple strategies to solve the problem. For example, a student might find 12 as a common multiple of 2, 3, and 4 by direct computation (2 X 6 = 12, 3 X 4 = 12, 4 X 3 = 12), and then found the answer by adding one to the common multiple. Another example is that a student listed the multiples of 2, of 3, and of 4; then identified the common multiple; and added one to find the answer, as shown below:

2, 4, 6, 8, 10, 12, 14, 16, ...
3, 6, 9, 12, 15, 18, ...
4, 8, 12, 16, 20, 24, ...
12 + 1 = 13, so the answer was 13.

Other students used guess-and-check strategies to solve the problem. Only a slightly larger percentage of male students (53%) than female students (48%) had a clear indication of using one of the solution strategies, with the difference being not statistically significant. For those who had a clear indication of using one of the strategies, female students tended to use common multiple strategies more frequently than males; while male students tended to use guess-and-check strategies more frequently than females ($\chi^2 (1, N = 116) = 4.93$, $p < .05$).

Male and female students made similar mathematical errors and used similar representations. The most frequent error (about 30%) made by both male and female students is that students manipulated numbers unreasonably. For example, they simply added the given numbers together to get the answer without any mathematical justification. The same percentage of male and female students (42%) used mathematical expressions to show their solution processes; 17% of female and 14% of male students used pictorial representation to show their solution processes; and 42% of female and 44% of male students used written words to show their solution processes.

Ratio and Proportion Problem. This problem assesses student problem-solving skills in a map-reading context that requires knowledge of ratio and proportion. In particular, students were given that 3 centimeters in a map represents 54 miles in actual distance, and they were asked to use proportional reasoning to determine the actual distance that 12 centimeters represents on the map. Each student's response was coded with respect to: (1) numerical answer, (2) solution strategy, (3) mathematical error, and (4) representation.

A larger percentage of male (69%) than female students (48%) had correct numerical answers ($z = 3.19$, $p < .01$). Similarly, a larger percentage of male (82%) than female students (60%) had a clear indication of using one of the identified solution strategies ($z = 3.68$, $p < .01$). However, both male and female students most frequently used a "unitary strategy" to solve the problem. One of the examples of using unitary strategy is like:

> 12 ÷ 3=4. Since 3 centimeters represents 54 miles, 4 X 54 = 216, so it is 216 miles.

Another example of using unitary strategy is like:

> Since 3 centimeters represents 54 miles, 54 ÷ 3 = 18, 18 X 12 = 216. So it is 216 miles.

Two male students used formal proportional reasoning strategy to solve the problem; no female students did so. Moreover, male and female students used similar representations in their solutions. In fact, over 90% of the male and female students used symbolic representations to show how they found their answer. The most frequent error made by both male and female students in solving this problem was that they manipulated numbers unreasonably; this finding is similar to what was found for the Number Theory Problem.

Brief discussion

This study used open-ended tasks to examine gender differences in solving complex mathematical problems. The results of this study suggest that overall, male students perform better than female students, but the gender differences vary from task to task. Gender differences appear to be significant on tasks requiring computation, but the difference dramatically decreases on tasks not necessarily requiring computation. In particular, in solving the division-with-remainder problem, males outperform females in the computation phase, but not in the sense-making phase

For those tasks showing significant gender differences, a more elaborate qualitative analysis of student responses was conducted. Although, a larger percentage of male than female students provided the correct answer, male and female students showed many similarities in the solution processes used to solve these open-ended problems, such as making similar types of mathematical errors and using similar strategies and representations. The results of this study suggest not only the complexity of the issue of gender differences in mathematics, but also the feasibility and usefulness of using open-ended problems to explore this issue.

References

Benbow, C. P. (1992). Academic achievement in mathematics and science between ages 13 and 23: Are there differences among students in the top one percent of mathematical ability? *Journal of Educational Psychology, 84*, 51-61.

Cai, J. (in press). *A cognitive analysis of U.S. and Chinese students' mathematical performance on tasks involving computation, simple problem solving, and complex problem solving.* (*Journal for Research in Mathematics Education* monograph series), Reston, VA: National Council of Teachers of Mathematics.

Cai, J., Magone, M., Wang, N., & Lane, S. (in press). Describing student performance qualitatively: A way of thinking about assessment. *Mathematics Teaching in the Middle School.*

Fennema, E. & Leder, G. C. (Eds.) (1990). *Mathematics and gender*. New York: Teachers College Press, Columbia University.

Hyde, et al. (1990). Gender differences in mathematics performance: A meta-analysis. *Psychological Bulletin, 107*(2), 139-155.

Lane, S. (1993). The conceptual framework for the development of a mathematics assessment for QUASAR. *Educational Measurement: Issues in Practice, 12*(2), 16-23.

Leder, G. C. (1990). Gender differences in mathematics: An overview. In E. Fennema & G. C. Leder (Eds.), *Mathematics and gender* (pp.10-26). New York: Teachers College Press, Columbia University.

Lindquist, M. M. (1989). *Results from the fourth mathematics assessment of the NAEP*. Reston, VA: NCTM.

Maccoby, E. E. & Jacklin, C. N. (1974). *The psychology of sex differences*. Stanford, CA: Stanford University Press.

Magone, M., Cai, J., Silver E. A., & Wang, N. (1994). Validating the cognitive complexity and content quality of a mathematics performance assessment. *International Journal of Educational Research, 21*(3), 317-340.

Marshall, S. P. (1983). Sex differences in mathematical errors: An analysis of distracter choices. *Journal for Research in Mathematics Education, 14*(4), 325-336.

Sells, L. W. (1973). High school mathematics as the critical filter in the job market. In R. T. Thomas (Ed.), *Developing opportunities for minorities in graduate education* (pp. 37-39). Berkeley: University of California Press.

Silver, E. A. (1993). Quantitative understanding: Amplifying student achievement and reasoning. Pittsburgh, PA: Learning Research and Development Center, University of Pittsburgh.

Appendix

Division Problem

Students and teachers at Miller Elementary School will go Spring sightseeing by bus. There is a total of 296 students and teachers. Each bus holds 24 people. How many buses are needed?

Show your work.
Explain your answer.

Answer:

Number Theory Problem

Yolanda was telling her brother Damian about what she did in math class.

> **Yolanda said, "Damian, I used blocks in my math class today. When I grouped the blocks in groups of 2, I had 1 block left over. When I grouped the blocks in groups of 3, I had 1 block left over. And when I grouped the blocks in groups of 4, I still had 1 block left over."**

Damian asked, "How many blocks did you have?"
What was Yolanda's answer to her brother's question?
Show how you found your answer.

Answer:

FAMILY CONFIGURATION AND MOTIVATION OF AFRICAN AMERICAN HIGH SCHOOL STUDENTS

Norma C. Presmeg, The Florida State University
Catherine Hill, Godby High School
Arthur R. Skelly, Jr., Rickards High School

Consonant with the current concern for achievement of minority students, the two action research projects described in this paper use interviews and case studies of ten African American high school students to address some issues relating to motivation in the learning of mathematics, particularly as this motivation concerns aspects of the family structures of which these students are a part. The data support the literature suggestion that high motivation to learn mathematics is more readily achieved amongst African American students from two-parent homes. However, the motivating factors are complex: amongst other factors, a negative role model effect is described, in contrast to the motivating effect of a relationship with a caring adult who values mathematics learning.

The Problem and its Significance

With an increasing concern that mathematics education should be fair, equitable, and accessible for all students, there is evidence that this is not the case in many American schools (Mathematical Sciences Education Board, 1990). In particular, African American students are not proportionally represented amongst high mathematics achievers, and in courses which prepare students for the study of mathematics at the tertiary level. The present research project investigated one aspect which has bearing on this issue (Banks & Banks, 1995), i.e., the influences of various family configurations and family members on the motivation of selected African American students in two schools.

Another issue is that research which is carried out in higher education institutions is sometimes perceived by teachers as not highly relevant to their day-to-day classroom activities and pedagogy (Lankford, 1993). The research described in this paper was carried out by two practising high school mathematics teachers in collaboration with a university mathematics teacher educator. The two related projects were chosen by the teachers as deeply significant to their work with African American students in their own mathematics classrooms. One veteran teacher, herself an African American, had achieved highly in mathematics although she came from a single-parent home. Her project, which we call the 'Family Project', investigated the family configurations (father-absent, mother-absent, and two-parent families) and related influences on the achievement and motivation of six of her students. In the second project, the 'Motivation Project', a teacher in his eighth year of teaching mathematics in another school in which 70% of the students are African American, investigated factors - including family configuration - which influenced motivation in the learning of mathematics.

Theoretical Framework

"It is generally accepted that school related achievement, attainment, personal and career aspirations, and eventual attainment, are functions of the direct and interactional effects of many factors. Among these are the individual, family and community" (Johnson, 1992, pp. 99-100). Research on single parent versus two parent families and the achievement of the children from these two configurations is mixed, but tends to support the idea of higher achievement among children from the two parent families. However, these findings are influenced by the socioeconomic status of the families involved, since single parent families usually are more economically disadvantaged than two parent families. Weissglass (1994, p. 69) wrote that "It is unwise and counterproductive … for reformers to ignore the fact that the current effort at reform is occurring at a time when schools are dealing with the effects on children of divorce, single-parent families, alcoholism, homelessness, violence, racial prejudice, sexual and physical abuse, and the widespread availability of drugs." He saw the effects of these conditions in classrooms as resulting in heightened stress levels for teachers. In seeking to understand the background experiences of some of our minority students and how these might influence their learning of mathematics, we adopted the theoretical position that motivation and achievement are influenced by a complex interrelationship of factors, including family configuration. We therefore chose African American students of both sexes from single-parent and two-parent homes for the purpose of learning more about aspects of their family life which influenced their learning of mathematics.

Methodology

Within this theoretical framework of personal and social factors, it was recognized that a qualitative methodology was appropriate, since it would provide the flexibility required to pursue unexpected issues as these arose. Data collection in both projects included transcripts of interviews with students, and classroom observation of students, as well as documents in the form of achievement and classroom tests.

The ten African American students in the two projects were as follows:

FAMILY PROJECT

NAME	SEX	AGE	GRADE	FAMILY
Evant	M	15	10	Two-parent
Jamella	F	15	10	Two-parent
Tim	M	18	12	Father-absent
Trivanna	F	14	9	Father-absent
Gerome	M	14	9	Mother-absent
Marie	F	18	12	Mother-absent

NAME	SEX	GRADE	CURRENT MATH COURSE	PARENTS
Kizzy	F	12	Explorations Math 2	Divorced
Mario	M	10	Explorations Math 2	Divorced
Tremecia	F	11	Geometry	Divorced
Trineshia	F	9	Algebra 1	Married

Data, interpretation and discussion

Our interview protocols tend to support the conclusion that high motivation to learn mathematics is more readily achieved amongst African American students from two parent homes. However, more importantly, our data suggest that the quality of a student's relationship with a caring and encouraging adult who values mathematics—whether in a single parent or a two parent home—is the crucial factor. In some cases our students were motivated to achieve in mathematics by negative or reverse factors: they did not want to grow up to be like a family member whom they did not admire. For instance, Tremecia spoke as follows.

> I look at my brothers and sisters and I get motivated. I do not want to grow up and be a bum. None of them has a regular high school diploma. Some have GED's, and some don't even have that. They don't do anything for themselves. If they need money or something, they mooch off of my parents. I hate that! I want better for myself.

This reverse role-model effect is similar to the phenomenon reported in Presmeg (1991), in which African American prospective teachers remembered poignantly negative experiences with mathematics teachers, which had the effect of causing them to aspire to be more caring and effective teachers than these negative role models were.

Career aspirations, sometimes based on family role-models, were also a strong motivating factor in the desire to achieve well in mathematics. Of the ten African American students in this research, all but one saw mathematics as the gateway to college studies and successful careers. However, their motivations were complex and individual. Mario, who eventually hopes to own his own business, had the following to say regarding his perceptions of college:

> If I don't go to college, I'll never own my own business. I mean I could, but I'd probably go broke because I would not have any formal training. That's why I'm in this program that helps students with their studies. It also provides us with experiences out in the business world, sort of like an internship. Before I entered this program, I did not like school and I had no incentive to do

> good. Now that I have a goal, I want to prove to myself that I can do this. I really like the challenge.

Trineshia had a related concern, as follows:

> I need to do good now so that I can get a scholarship. My parents make a good salary, but I don't think they could afford what a college education costs these days. I just want to try my best so that the rewards could possibly help me and my parents in the long run.

With regard to motivation to do well in mathematics, whether or not a student's parents were divorced appeared to be less important than the quality of a relationship with a parent who cared and valued learning. Tremecia, who lives with her mother, spoke as follows:

> My mother tries to keep on me about my studies. I think that has more to do with the fact that she is now back in school herself. She has held many different jobs, and I think that she wants to get a degree to get a steady job. She wants me to get my education now so that I won't have a hard time in the future. She knows that math is not my best subject. She is sympathetic because she did not do very good in math as a child. However, it is not an excuse to do poorly. She just tells me to try harder, and not to give up.

With regard to achievement, analysis of data from the family project suggests that the two students in this study who live with their fathers appeared to be underachieving in mathematics, while the other four students were achieving satisfactorily. From these data alone, no generalizations can be made. However, from the interviews in both projects it appears that single parents who work because of economic necessity often find it more difficult to devote the time to take an interest in the quality of their child's learning in mathematics. According to research by Nieto (1992), the family as a unit is a significant asset in successful learning by minority students. Having a close-knit family that encourages a child, allows for open and meaningful conversations that facilitate the desire to achieve. Our data support this conclusion.

References

Banks, J. A. & Banks, C. A. M. (Eds.) (1995). *Handbook of research on multicultural education.* New York: Macmillan.

Johnson, S. T. (1992). Extra school factors in achievement attainment and aspiration among junior and senior high school-age African American youth. *The Journal of Negro Education, 61*(1), 91-116.

Lankford, N. K. (1993). Teacher as researcher: What does it really mean? In P. S. Wilson (Ed.) *Research ideas from the classroom: High school mathematics*. New York: Macmillan.

Mathematical Sciences Education Board (1990). *Making mathematics work for minorities: Framework for a national action plan, 1990 - 2000*. Washington, D.C.: National Academy Press.

Nieto, S. (1992). *Affirming diversity: The sociopolitical context of multicultural education*. New York: Longman.

Presmeg, N. C. (1991). *Black students and university mathematics: Report on a research project to investigate factors underlying difficulties experienced by Black prospective teachers in pure mathematics courses*. Center for the Study of Teaching and Learning, Florida State University, Summer, 1991.

Weissglass, J. (1994). Changing mathematics teaching means changing ourselves: Implications for professional development. In D. B. Aichele & A. F. Coxford (Eds.) *Professional development for teachers of mathematics* (pp. 67-78). Reston, Virginia: National Council of Teachers of Mathematics, 1994 Yearbook.

GENDER-RELATED DIFFERENCES IN INTERACTION PATTERNS IN ELEMENTARY SCHOOL INQUIRY MATHEMATICS CLASSROOMS

Diana Underwood Gregg, Purdue University Calumet

The purpose of this study was to identify interaction patterns that emerged during mathematics instruction in elementary school classrooms which established an "inquiry" mathematics tradition, to describe any gender-related differences in these patterns, and to attempt to account for the presence or the absence of such differences. Preliminary analysis suggests that aspects of an inquiry approach to mathematics instruction may have had a positive impact in providing gender-equitable learning opportunities for boys and girls.

Gender differences in mathematics teaching and learning have been studied by numerous researchers over the past twenty years (Eccles & Blumenfeld, 1985; Fennema & Sherman, 1978; Hart, 1989; Jungwirth, 1991; Leder, 1992). In general, these studies have evolved from findings which indicate that males tend to outperform females in mathematics on standardized measures, and that females are less likely than males to take non-compulsory courses in high school mathematics.

In an effort to account for these phenomena, researchers have studied gender-related differences in males and females' beliefs about their mathematics abilities, differences in behaviors that females and males exhibit as a result of their beliefs, the influence of social interaction on their beliefs and behaviors, and gender-related differences in classroom interaction patterns. However, the vast majority of the studies that have noted differences in beliefs, behaviors, and/or interactions have taken place in classrooms characterized by what Cobb, Wood, Yackel, and McNeal (1992) call the *school mathematics tradition*. In this tradition, students are typically expected to learn and become proficient at solution methods and procedures that are presented to them by the teacher and their textbooks.

This tradition is in contrast to the *inquiry mathematics tradition* advocated in the current reform movement in mathematics education. In an inquiry mathematics classroom, the emphasis is on figuring out personally meaningful solutions and engaging in mathematical reasoning, explanation, and justification. Since the activities of explaining and justifying are central aspects of inquiry instruction, but not of traditional school mathematics instruction, the interaction patterns that occur in inquiry classrooms contrast dramatically with those of school mathematics classrooms (Cobb, et al., 1992). This suggests that inquiry classrooms are potentially rich sources for studying gender-related differences in attitudes, beliefs and motivations.

An important question that arises then is the following: How might a qualitative change in mathematics instruction influence the patterns of interaction that arise in the classroom and thereby influence students' learning opportunities with respect to gender? Will the gender-related differences recorded in the past continue to perpetuate inequitable learning opportunities for males and females or could an inquiry mathematics classroom tradition provide a more gender-equitable environment? This study begins to address this question.

Theoretical Focus

The theoretical framework that guides this research stems from symbolic interactionism (Blumer, 1969) and ethnomethodology (Garfinkel, 1967). Under these assumptions, individuals in a group construct subjective meanings for things by interpreting each others' actions and adjusting their interpretations in the course of their interactions. Although meanings are subjective, they are experienced as universal truths by the participants within an interaction. Therefore, when applied to the analysis of mathematics lessons in schools, this point of view assumes that the individuals' mathematical activity is reflexively related to the classroom microculture (Voigt, 1992).

Guiding Research Questions

The research questions that guide this study are the following: 1) What are the typical patterns of interaction that emerge during whole class mathematics instruction that follows the inquiry tradition? 2) What, if any, are the gender-related differences in the interaction patterns during whole class, teacher-led activities? 3a) If there are gender-related differences in the patterns of interaction, then how are the gender-specific interaction patterns interactively constituted? and 3b) If there are no gender-related differences in the patterns of interaction, then are there aspects of the *inquiry* approach to instruction that can account for this lack of gender-related differences?

Data Collection

The mathematics instruction in two second and two third grade classrooms was video-recorded for two consecutive weeks during the last two months of the 1992-1993 school year. The classroom teachers conducted all the lessons using instructional activities and strategies that had been developed by the Purdue Problem-Centered Mathematics Project. Interviews were conducted with eight children from each of the 4 classes to gain information about their mathematical conceptual understandings as well as their perceptions of classroom events, their motivation for participating in whole-class discussions, and their views about their mathematics ability.

Methods of Analysis

In the first stage of the data analysis, transcripts of the mathematics lessons are currently being analyzed using a constant comparative approach (Glaser and Strauss, 1967). After transcribing each lesson, *theoretical memos* are written for each interaction sequence. These memos contain my interpretations and hypotheses about the meaning of events to the participants and serve as the basis for interpretation of subsequent lessons. With each analysis of subsequent episodes, the conjectures previously made regarding comparable sequences are tested, refined or set aside. In the second stage of the analysis, the knowledge gained from the interviews with the target students and their teachers will be used as a means of triangulating or refuting assertions made in the first stage of the analysis.

Findings

I will discuss the preliminary findings from two of the four classrooms—Mary's and Josette's (both names are pseudonyms). The typical smooth flow of discourse in interactions in traditional classrooms has been described by Mehan (1979) as the initiation-response-evaluation scheme (i.e., the teacher asks a known-answer question, a student answers the question, and the teacher evaluates the answer). In contrast to this pattern, the smooth discourse in Mary's class when there was no disagreement about the answer to a task could be described as initiation-response-evaluation-echo, response-evaluation-echo, response-evaluation-echo, etc. Mary began by posing a task for which no precursory instruction had been given (initiation). A student, usually a volunteer, directed his/her response in the form of an answer to the task followed by his/her solution (response). Whereas in traditional school mathematics instruction in which the teacher assumes the role of the sole evaluator of the students' answers, in this class the students became a community of validators by calling out something that would suggest whether or not they agreed with the answer/solution that was given (evaluation). The teacher followed this by repeating the student's answer/solution back to the other members of the class, or by helping the student express what she thought the student was trying to say. Because the teacher only contributed to helping the student clarify his/her solution but never intentionally altered the nature of the student's solution, the last part of this recurring pattern is described as an "echo" of the student's answer/solution. Following her echo, Mary called on several other students for their responses to the original question. These were also evaluated by the students and then echoed by the teacher. Thus, instead of the typical school mathematics role as trainer and evaluator, Mary's role could be likened to that of a moderator.

When a student gave an answer or solution which was evaluated as incorrect by his/her peers, the class discussion would "breakout" as students would simultaneously begin calling out their arguments against the answers/solutions that were given. The breakout ended when the teacher reassumed the role of turn taking monitor and gave the floor to a student, thus ending the simultaneous talk. Although she typically would return to the student whose answer was disputed to see if s/he had, in Mary's words, "changed his/her mind", Mary never pressured students into changing their answers. In fact, just the opposite situation appeared to be the norm in this class. Even though she allowed the breakout in the classroom discourse to occur (and I would argue that she needed the breakouts to occur because she would not openly evaluate the answers herself), Mary unwaveringly upheld the students' right to state their solution without being interrupted, and to not be obligated to change their answer for *any* reason.

Based on the analysis of the patterns of interaction in Mary's class and previous literature, several gender-specific questions arose. Only one will be discussed here:

- What role does gender play in teacher-male vs. teacher-female interactions that involved attempts to gain the floor?

Both females and males used similar strategies to get a turn, such as calling out "I got something different" or "I did it a different way." Occasionally, both males and females would call out at inappropriate times, making it difficult for them to get a turn. However, there were a small number of incidents in the data in which the girls, in order to provide a rationale for why they should be given a turn to present their solution to a specific problem, would proclaim that they had "problems", it was "hard" for them to get their answer, or that their solution was "confusing". What makes this interesting is that the girls were not using this strategy as a means of getting help to solve problems that they perceived to be too difficult for them to solve. In other words, they were not exhibiting learned helpless behaviors. All of the girls who used this strategy to get a turn had invented viable solutions for the problems which they subsequently presented to the class.

A few years ago, the Mattel Toy Company, makers of the Barbie doll, came under fire when one of the phrases that their talking Barbie had been programmed to say was that "math is hard". By saying that "math is hard", Barbie was supposedly reinforcing the stereotype that for girls, math was too difficult. In the school mathematics tradition, if one believes that math is hard, this implies that he/she is probably not able to easily solve school math problems. This indicates that he/she has limited mathematical abilities. In Mary's class, it was taken-as-shared that math was sometimes "hard". However, "hard" had a different meaning for the students in this classroom than for those who have experienced traditional school mathematics instruction. When students or Mary described a problem as "hard", it did not mean that the participants believed that the problems were beyond their ability to solve it. Problems that were described as being "hard", "difficult" or "confusing" meant that students had to work harder at figuring them out. When the girls in this class described a task as "hard", they were not lowering their status by indicating a lack of competency (which might be the case in traditional school mathematics classrooms).

One of the reasons for the lack of gender-related differences in the patterns of interaction in Mary's class might be her role as moderator rather than evaluator and trainer. Mary did not have a mathematical agenda that she was trying to get the students to "see." Jungwirth (1991) found that, in traditional classrooms, boys may be more apt than girls to play along with the teacher's agenda and thus appear more competent. In Mary's case, her role as moderator promoted an atmosphere of trust in which the students knew that Mary would value all of their responses. This atmosphere contributed to a situation in which the students were not afraid to accept the challenge of problems that they were not sure they would be able to solve.

In contrast to Mary's role as moderator, Josette was not so equally accepting of all students' answers/solutions. Less sophisticated solutions typically received less recognition from the teacher. Whereas Mary "echoed" all solutions, Josette did not "echo" solutions in which students counted by ones to solve a problem. Furthermore she often elicited solutions that students figured out "without counting" and highlighted more sophisticated solutions when they appeared to fit with her agenda for the task. When students gave an answer that Josette considered to be wrong, she did not directly tell them that they were "wrong", but the rising inflection in her voice that she often used when she repeated their answer/solution was typically interpreted by students as an indicator that their answers were wrong. Thus, the typical pattern of interaction in Josette's class more closely resembled the tradition initiation-response-evaluation pattern.

Josette's subtle evaluation of students' answers was often followed by a modification in the typical pattern of interaction that corresponded to the "breakout" pattern in Mary's class. However, whereas the "breakout" pattern was played out in Mary's classroom interactions when students judged an answer to be incorrect, this open forum for calling out one's disagreement with a peer's answer/solution was not practiced in Josette's class. In Josette's class, if a student disagreed with an answer/ solution given by his/her peers, norms had been established prior to the data collection period in which the student who disagreed was under the obligation to ask the student who provided the solution questions about the aspects of the answer/solution which he/she disagreed with. Unlike in Mary's class in which students who gave a "wrong" answer were not obligated to address arguments regarding their solutions, the students in Josette's class could not hold on to an answer/solution without addressing these arguments.

Although no gender differences were noted in interactions when students questioned each others' solutions, the following gender-specific question arose:

- What role does gender play in the teacher-student interactions when students' answers/solutions are judged to be incorrect?

When Josette judged an answer/solution to be wrong, she was much more directive in her interactions with some students than she was with others. However, the data analysis indicated that the differences in these interactions were not related to the student's gender, but to Josette's perception of the different cognitions of the students. Josette interacted similarly with strong females and strong males. When a strong student gave an atypical answer which might have been judged as being incorrect, Josette would indicate that *she* was having trouble figuring out how they had solved the task and would ask the student for a clarification. When a mathematically less able student began to present a solution that did not fit with a solution that she expected to hear, Josette would cut off the student's explanation before s/he could complete it and steer her/him to what she considered an acceptable solution for that task. There were also not any gender differences in the ways in which males and females responded to this steering. For instance, both weak females and weak males accepted the funneling in the same manner.

Neither boys nor girls would contradict the teacher's negative evaluation and steering to a different solution even when their solutions were correct.

A possible reason why there were not any gender related differences in the patterns of interaction in Josette's class might be because Josette's focus was on how she was supposed to "do" the instructional activities so that students would learn. Josette seemed to have a "script" that she followed for each activity that she and the students engaged in. It seems possible that she was so preoccupied with following the "script" for the instructional task that she did not pay much attention to the students' mathematical conceptions unless their solutions did not fit her script. For those cases when things did not go smoothly, she had take action to get students back on track with the script. As noted, these actions differed according to Josette's perception of students' abilities, but not according to their gender.

References

Blumer, H. (1969). *Symbolic interactionism.* Englewood Cliffs, NJ: Prentice-Hall.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29,* 573-604.

Eccles, J. S., & Blumenfeld, P. (1985). Classroom experiences and student gender: Are there differences and do they matter? In L. C. Wilkinson & C. B. Marrett (Eds.), *Gender influences in classroom interaction* (pp. 79-114). New York: Academic Press.

Fennema, E., & Sherman, J. A. (1978). Mathematics achievement and related factors: A further study. *Journal for Research in Mathematics Education, 9,* 189-203.

Garfinkel, H. (1967). *Studies in ethnomethodology,* Englewood Cliffs, NJ: Prentice Hall.

Glaser, B. G. & Strauss, A. L. (1967). *The discovery of grounded theory.* New York: Aldine.

Hart, L. E. (1989). Classroom processes, sex of student, and confidence in learning mathematics. *Journal for Research in Mathematics Education, 30,* 242-260.

Jungwirth, H. (1991). Interaction and gender: Findings of a microethnographical approach to classroom discourse. *Educational Studies in Mathematics, 22,* 263-284.

Leder, G. C. (1992). Mathematics and gender: Changing perspectives. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 597-622). New York: Macmillan.

Mehan, H. (1979). *Learning lessons.* Cambridge, MA: Harvard University Press.

Voigt, J. (1992, August). *Negotiation of mathematical meanings in classroom processes.* Paper presented at the International Congress on Mathematics Education, Quebec City.

MATHEMATICS EDUCATION RESEARCH ON MINORITIES FROM 1984 TO 1994: FOCUS ON AFRICAN AMERICAN STUDENTS

Sunday A. Ajose, East Carolina University

Judged by its results, the current system for educating African American students in mathematics is a distinct failure. For years, educators either ignored this problem or simply blamed the failure of the system on its victims. Eleven years ago, the Journal for Research in Mathematics Education drew national attention to the problem by devoting an entire issue to articles and reviews of available research on how America's ethnic minorities learn, and perform in mathematics. Twenty-four studies involving African American students were reviewed, and several factors that might influence performance and participation in mathematics, identified. Following up on that work, this paper reviews studies that have been done on this subject since 1984. Most were correlational studies.

Why are African Americans (blacks) so seriously underrepresented in mathematics? What causes the persistent under-achievement of black students in this subject? How can schools rectify the gross racial inequity in mathematics education outcomes? These and similar questions have been begging for answers for decades. The Journal for Research in Mathematics Education (JRME) drew attention to this neglected problem when, in 1984, it devoted an entire issue to research on factors which influence how America's ethnic minorities, including blacks, learn and participate in mathematics (Matthews, 1984). This paper is a follow-up to that seminal work. Its purpose was to review some of the research that had been done on this subject since 1984, and give ideas that may guide further research in the area. Only studies involving black students were considered.

Selection of Research Studies. Because of space limitations, studies reviewed for this paper were identified from lists of mathematics education research which appeared in July issues of the JRME from 1984 to 1994. Only a few of these studies were cited. Items were selected for their potential to contribute to our meager knowledge base on the subject of blacks and mathematics. Some of the studies focused exclusively on black students; most included a significant number of black students in their samples.

Framework for this Review

Matthews' (1984) review identified several parent-, student-, and school-related factors thought to influence the quality as well as the outcomes of the mathematics education of minority students. These factors were classified into three groups, and presented as in Table 1. No claim was made about the comprehensiveness of the list of factors. At the time of this initial review, there was very little empirical evidence linking some of the variables to the mathematics performance of black students. However, Matthews' framework was used for the current review because of its logical structure and because it was sufficient to accommodate all the factors that were cited in the studies reviewed.

Table 1

Parent, Student, and School Factors That Affect the Performance and Participation of African Americans in Mathematics

Parent	Student	School
Ascribed Characteristics race, sex, age	*Ascribed Characteristics* race, sex, age	*Climate* discipline, attendance
Cognitive past and present education & occupation	*Cognitive* past and present mathematics performance and enrollment	*Organization* course offerings, sequence and prerequisites, curriculum placement, class size
Affective expectations and aspirations for child; support for mathematics performance.	*Affective* achievement orientation, self-concept, locus of control, stereotyping, perceived utility of mathematics	*Resources* facilities, materials
Cultural communication style, primary language spoken at home	*Cultural* cognitive learning style, language proficiency	*Personnel* ascribed characteristics, professionalism, instructional methods, attitudes and perceptions, student interaction.

Parental Influences On Mathematical Achievement

A search of the literature turned up no study of the effects of parents' ascribed or cultural characteristics on student performance in mathematics. However, the effects of cognitive and socioeconomic factors were examined in several studies. Most of these studies found parental occupation, income and education to be positively correlated to children's mathematical achievement (Blackwell, 1984). One study (Kaaya, 1989) found no such relationship.

Many mathematics educators believe that high but realistic expectations can exert a positive influence on mathematical achievement. Researchers who explored this issue in black student populations found supporting evidence for the belief (Kaaya, 1990; Rhone, 1990). The two studies also suggest that the educational and vocational aspirations which black parents have for their children can predict how well these students will do in mathematics. It appears that parental expectations communicate to children in a very powerful way how much their parents *believe* in them. This, in turn, strengthens the children's *belief* in themselves.

Other correlates of improved mathematical performance among black students are success-related behaviors of parents such as developing partnerships with their children's schools, having positive attitudes towards mathematics, engaging in mathematics-related activities with their children, scheduling homework routines and making sure that children get help with schoolwork when necessary.

Student Factors

It is widely accepted that the characteristics which a learner brings to any learning situation will affect the outcome of the event. In the past, scholarly discourse on differential achievement in mathematics often centered on the minority student's race, risk factors and assumed cultural deficits. During the past ten years, however, the research focus appears to have shifted to the student's cognitive, affective and unique cultural characteristics.

Reyes' (1984) review of the literature on affective variables showed a consistent pattern of strong, positive correlations between self concept of mathematical ability and achievement in the subject. Similar results were obtained from studies of black students ranging from elementary grades through college (Groman, 1989; House, 1993). Groman's study further showed that measures of academic self-concept could be used to predict the number of mathematics courses female black students might take in the future. It seems that self-confidence is very influential in decisions black students make concerning their mathematics education.

The concept of cognitive style recently emerged as a means of explaining some of the persistent racial differences in mathematics performance and participation. Hale-Benson (1986), for example, argued that most African American students process information in ways that are very different from those of whites. According to this perspective, black students tend to be field-dependent, intuitive rather than analytical, holistic and people oriented; they have relational learning styles which differ from the analytic styles that schools prize. As a result, racial disparities in academic achievement may be partly due to the failure of schools to appreciate and adapt to the unique characteristics of African American learners. These claims have, so far, received only limited support from studies of attributional and learning style preferences of African American students.

The results of some recent studies indicate that many black students have affective profiles similar to what Hale-Benson described (Rech, 1991). The black students in these studies tended to have an external locus of control, and be field-dependent—a trait generally associated with low mathematics achievement. However, other studies found no significant positive correlation between mathematics achievement and these traits. In fact, the data from one study flatly contradict the claim that black students are predominantly holistic (Roberts, 1991). There is clearly a need for more research in this area.

School Factors

There is no question that, in general, black students under achieve in school mathematics. They are subject to more disciplinary measures, and are labeled as slow learners in disproportionately large numbers. What has always been at issue is whether schools contribute to the learning problems which these students have, and if so, how. Findings from some studies suggest that school policies such as tracking and grade retention are detrimental to black students. The common practice of tracking and putting academic labels particularly on black students is supposedly based on objective scientific measures. Yet, Johnson (1986), found that racial and socioeconomic considerations strongly influence and contaminate the process. Academic tracking is often denounced, with justification, because bright black students are sometimes mislabeled, and because the classification creates lower expectations of the students involved. There is some indication, however, that factors other than the mere act of tracking itself may be responsible for the plight of academically tracked black students. The results of a study involving black students in 8th grade found no effect due to ability grouping, on the mathematical performance of these students (Meeks, 1994).

Like tracking, grade retention and emphasis on discipline may be counterproductive school practices. It seems quite reasonable not to promote a student until basic essential knowledge for his or her current grade has been acquired. Moreover, school effectiveness research recommends that schools maintain safe and orderly learning environments. Nevertheless, grade retention and an undue focus on discipline tend to correlate with low mathematics performance by affected black students.

School climate and social structure also seem to influence how black students perform in mathematics. Where classroom tension is high, differential mathematics achievement between black and white students tends to be high; where cohesiveness and satisfaction prevail, achievement differences between the races tend to be small (Deng, 1992). Furthermore, hostile school environments alienate black parents and prevent the development of beneficial parent-school partnerships.

Studies of the effect of expectations, reviewed earlier in this paper, point to a strong link between mathematics performance and achievement-related expectancies of students and parents . Teacher expectations also appear to have a strong impact on the performance and participation of black students in mathematics.

To begin with, there is some evidence that teachers' expectations are influenced by student characteristics. Unfortunately, teachers seem to value qualities of obedience, dependence, and conformity rather than assertiveness and independence in black students. Although there is no measure of the power of teacher expectations, its strength is indicated by the finding that the mere perception of high teacher expectations evokes in black students greater task orientation and performance.

Curriculum and Instruction also turn out to be strong factors in the mathematical performance of black students. Variables found to have a positive effect on performance include early intervention with culturally relevant curriculum materials, and the use of problem solving and constructivist instructional strategies. Cooperative Learning structures benefit some, but not all, black students; Computer Assisted Instruction can improve performance by low achievers. However, remedial strategies which are used with students in lower mathematics tracks correlate only with poor performance.

Concluding Remarks

Research on the mathematics education of African Americans has taken a turn for the better. For one thing, the research is being done with black children, and increasingly by people with a direct experience of some of the issues involved. For another, theoretical underpinnings of the studies, where they exist, are shifting from those that view the unique characteristics of black children as deficits, disadvantages, and pathologies to those that see the educational problems of these children primarily as issues of inequity. The research is clearly in its infancy, and lacks focus for the most part. We now have some useful information in the substantial number of positive correlations that have been found between various factors and the mathematical performance of black students. However, solutions to the persistent problem of under achievement have yet to be found.

To make progress on this front, researchers need to heed some recommendations of the Research Advisory Committee (RAC) of the National Council of Teachers of Mathematics (RAC, 1989). The committee recommends that researchers:

1. Develop useful conceptualizations of how minority (black in this case) students learn mathematics.
2. Enlarge current research efforts to focus in a systematic way on the mathematics learning of underrepresented and underserved groups.
3. Conduct school-based research that addresses teacher-minority student interactions, and how to change the classroom teaching and learning environment.

There are several useful conceptualizations like the ones called for in item #1, most notably a model by Reyes and Stanic (1988). These ideas have not had much impact on current research however. When they do, research will probably begin to yield the kind of information that will help make school mathematics work for African Americans and other underrepresented ethnic minorities.

References

Blackwell, Donna L. (1984). The quantitative orientation of black high school students. (Doctoral dissertation, Washington University, 1983). *Dissertation Abstracts International, 45A*, 442.

Deng, Bing. (1992). A multilevel analysis of classroom climate effects on mathematics achievement of fourth-grade students. (Doctoral dissertation, Memphis State University, 1991). *Dissertation Abstracts International, 53A*, 53.

Groman, Cynthia L. (1989). Math achievement, attitudes, and plans of black and white, male and female high school students. (Doctoral dissertation, Indiana University, 1988). *Dissertation Abstracts International, 50A*, 101.

Hale-Benson, Janice E. (1986). *Black Children: Their Roots, Culture and Learning Styles*. (revised edition). Baltimore: The Johns Hopkins University Press.

House, J. Daniel. (March 1993). Achievement-related expectancies, academic self-concept, and mathematics performance of academically under prepared adolescent students. *Journal of Genetic Psychology,* 154, 61-71.

Johnson, Ruth S. (1986). An exploratory study of academic labeling, student achievement, and student ethnographic characteristics. (Doctoral dissertation, Rutgers University - New Brunswick, 1985). *Dissertation Abstracts International, 46A*, 2656.

Kaaya, Mathias I. (1989). Identifying variables supportive to achievement in mathematics: a comparison of African-American students in San Francisco Bay area. (Doctoral dissertation, University of San Francisco 1987). *Dissertation Abstracts International, 50A*, 92.

Matthews, W. (1984). Influences on the learning and participation of minorities in mathematics. *Journal for Research in Mathematics Education*, 15, 84-95.

Meeks, Wendell A. (1994). The effects of classroom ability grouping on eighth-grade student achievement in mathematics. (Doctoral dissertation, Southern Illinois University, 1993). *Dissertation Abstracts International, 54A*, 2933.

Rech, Janice F. (1991). The relationships between mathematics attitude, self-concept, learning style, socioeconomic status, gender, and mathematics achievement among fourth and eighth-grade (fourth grade) (Doctoral dissertation, University of Nebraska - Lincoln, 1990). *Dissertation Abstracts International, 52A*, 457.

Research Advisory Committee of the National Council of Teachers of Mathematics. (1989). The mathematics education of underserved and underrepresented groups: A continuing challenge. *Journal for Research in Mathematics Education*, 20, 371-375.

Reyes, Laurie H. (1984). Affective variables and mathematics education. *Elementary School Journal*, 558-581.

Reyes, L. H. , & Stanic, G. M. A. (1988). Race, sex, socioeconomic status, and mathematics. *Journal for Research in Mathematics Education*, 19, 26-43.

Rhone, Lorna, M. (1990). Relations between parental expectation, mathematics ability, mathematics anxiety, achievement on mathematics word problems, and overall mathematics achievement in black adolescents. (Doctoral dissertation, New York University, 1989). *Dissertation Abstracts International 50A*, 3902.

Roberts, Veronica A. (1991). Relationships among processing behavior mode, cognitive development level and mathematics achievement in black metropolitan high school students (black students). (Doctoral dissertation, Howard University, 1991). *Dissertation Abstracts International, 51B*, 5054.

AUTOBIOGRAPHIES OF WOMEN AS TEACHERS/LEARNERS IN THE ELEMENTARY MATHEMATICS CLASSROOM

Diana Erchick, The Ohio State University

This study was conducted in an elementary school in a large urban district in the midwest. Data collection was completed at the end of a two-year pilot in which the researcher was the Mathematics Teacher Leader in the school. The collected data for each of the six female teacher participants were in the form of a mathematical autobiography, an interview discussion of teacher instructional choices, a videotaped mathematics lesson, and a follow-up interview reflecting upon the videotaped lesson. The following research questions organized the study: 1) What are the ways of knowing mathematics owned by these teachers? 2) What role do past experiences play in the development of their ways of knowing mathematics? 3) What voices of authority do these women choose to inform their instructional choices in mathematics?

The teachers in this study displayed evidence of a variety of perspectives of ways of knowing mathematics. With *Women's Ways of Knowing* (Belenky, Clinchy, Goldberger and Tarule, 1986) as a theoretical frame, perspectives on knowing mathematics were identified in these teachers. For instance, as one teacher described her college mathematics experience, she revealed herself then to be a "received knower" who expected that attendance and listening would assure that learning mathematics would happen for her.

Narratives describing past experiences included exemplars cited by the teachers as singular moments having a particular impact on their perspective on the learning of mathematics. The teachers revealed preferences for ways of knowing mathematics. Some teachers identified what they believed to be inadequate instruction from their experiences as students, or methods of instruction that became less adequate as the participant developed new perspectives on learning mathematics.

In making instructional choices, the teachers chose from among many voices of authority in mathematics education reform. These voices included state, local, and university authorities; the NCTM; and the women's own internalized voices of experience. Of the voices heard, it was the teacher's own voice of experience that was most prevalent in the decision-making processes guiding mathematics instruction.

In reflection upon experiences in mathematics, ways of knowing mathematics surfaced as a part of the internal voices of authority. The experience of reflective practice may further professional development by exploring and understanding ways of knowing mathematics as they surface within teaching.

Reference

Belenky, M.F., Clinchy, B. M., Goldberger, N.R., & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice and mind.* New York: Basic Books.

GENDER AND ETHNIC DIFFERENCES ON A DIVERSE URBAN CAMPUS IN REQUIRED MATHEMATICS COURSES

Shirley B. Gray, California State University

The presentation focuses on two research investigations conducted in mathematics courses required for graduation on a diverse campus. The discussion will conclude with recommendations based on in-depth study and experience. The subjects in these intense and on-going investigations constitute a subset of a campus population of 20,800 students that is 11% African-American, 17% White, 38% Latino, 33% Asian, and 1% Native American. These ratios change each year, with Asian and Latino percentages increasing, and White and African-American numbers decreasing. Broad variability exists in the Latino subjects (e.g., Chicanos, Mexicanos, Central and South Americans, Cubans, Puerto Ricans) and the Asians (e.g., Taiwanese, mainland Chinese, Japanese, Thai, Vietnamese) as well as combinations (Japanese born and raised in Peru). In particular, 40% are residents of the inner city. The president of our university, an African-American, has proudly called our student body the most diverse in the United States.

Summary Data Analysis and Conclusions. Several points have come through clearly: There is no significant difference either in mathematics achievement or attitude between males and females (and this finding was independent of the broad age group examined); importantly, there was no indication that mathematics anxiety is higher in preservice teachers than in other non-mathematics majors; and no "ethnic factor" was revealed in mathematics attitude—indeed, all four ethnic groups examined were clustered closely together in mathematics self-concept scores. In addition, results of data analyses (N = 1,000) focused on the effectiveness of computer aided instruction (CAI) will be summarized.

Thus, with statistically derived outcomes, which might be somewhat intuitively counter to the expectations of instructors on other campuses, the reader might ask why these results should be taken seriously. The reader might also ask what practices we, as an experienced faculty members, have found useful. Suggestions have been garnered from several sources: our older graduates reflecting on what they considered important about their campus experiences; our campus Center for Effective Teaching; other student support services; and our faculty. Three categories will be discussed:

- Suggestions for Administrators Based on Our Experience.
- Suggestions for Research Investigations.
- Suggestions for the Mathematics Classroom.

CAT'S CRADLE: MAKING CULTURAL CONNECTIONS IN THE MATHEMATICS CLASSROOM

Rita L. Eisele, Indiana State University

The mathematics classroom seems to be an unlikely place for multicultural activities. Mathematics is usually taught without any historical framework or appreciation of the contributions made by different cultures. Many cultures do not have professional mathematicians, but the people have clearly shown a mathematical ability in everyday activities. Some of these activities include art, architecture, religious rituals, and games (Ascher & Ascher, 1981).

Making string figures, or a "cat's cradle" as it is commonly known, is one of the oldest games in existence, and is played by about every culture in the world. Charles Moore (1988), Jearl Walker (1985), and other mathematicians have discussed the link between string figures and mathematical thought.

I teach a mathematics content course for preservice elementary teachers and have developed and studied students' reactions to a unit on string figures. My goals in teaching this unit were to develop an appreciation for the mathematical abilities of people from different cultures and to provide a model for incorporating multicultural education into their own mathematics classrooms. Data were collected primarily through surveys, student journal writing, and work samples.

In my poster session I propose to (1) demonstrate several of the string figures from the Navajo culture, (2) discuss some of the mathematical concepts involved, and (3) share the results of my investigation of student reaction to a unit on string figures.

References

Ascher, Marcia, & Ascher, Robert (1981). *Code of the Quipu*. Ann Arbor: The University of Michigan Press.

Moore, Charles G. (1988). The Implication of String Figures for American Indian Mathematics Education. *Journal of American Indian Education, 28*(1), 16-26.

Walker, Jearl (1985). Cat's Cradles and Other Topologies Formed With a Two-Meter Loop of Flexible String. *Scientific American, 252*(5), 138-43.

STATISTICAL STORYTELLING AND ITS IMPACT ON ADOLESCENT FEMALE STUDENTS

Susan Prion, University of San Francisco
Mathew Mitchell, University of San Francisco

The poor performance of adolescent girls in mathematics classes has been the study of many interesting research studies. Theories abound to explain why the interest or mathematics ability of female teens seems to drop so dramatically during high school. The small number of women majoring in mathematics at the undergraduate and graduate level seems to further emphasize this problem.

We discovered an interesting and exciting effect contrary to previous studies during the assessment and evaluation of an experimental curriculum titled *Statistical Thinking*. The primary tool used in the curriculum was the *Teaching Sheet*, a statistical storytelling device used to helping students construct their own meanings and understanding for statistical selected techniques. *Teaching sheets* were theorized to be effective because students learn more when they use their new knowledge to teach others.

The *teaching sheets* seemed to be particularly effective for young female students who had not been previously successful in mathemtics classes. They exceeded their male contemporaries in the quality and quantity of work produced during the class session. Their instructors and their parents spontaneously commented on the excitement and interest for mathematics that had been awoken in these students. The reasons for the surprising changes in performance were unclear to the instructors or the researchers.

At the end of the course, a focus group interview was conducted with seven of these young female students to collect the students' perspectives about the success of the *teaching sheets* for them. Six were students of color. This poster session presents the major points of the content analysis of that focus group, and provides an interesting insight into the problem of young girls and mathematics as told through the words of the students themselves.

Student Beliefs and Attitudes

MATHEMATICAL DISPOSITIONS OF STUDENTS ENROLLED IN FIRST-YEAR ALGEBRA

Daniel J. Brahier, Bowling Green State University

Dispositions of eighth graders accelerated into first-year algebra were described in this study. Data were collected through surveys, observations, interviews, and cumulative academic files. The most frequently reported reasons for enrolling in algebra were for acceleration of course-taking and preparation for high school. Males demonstrated a higher level of self-efficacy to perform in algebra and secondary mathematics. Students showed a high level of perseverance in terms of sacrifices made to take the course, but classroom performances indicated negative dispositions toward mathematics. Students were driven by a desire to please the teacher and earn grades rather than out of natural curiosity and interest. Neither students nor their parents recognized the real-world applications of algebra. Certain teaching methodologies appeared to evoke positive dispositions.

When students reach the eighth grade, they often either make a decision or have a decision made for them regarding the study of mathematics—whether to maintain a program that includes a general mathematics "survey" course or to pursue the study of first-year algebra. Traditionally, students have enrolled in a first-year algebra course in the ninth grade. However, over the past ten years, there appears to be a growing interest in eighth graders studying first-year algebra. In the 1991-1992 school year, 13% of the eighth graders in the United States were enrolled in a first-year algebra course (Blank & Gruebel, 1993), and the percentage was similar in 1990 (Mullis, Dossey, Owen, & Phillips, 1991). According to Epstein and Mac Iver (1992), 67% of all public and 47% of Catholic schools in the U.S. reported offering a full year of algebra to eighth graders.

Being a teacher of eighth grade algebra, I became curious as to why they were taking the course and what their attitudes and dispositions were toward the study of mathematics in general. In the Fall of 1994, I conducted a study to address these issues. While several research questions were pursued in the current study, the primary focus of this paper is on the following two questions: (1) Why do the students choose to take an algebra course, and what are the students' attitudes and classroom behaviors in the study of mathematics? (2) To what degree, if any, do the student dispositions as defined by the NCTM differ from the student simply having a "positive attitude" toward mathematics (e.g., "I like the subject," or "I think that math is useful")?

Conceptual Framework and Definitions

When the National Council of Teachers of Mathematics released the *Curriculum and Evaluation Standards for School Mathematics* (1989), they proposed an Evaluation Standard which they referred to as "mathematical disposition." Disposition, they explained, has several components: (1) interest and curiosity, (2) perseverance, (3) confidence, (4) flexibility, and (5) valuing the application of mathematics. The NCTM stated that "disposition refers not simply to attitudes but to a tendency to think and to act in positive ways" (p. 233). Motivation literature typically focuses on emotions (interest and curiosity), as well as confidence and goals. Definitions and descriptions follow.

Interest: Interest, as described by Dewey (1913), refers to one's desire to pursue some object because the person recognizes that it will promote personal growth. Research on interest that was particularly relevant to the current study was conducted with ninth graders by Harter (1981). She concluded that as students progressed through the grades, they showed more of a preference for easy work instead of a challenge and worked more for teacher approval and grades than out of curiosity and interest.

Perseverance: Perseverance can be described as the willingness of an individual to remain on task until completion of a difficult problem or situation. Perseverance has been linked to the dispositions of interest (Hidi, 1990) and self-efficacy (Collins, cited in Bandura, 1993; Multon, Brown, & Lent, 1991).

Confidence (Self-Efficacy): Personal agency beliefs have perhaps been most fully investigated under Bandura's (1977) term "self-efficacy." Bandura (1986) defined self-efficacy as "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (p. 391).

Flexibility: Flexibility was defined by the NCTM (1989) as the student's tendency for "exploring mathematical ideas and trying alternative methods in solving problems" (p. 233). In the Standards document, they stated that classroom discussions can reveal information about student flexibility by the teacher reflecting on questions such as, "How willing are students to explain their point of view and defend that explanation?" (p. 234).

Research Methods

I conducted this research by using a sample of eighth grade students enrolled in first-year algebra in Catholic schools in a Midwest state. I chose this population, in part, because most of the students elected to take algebra instead of the traditional eighth grade mathematics course. The reader should not assume that any generalization of the data to other students in another Diocese, public school district, or state is necessarily implied.

Of the schools having eighth year classes in the Diocese studied, 45 had an algebra course available. The schools were located in diverse settings. Twelve of the classes were selected by using a stratified purposeful sample, as suggested by Patton (1990). These 12 classes served 19 schools, since some classes contained students from two or three different buildings. Participating schools were deliberately chosen to ensure a mix of demographics and a reasonable sample size. A majority of the algebra courses in the Diocese were taught in urban and suburban settings, so most of the participants were selected from this group. A total of 107 males and 96 females were included in a survey.

After issuing a 20-item, paper-and-pencil survey to these students, four of the schools were selected for additional study. They were purposefully selected to include large and small classes, male and female teachers with varied experience levels, and a demographic mix. The students from these four selected schools were observed over a six-week period, and then six students were purposefully selected from each class for a student interview and an interview with their parents. The cumulative files of each of these 24 students were also examined. Field data were collected in the classroom and in interviews. Collection of data from multiple sources allowed for triangulation of the data.

Results

Reasons for Taking Algebra in Eighth Grade

On the survey, respondents were asked why they took algebra as eighth graders, and this issue was pursued in interviews with students and their parents. The survey responses to the question were categorized and are summarized in Table 1. In the subsequent interviews, acceleration and preparation for high school were also cited as primary reasons for taking algebra. Overall, students had primarily extrinsic reasons for taking algebra. Particularly prevalent was the attitude of interviewed students that failure to take the course might allow other students to "get ahead" of them. This finding suggests that most of the students held what Ames & Archer (1988) referred to as an ego goal orientation, in which individuals are motivated by their desire to outperform their peers.

Table 1.
Rank-ordered survey responses on why eighth graders took algebra.

Reason	Responses Male (n = 107)	 Female (n = 96)	 Total (n = 203)
Acceleration of High School Mathematics	73	71	144
High School Preparation	19	19	38
Perceived Ability	6	13	19
Challenge	10	8	18
Like/Enjoy Mathematics	6	7	13
Coerced/Forced to Take the Course	7	3	10
Opportunity or Desire to Learn More	5	5	10

Note. Students frequently provided multiple reasons for taking algebra.

Dispositions

Some of the most interesting results of the study were in the area of self-efficacy of students. The respondents were asked to report past grades in mathematics and to project grades in algebra and beyond on the survey. The survey results are reported in Table 2.

Table 2.
Survey responses to 7th grade mathematics grades and predicted algebra grades.

	Responses		
Course	Male (n = 106)	Female (n = 96)	*t*
7th Grade (Reported)			
M	11.47 (A-/A)	11.65 (A/A-)	1.00
SD	1.34	1.19	
Algebra (Projected)			
M	10.43 (B+/A-)	9.83 (B/B+)	2.61*
SD	1.55	1.71	
Repeated Measures *t*	6.16*	8.48*	

Note. In computing the mean, an F = 1 point, D- = 2 points, D = 3 points, ... , and A+ = 13 points. Therefore, the higher the mean score, the higher the reported or projected grade in that course. * $p < .05$.

Overall, females in the survey reported slightly higher grades in seventh grade mathematics than their male peers, but they predicted significantly lower grades in algebra. Likewise, both males and females predicted that their algebra performances would be at least one letter grade lower than their previous mathematics grades. Successful past performances in mathematics did not generally result in a high level of self-efficacy to perform in algebra and high school mathematics courses in general. Phillips (1984) and Dweck (1986) reached similar conclusions in their research. In the interviews for the current study, students described their perceptions of algebra and high school mathematics as being "different" and more difficult, citing cases of friends and siblings who struggled with these advanced courses as influencing their attitudes.

Since most of the students took the algebra course by choice, they described the types of sacrifices that they were making to accommodate the course. Some students were taking two mathematics courses at once, while others were involved in athletics, activities, and part-time jobs that kept them up late at night to complete homework, which concerned their parents. However, classroom observations showed that students showed little signs of perseverance on day-to-day tasks. When confronted with a difficult task in class, they tended to become quickly frustrated and immediately sought assistance from the teacher.

Interviews with students about their classroom behaviors indicated that most eighth graders found it extremely important to please the teacher by following classroom rituals, such as properly correcting errors on papers and writing solutions the way that the teacher had modeled. They appeared much more interested in impressing the teachers and earning high grades than learning for the sake of learning.

Many classroom observations in this study were disappointing in terms of collecting data on dispositions because of the nature of the lessons being taught.

Since most of the lessons followed a traditional path of checking homework, "showing" new sample problems, and allowing students to begin their homework, there was little opportunity for students to demonstrate positive dispositions. A problem posed in the classroom needs to be rich enough to evoke curiosity or to make the students feel that it is "worth" pursuing. Though very infrequent in my observations, the classroom experiences that involved teamwork, calculators, and "real-life" problems appeared to evoke positive dispositions.

Finally, students and their parents saw little use or real-world applications for algebra. They felt that the course was important, but only as a prerequisite for other classes. Students felt that the individuals who most needed an algebra course were future algebra teachers. This is similar to the circular argument that the reason we study algebra is to prepare for geometry, which prepares us for more algebra and so forth. Misconceptions about the nature of algebra appeared to stem from teacher-directed classroom experiences that emphasized rote, mechanical symbol manipulation over problem solving.

Discussion

Prior to an algebra course, it appears that a student's dispositions are affected by experiences and perceived ability, as suggested by Schunk (1991). In addition, a model such as a peer or sibling may affect an initial disposition. After the class begins, however, feedback and modeling from the teacher, as well as peer models and parents begin to interact on the student. When faced with peers, siblings, parents, and even teachers who have poor dispositions in algebra, the student becomes part of a recurring cycle of negativism toward mathematics. The only way that a student can get out of this cycle and develop a positive disposition, therefore, is for the teacher to instruct in a way that would assist the student in appreciating that algebra is worth knowing in and of itself. This type of instruction, however, depends upon a strong curriculum and relevant curricular materials, and it assumes that the teacher understands the relevance of algebra and uses effective instructional techniques.

The NCTM (1991) established a teaching Standard that educators should "promote mathematical disposition." They stated that it is the teacher's role to model positive disposition. The assumption is that the classroom environment established by the teacher will affect student dispositions and general beliefs about mathematics, similar to Schunk's (1991) assertion that positive classroom models have a direct effect on student self-efficacy and persistence. When teachers model positive attitudes, they have the potential to create a learning environment that fosters inquisitiveness and curiosity.

Additional research is needed on the development of student dispositions toward mathematics. For example, theory-driven issues such as the relationship between self-efficacy and persistence of eighth graders could be pursued, as well as a comparison of goal orientations by gender. In addition, longitudinal work with students involved in the survey and interviews described in this study could shed light on how many of these students actually move on to take four more years of high school mathematics, as well as how and why their dispositions toward mathematics evolve over the next several years.

References

Ames, C. & Archer, J. (1988). Achievement goals in the classroom: Students' learning strategies and motivation processes. *Journal of Educational Psychology, 80*(3), 260-267.

Bandura, A. (1993). Perceived self-efficacy in cognitive development and functioning. *Educational Psychologist, 28*(2), 117-148.

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Englewood Cliffs, NJ: Prentice-Hall.

Bandura, A. (1977). Self-efficacy: Toward a unifying theory of behavioral change. *Psychological Review, 84*, 191-215.

Blank, R.K. & Gruebel, D. (1993). *State indicators of science and mathematics education 1993*. Washington, DC: Council of Chief State School Officers.

Dewey, J. (1913). *Interest and effort in education*. New York, NY: Houghton Mifflin.

Dweck, C.S. (1986). Motivational processes affecting learning. *American Psychologist, 41*(10), 1040-1048.

Epstein, J.L. & Mac Iver, D.J. (1992). *Opportunities to learn: Effects on eighth graders of curriculum offerings and instructional approaches* (Report No. 34). Baltimore, MD: The Johns Hopkins University.

Harter, S. (1981). A new self-report scale of intrinsic versus extrinsic orientation in the classroom: Motivational and informational components. *Developmental Psychology, 17*(3), 300-312.

Hidi, S. (1990). Interest and its contributions as a mental resource for learning. *Review of Educational Research, 6*, 549-571.

Mullis, I., Dossey, J., Owen, E., & Phillips, G. (1991). *The state of mathematics achievement*. (Report No: 21-ST-04). Washington, DC: U.S. Department of Education.

Multon, K.D., Brown, S.D., & Lent, R.W. (1991). Relation of self-efficacy beliefs to academic outcomes: A meta-analytic investigation. *Journal of Counseling Psychology, 38*, 30-38.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Patton, M.Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage Publications, Inc.

Phillips, D. (1984). The illusion of incompetence among academically competent children. *Child Development, 55*, 2000-2016.

Schunk, D.H. (1991). Self-efficacy and academic motivation. *Educational Psychologist, 26*, 207-231.

ALGEBRA FOR ALL IN A CULTURALLY DIVERSE SETTING: ATTITUDES OF THE PARENTS AND STUDENTS

Billie F. Risacher, San Jose State University

Insufficient mathematics preparation is widely cited as a factor contributing to under-representation of non-Asian minorities in scientific and engineering fields (*Everybody Counts*, 1989). This concern is reflected by recommendations that all students study a core curriculum of mathematics sufficient to enable all to pursue higher education (*NCTM Standards*, 1989), and by Equity 2000 sites requiring all high school students to study Algebra and Geometry. However, numerous studies indicate that beliefs and attitudes of students and parents are contributing factors to student achievement and career paths. This paper examines the beliefs and attitudes of the parents and students in an "Algebra for All" program aiming to increase student success by using an innovative curriculum and providing additional support services. The program is supported by a state grant and includes a local university, a community college, business partners, and the high school. The location is a large inner-city California high school with a diverse ethnic student population of which 60% are Hispanic and African American, the target students.

Methodology & Results

All students entering the program responded to a written survey on attitudes and beliefs about mathematics and family background. A written survey in the second year was distributed to all parents and students. Personal interviews and school records document other factors of interest, i.e., career goals, attendance, number of extra tutoring sessions attended, and parental involvement.

Results indicate that students feel that succeeding in mathematics is important; intend to go to college; and cite career goals of doctor, lawyer, engineer, and scientist. Few target students voluntarily use the after school tutoring service and their school attendance was very poor. Parents and students in the second year reported they were satisfied with the student's mathematics class, despite the high failure rate, which was worse for the target students. Parents indicate frequent interaction with their children about their mathematics work, but have little contact with the teacher or school personnel and have little knowledge about colleges or entrance requirements. Inconsistencies between goals, beliefs, effort and success appear to be exacerbated by language differences between the students and teachers and between school personnel and parents. It appears that the students and parents value mathematics but have a naive optimism about achieving in mathematics and career goals.

DEVELOPMENTAL STUDENTS' CONNECTIONS: INTERPERSONAL, MATHEMATICAL, AND PERSONAL

Sue Tinsley Mau, Indiana University Purdue University Indianapolis

Attribution theory suggests that students answer current questions based on their past experiences. The beliefs and attitudes literature suggests that students operate under a set of beliefs that motivates their current behavior. Within the mathematics classroom, constructivists suggest that students have reasons for their answers that make sense to them. Bruner (1990) suggests that people develop their own stories as agents that help reconcile outcomes with expectations. It is in the examination of individuals' stories that we can begin to understand concerns that strongly influence individuals' actions.

This study examined the stories of nine students enrolled in a no-credit mathematics course at Indiana University Bloomington (IUB) in an effort to understand the personal histories of students and how that influenced their current mathematical experience. *Women's Ways of Knowing: The Development of Self, Voice, and Mind* (Belenky, Clinchy, Goldberger, & Tarule, 1986) provided a framework for analysis. Key results indicated that students desire at least three types of connections: interpersonal, mathematical, and personal. Students desiring interpersonal connections cited the importance of developing a relationship with the teacher. Students voiced a very real need to have the mathematics they were learning somehow connected to previous topics from other classes. Finally, students wanted a personal connection; they wanted mathematics connected to their lives.

The paper concludes with an examination of current calls for reform and how pedagogical strategies may influence change.

References

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice, and mind.* New York: Basic Books.

Bruner, J. (1990). *Acts of Meaning.* Cambridge, MA: Harvard University Press.

ENHANCING STUDENT INTEREST IN STATISTICS

Susan Prien, University of San Francisco
Mathew Mitchell, University of San Francisco

This paper reports on the effects of an innovative statistics curriculum on student attitudes towards mathematics. The purpose of the study was twofold. First, it represented an attempt to test the plausibility of systematically enhancing student interest within the mathematics classroom. Second, it sought to better explain the relationship between personal interest in mathematics, situational interest in the classroom, mathematical anxiety, and student achievement within the context of an experimental statistics curricula.

The essential problem necessitating this study is poor student motivation to learn in the mathematics classroom. Perhaps the most direct variable for explaining student motivation may be the construct of "interest" (Hidi, 1990). One of the fundamental distinctions made in interest research is the difference between personal and situational interest. *Personal interest* refers to an interest that people *bring to* a situation. It is considered fairly stable and difficult to influence in the short-term. *Situational interest* refers to an interest that people have when *participating in* a situation. It can vary often and dramatically in a short period of time. A mathematics classroom that is consistently high in situational interest may significantly alter an individual's personal interest towards mathematics.

The authors investigated the possibility of enhancing situational interest by systematically designing instruction that would have a high level of student involvement and meaningfulness (Mitchell, 1993). The statistics curriculum was primarily designed around the use of Microsoft EXCEL®. For each key statistical concept, students created "teaching sheets," a worksheet that would teach a "novice" about the data analysis technique under study. Students needed to include three key ingredients in their teaching sheets: (1 a *storyline* which explained the purpose and importance of the statistical concept, (2) a well organized *number playground* where the user could try out various combinations of data, and (3) a *visual representation* (or graph) of the statistical concept.

The survey results and student products from this curriculum were used to understand how, and how well, the curriculum enhanced student interest in mathematics. This report will present our initial analyses of the data collected.

References

Hidi, S. (1990). Interest and its contribution as a mental resource for learning. *Review of Educational Research, 60*, 549-571.

Mitchell, M. T. (1993). Situational interest: Its multi-faceted structure in the secondary school mathematics classroom. *Journal of Educational Psychology, 85*(2), 424-436.

STUDENTS' CONCEPTIONS OF MATHEMATICS, LEARNING, KNOWING, AND SELF

Lionel N. LaCroix, John Fraser Secondary School

This study examined conceptions relating to the learning of mathematics held by two typical students in an in situ Algebra 12 class. Their learning of logarithms served as the context for this inquiry.

The students' conceptions were integrated, containing both similarities and differences. The following views were similarities: mathematics was a set of truths, handed down to them by their teachers, in the form of rules and procedures to get answers for the questions encountered in class; views about knowing and learning in mathematics, i.e. the memorization of unrelated facts, were undifferentiated from those in other subjects at school; it was the teacher's job to make learning easy for students by presenting new material slowly and in a step-by-step manner; students played a passive role in the learning process; and students "understood" mathematics when they got the right answers for the questions encountered in class. The students also lacked confidence in themselves in mathematics and believed that their learning and success depended upon factors which were largely beyond their control.

The following conclusions were drawn from this analysis:

- students' views on the nature of mathematics and on learning and knowing mathematics shape the learning objectives that they set for themselves;
- subtle but profound failures in communication can occur between teachers and students in mathematics;
- students' views about mathematics, learning and knowing limit or enhance their participation within the classroom, thereby affecting their success;
- unresolved conflicts in students' conceptions may be a detriment to their learning; and
- students may hold views about knowing and learning, constructed outside of the mathematics classroom, which they apply inappropriately within it.

These results indicate that educators must address the full range of conceptions which students construct and apply within the classroom to affect significant improvement in students' learning of mathematics.

A STUDENT'S PERSPECTIVE ON TWO MATHEMATICS CLASSROOMS: PROBLEM CENTERED VS. LECTURE

Sandra Davis Trowell, The University of Alabama

The focus of this study is one student in a college mathematics problem solving course in which a problem centered learning environment (Wheatley, 1991) was established. By examining from an interactionist and constructivist perspective (Blumer, 1969; Bauersfeld, 1988; von Glasersfeld, 1989) this mathematics classroom in which making sense of mathematics was expected and encouraged and listening to students in such classrooms, we can begin to establish frameworks for creating such environments.

Joy, a student who was interviewed throughout this problem solving course, describes her experiences in this course. Throughout her interviews, Joy proved to be extremely reflective as she discussed her problem solving class as well as her experiences in other mathematics classes. She described problem solving as a personal sense making activity and valued her classmates mathematical ideas. Joy frequently discussed the differences between her problem solving class and her other mathematics class. She described her other mathematics class as teacher centered and intimidating and one in which the teacher always remained in "control." Joy was not judgmental as she painted contrasting pictures of her mathematics classes.

Joy was able to function autonomously in this mathematics classroom. She made powerful mathematical constructions and became a part of an intellectual community. Given Joy's experiences, we can assert in a problem centered mathematics classroom, students have more potential to function autonomously while engaging in rich mathematical activities.

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In D. A. Grouws and T. J. Cooney (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics.

Blumer, H. (1969). *Symbolic interactionism: Perspective and method*. Englewoods Cliffs, NJ: Prentice-Hall, Inc.

Von Glasersfeld, E. (1989). *Radical constructivism in mathematics education*. Boston: Kluwer Academic Publishers.

Wheatley, G. H. (1991). Constructivist Perspectives on Science and Mathematics Learning. *Science Education*, 75(1), 9-21.

ASSESSING STUDENTS' CONNECTIONS BETWEEN MATHEMATICS AND OTHER DISCIPLINES

Angeles Domínguez, Syracuse University

This study examines students' connections between mathematics and other disciplines. Further, it analyzes how students' attitudes toward mathematics and their beliefs about this subject are related to those connections.

Recent research on mathematical connections to other disciplines focuses on curriculum activities for students and teaching changes to promote the integration of mathematics and other areas (Roth, 1993; Sambs, 1991). However, so far I have not found research that relates students' mathematical connections to their affective domain toward the subject.

This study used (a) an ethnomathematics approach as defined by Pompeu (1994) to analyze students' connections, and (b) McLeod's (1992) affect framework as the basis for the analysis of students' beliefs and attitudes toward mathematics. This study attempts to relate the affective factors to students' mathematical connection to other subjects.

Fifteen undergraduates and two graduate students, all of whom were enrolled in a mathematics course participated in this study. They were all nonmathematics majors. Data collection consisted of a sequence of two interviews. The data collected from the first interview was reviewed and served as a basis to look through students' textbooks. Specific situations from the textbooks–including graphs, functions, tables, content of a paragraph, and pictures–were selected and presented to the students in the second interviews.

The analysis focused on (a) students' disposition toward mathematics, (b) students' conception of mathematics and the kind of mathematical ideas they relate to their disciplines, and (c) how students visualize the need and use of mathematics in their majors. Preliminary data analysis suggests that students' perception of mathematics is limited to its computational aspect, reducing the visualization of mathematical ideas in other subjects. Also, students' attitude influences their recognition of mathematical ideas in other areas.

References

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575-596). New York: Macmillan Publishing Company.

Pompeu, G. (1994). Another definition of ethnomathematics. *Newsletter of the International Study Group on Ethnomathematics*, 9(2).

Roth, W. M. (1993). Problem-centered learning for the integration of mathematics, and science in a constructivist laboratory: A case study. *School Science and Mathematics*, *93*, 113-122.

Sambs, H. (1991, April). *Problem solving in physics and mathematics–What do they have in common?* Paper presented in the Annual Meeting of the National Association for Research in Science Teaching, Lake Geneva, WI.

COLLEGE STUDENTS' EARLY IDEAS ABOUT MATHEMATICAL PROOF

William E. Geeslin, University of New Hampshire

Learning to construct a mathematical proof is an important goal for students majoring in mathematics, mathematics education, and related disciplines. The University of New Hampshire attempts to achieve this goal in part by enrolling students in a sophomore level course aimed at teaching students how to read, comprehend and write proofs appropriate for that level. With the exception of the statement/reason type proof in the typical high school geometry course, students at this level have had little or no experience with proof. Mathematicians often state that the particular mathematical topics in this course can vary widely as long as students practice writing proofs. Proof is seen as a critical skill that can be applied equally across mathematical topics given the appropriate prerequisite content knowledge (axioms, postulates, definitions, etc.) in that area. Students on the other hand view proof as a difficult and perhaps useless activity not connected directly to understanding mathematics. Selden and Selden(in press) provide a technical analysis of students proofs and conclude that students understand even less than expected .

Student writings concerning mathematics often provide insight into student conceptual development that is not revealed in achievement tests (Geeslin, 1977). Twenty-five students enrolled in a sophomore course on proof were asked to provide written answers concerning what they thought mathematics was, what they thought mathematical proof was, and related questions at the beginning of, during, and at the end of the semester. These written responses were not used as part of the course grade. Traditional tests and homework were used to determine students' grades. Summary teacher evaluation data was available as well. Following the course, four graduate students in mathematics education were asked to view selected responses and identify which students received a high, average or low grade in the course. Qualitative differences in students' responses related to achievement do exist, but are difficult to describe quantitatively.

References

Geeslin, W. (1977). Using Writing About Mathematics as a Teaching Technique, *Mathematics Teacher*, *70*, 112-115.

Selden, J. & Selden, A. (in press). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*.

COMPUTATIONAL PREFERENCES: FIFTH & SIXTH GRADE STUDENTS

Mary Ellen Schmidt, The Ohio State University

Ninety-nine fifth and sixth grade students from a suburban school district in North Central Ohio participated in a study to assess how they selected a method of computation — using a calculator, paper and pencil, or mental math. After completing a questionnaire which probed their beliefs about calculators and their choice of computational method on various types of problems, students were interviewed to gain additional information about their thinking. The results suggested that 1.) students hold positive beliefs about calculators and 2.) use primitive visual level reasoning when selecting a computational method.

In 1989, NCTM's *Standards* recommended a refocus in K-8 mathematics away from the paper and pencil computation which had dominated the mathematics curriculum in the early grades. Consequently, students' computational experiences needed to focus less on practicing algorithms and focus more on mental mathematics, calculators, and other related skills such as estimating and judging the reasonableness of answers. This paper will present the results of a study designed to assess students' computational preferences.

Materials and Research Methods. Fifth grade (n=49) and sixth grade (n=50) students from a suburban school in North Central Ohio completed a three part questionnaire and were interviewed. The assessment instrument was developed using questions from research (Reys, Reys, & Hope, 1993), questions adapted from other research studies, and content questions pertinent to what was taught in the students' classroom. Students were asked to (a) provide demographic data, (b) respond to open ended questions concerning their beliefs about calculators, (c) respond to questions (both equation and word problem formats) by selecting their computational preference and, (d) write a word problem that they would use a calculator to solve. The interview further probed selected student responses.

Responses to open ended questions were categorized. Data from Likert type questions were entered into a statistics program and analyzed.

Results. Students selected paper and pencil calculations based on the algorithm they worked on most recently. Little attention was given to mental math computation. Calculators were selected based on the number of digits in the numbers to be operated on.

Conclusion. While students tended to have positive beliefs about calculators, their selection of computational method was often at a more primitive visual level. The results suggest that more attention needs to be given to selecting computational methods in the elementary and middle grade course-of-study.

Reference

Reys, B.J., Reys, R.E. & Hope, J.A. (1993). Mental computation: A snapshot of second, fifth, and seventh grade student performance. *School Science and Mathematics*, *93*(6), 306-315.

LEARNED HELPLESSNESS: AN EXPLANATION FOR AFRICAN AMERICAN ACHIEVEMENT IN MATHEMATICS?

Vincent T. Snipes, Florida State University
Bradford F. Lewis, Florida State University

In this research study we investigated the mathematics achievement of five African American female college students and its relationship to attribution theories. After reviewing reports on the mathematics achievement of African Americans in mathematics, learned helplessness became the area of focus for further investigation.

The purpose of the study was to determine whether or not African American female college students exhibited symptoms of learned helplessness in mathematical achievement. According to their "like" or "dislike" of mathematics, five participants were chosen for their proximity and availability to the researchers. Only females were chosen, so we would not have to account for gender differences in our study. Each student was interviewed for approximately one hour. All interviews were videotaped by the researchers. After data analysis, follow up questions were asked for clarification if necessary. The data from the interviews were analyzed from the framework of attribution theory.

Patterns that emerged from the data indicate that students who did not like mathematics, who were not motivated to study mathematics, or who actively avoided mathematics all had negative experiences in mathematics. Additionally, students' motivation was increased by the presence of incentives. According to our data, learned helplessness is not a viable theory for explaining why African Americans avoid mathematics related careers.

STUDENT PERCEPTION OF THE AUTHORITY OF THE COMPUTER/ CALCULATOR IN THE CURVE FITTING OF DATA

Douglas A. Lapp, Morehead State University

The study examined the perception of students with regard to the authority of technology in the practice of data analysis through curve fitting. The psychological basis for the study was cognitive conflict theory. The study was qualitative in nature and sought to describe the students' reaction to the nature of technological authority. Conflict was introduced through multiple representations and student reactions were observed as they sought to make sense of the conflicting and unexpected results.

Five students involved in a beginning statistics course at a liberal arts university during Autumn Quarter 1994 were selected from a class of 33 students. Ten students volunteered to participate and selection was based on observations by the researcher and consultation with the course professor to obtain a varied range of achievement for the 5 students based on grades from the first week and a half of the course.

Data were collected through observations in both the classroom and computer laboratory. In addition to student observations, the students were interviewed four times with the exception of one student who was interviewed three times.

The interviews included both philosophical and task-oriented questions. During the final interview, the researcher programmed the calculator to shift the regression line for each of the four task-oriented questions by increasing factors as the students worked through the problems. The purpose for the regression line shifting was to see when the students would first doubt the solutions being given by the machine. The interviews were audiotaped and videotaped, and then transcribed and analyzed for content using a coding scheme.

Several factors with respect to the authority of the machine were discovered. (1) The calculator's authority seemed to be heightened when the students had repeatedly solved similar problems using the technology. (2) The ability of students to match answers given by the machine in several different representational forms lent a greater credibility to the calculator. (3) A connection was present between the authority of the instructor and the authority of the calculator. (4) A higher mathematics background played a role in the increased willingness to challenge the authority of the machine. (5) Students assessed their perception of the authority of the machine based on their perception of the authority of the people who created the technology.

Teacher Beliefs and Attitudes

ACHIEVING COGNITIVE EQUITY IN THE MATHEMATICS CLASSROOM

Marcia L. Tharp, Old Dominion University
Chris Lovell, Old Dominion University

A preliminary investigation of patterns of teacher thought about student reasoning and learning involved presenting 23 preservice teachers with a "dilemma of practice about equity." Resulting stages of teacher development concerning equity in reasoning were found to correlate significantly with Perry's scheme ($r=.39$, $p<.10$). However for those who hold the view that mathematics learning is mostly rule-based no significant correlation was found with these stages of teacher thought about reasoning and equity. Yet, holding a rule-based viewpoint of mathematics learning was found to correlate significantly in the expected direction with Perry's scheme ($r=-.43$, $p<.05$).

In an effort to achieve equity, the Professional Standards For Teaching Mathematics (NCTM, 1991) call for mathematics teachers to develop an extensive multicultural knowledge base and especially to know how students' linguistic, ethnic, racial, gender, and socioeconomic backgrounds influence learning of mathematics. As a result, researchers have begun to examine how teachers' conceptions of equity influence pedagogical practice. Most school-based conceptions of equity (Secada, 1994) focus on how teachers work with differences among students such as gender, race, ethnicity, and class. In the everyday classroom, these differences recurrently manifest themselves as differences in reasoning, and preservice teachers are enjoined to attend to such differences by honoring each child's reasoning process through careful probing and non-negative critical questioning (NCTM, 1991). What is missing in this general research line on cognition in the classroom, however, is a description of differences among the teachers, *themselves*, in their beliefs about learning and equity and how such beliefs might be, in turn, related to the cognitive developmental levels of the teachers.

Since it has been recognized that how teachers view reasoning and mathematics is a key determinant of how they teach mathematics (Simon and Schifter, 1991), it is imperative that educators examine how equity may be engaged in the classroom by teachers who hold varying conceptions of reasoning and learning mathematics. Thus the question arises: do differences among teachers on the dimension of cognitive development relate to how they reason about student differences in reasoning? When teachers invoke "equity" as a basis for instructional moves, does such an equity position relate to cognitive stage? The purpose of this research was to answer these questions and to investigate more generally teacher thinking about instruction, mathematics learning, and equity.

Method

A preliminary investigation of patterns of teacher thought about student reasoning and learning involved presenting 23 graduate-level, preservice teachers with a "dilemma of practice about equity" (Table 1), a dilemma which was selected to elicit strategic instructional thinking along with reasoning about possible ethical action. Respondents were asked to give their reasoning on the dilemma in the form of a short essay protocol. The resulting projective protocols were then subjected to a process of content analysis (Miles and Huberman, 1994) where certain qualitative themes, or "structures of knowing" relating to adult development were identified. Theories of increased sophistication in perspective taking (Selman, 1980; Commons, & Richards, 1984), of moral and ethical development (Kohlberg, 1984; Gilligan, Ward, Taylor, & Bardige, 1988), and of intellectual development (Perry, 1970; Belenky, Clinchy, Goldberger, & Tarule, 1986) guided the classification of the 23 master's level preservice teachers into stages of cognitive development.

Table 1. *A Dilemma Of Equity In Practice*

The following is a description of a challenge encountered in the life of a teacher. Please take time to give your reasoning on this incident.

> A white female elementary school teacher in the United States posed a math problem to her class one day. "Suppose there are four blackbirds sitting in a tree. You take a sling shot and shoot one of them. How many are left?" A white student answered quickly, "That's easy. One subtracted from four is three." An African immigrant youth then answered with equal confidence, "Zero." The teacher chuckled at the latter response and stated that the first student was right and that, perhaps, the second student should study more math. From that day forth, the African student seemed to withdraw from class activities and seldom spoke to other students or the teacher.

What are your thoughts on this matter?

Results

Four patterns of response were discerned, each corresponding to a theoretical stage of teacher development (Stage 1, a hypothesized level "silent knowing," was not evident in the sample.):

Stage 2 (authority centered/self-protective). The preservice teacher-respondent gives or implies "higher authority" as a motive for probing student reasoning. Equity issues are either omitted or couched in reactive terms. Example: "The teacher should have asked the African youth what her reasoning was. We have been learning how important it is for teachers to do this." Further, the respondent, projecting herself or himself into the dilemma, often stresses themes of defensiveness. Equity action is sometimes seen as a punitive move. Example: "The negative response to the child's answer was insensitive to say the least. She should have asked him why he gave that answer. He, I'm sure, had his own logical reasons for his conclusion, but due to her prejudice he was not able to explain. She should be relieved of her job."

Stage 3 (mutualism). Equity action is viewed as an extensive, distributive process; undifferentiated equity ("fairness") demands that all students' reasoning be valued equally. Example: "The teacher seems to consider there is only one answer to the question. She/he didn't think there might be more than one answer to the question. From my point of view, the second child is equally correct. Perhaps that is not relevant to what the teacher had in mind, but nonetheless, the second child should not be put down for his line of thinking."

Stage 4 (autonomy/proceduralism). Stress is given to the teacher's autonomous ability to select and teach procedures for knowing and valuing (mathematical operations, ethical guidelines, and so forth). Equity is differentiated along lines more complex than simple rule-based heuristics; comparison of reasoning methods is emphasized. Equity is viewed as instrumental to instructional ends. Example: "What a *great* answer from that child! Of course it would be zero because all the birds would have flown away! His answer was based on real-world experience. The other, 'one subtracted from four' was textbook in nature. I would show the class both points of view."

Stage 5 (contextual relativism/constructivism). The teacher at this stage sees that instruction is a complex process, full of contingencies and resonating with many voices constructing knowledge together. Opportunities to capitalize on "the found curriculum," the "teachable moment," and classroom co-constructive possibilities are emphasized. Likewise, equity decisions are viewed as complex, principled, and interactive processes. An example: "The African was talking about his real life observation of birds. The other was answering a math question, thinking of a mathematical operation. The teacher's thinking was narrow in scope. She should have *not* done any comparing of students' answers. She could have had success with including the African by simply asking, 'Why do you say that is the answer?' Then the African, still confident, would have enlightened her and the rest of the class."

Two respondents showed evidence of stage five (9%) thinking in which classroom co-construction and non-judgmental comparison of reasoning are emphasized. The remaining 91% were categorized in lower stages of autonomy/proceduralism (22%), mutualism (39%), and authority centered/self protection (30%). This distribution of respondents accords with research investigations where cognitive developmental schemes were applied to higher education (Pascarella & Terenzini, 1991). As a check on this classification, a quantitative measure according to the Perry Scheme (Moore, 1989) was administered to the respondents, revealing a moderate correlation to a general index of cognitive development ($r = .39$, $p < .10$). As a further check on the validity of the classification, correlation was sought with a measure of teacher perspective on mathematics learning, the View of Learning Math as a Rule-Based Subject (VLMRBS). This instrument (Tharp, 1992), assesses the degree to which a teacher adheres to the view that mathematics learning is mostly oriented toward processes which involve the manipulation of symbols and memorization of facts as opposed to the view that learning mathematics is based on reasoning about relationships and patterns. Since the differences measured by the VLRMBS strongly suggest a cognitive developmental sequence, the researchers turned to this "rule-based" measure. Correlation with the hypothesized teacher stages did not reach significance, although the VLRMBS did itself correlate significantly in the expected direction with the general index on the Perry measure ($r = -.43$, $p < .05$). Correlations between the "rule-based" measure (VLMRBS) and the four Perry levels again confirmed the inverse relationship between teacher view of mathematics learning as a rule-based process and teacher level of cognitive development (Table 2).

Table 2

Pearson Correlations Between Preservice Teacher View of Mathematics as a Rule-Based Subject (VLRMBS) and Level of Cognitive Development According to the Perry Scheme (N=23)

	Correlation to VLRMBS	
Perry Level	*r*	*p*
Position 2	.30	.17
Position 3	.37	.09*
Position 4	-.46	.03**
Position 5	-.19	.40

Bartlett _^{2} = 141.973, p < .01, df = 10

*Significant at the .10 level or better. **Significant at the .05 level or better.

Conclusions

According to Secada (1991, p. 49) an equitable mathematics education would include, "real contexts that reflect the lived realities of people who are members of equity groups." While Secada (1991) has argued that all children should see themselves as part of a mathematics curriculum regardless of their background, these results show that preservice teachers are not always ready to provide instruction that fully honors that background. Only nine percent of the preservice teachers at stage five hold a radically different view from those at lower stages that allows for the practice of "active equity", where individual reasoning is sought out to magnify the growth in understanding of an entire group. Since this sample is representative of other research on cognitive development in higher education (Pascarella & Terenzini, 1991), one may expect similar results would be reflected in classroom practice. Though researchers have begun to recognize learning as a process of "shared knowings" that involves an entire mathematics community (Simmon, 1995), the present study calls into question the easy assumption that most teachers are cognitively disposed to facilitate mathematics community learning. Because NCTM reforms call for teachers to succeed in reaching all learners, the researchers conclude teachers will need to be assisted in moving to a stage of cognitive development that allows them to recognize the value of fully honoring diverse perspectives in the classroom as a tool for learning.

While this stage model can serve as a way of conceptualizing how teachers view their role in regard to reasoning about honoring the expression of student thinking and equity, interplay with specific student and teacher beliefs about mathematics learning that may run counter to constructing "shared knowings" cannot be ignored. Approximately 50 percent of students hold a view that learning mathematics is rule-based, i.e. mostly process-oriented and memorization (Kouba et. al., 1988). Thompson (1992) has suggested that some communication is effected between the beliefs of students and those of teachers. Given that preservice teachers' view of learning mathematics as "rule-based" is correlated positively with Perry stages 2 and 3 and negatively with higher stages, connections to this viewpoint should be investigated further and considered by educators who desire to move teachers to a stage where "active equity" is practiced and "shared community knowing" is sought.

References

Belenky, M.F., Clinchy, B.N., Goldberger, N.R., & Tarule, J.N. (1986). *Women's ways of knowing: The development of self, voice, and mind.* New York: Basic Books.

Commons, M. L., & Richards, F. A. (1984). A general model of stage theory. Applying the general stage model. In M. L. Commons, F.A. Richard, & C. Armons (Eds.). *Beyond formal operations: Late adolescent and adult cognitive development* (pp. 120-157).

Gilligan, Ward, Taylor, & Bardige (1988). *Mapping the moral domain.* Cambridge: Harvard University Press.

Kouba, V., Brown, C., Carpenter, T., Lindquist, M., Silver, E., and Swafford, J., (1988). Results of the fourth NEAP assessment of mathematics: Measurement, geometry, data interpretation, attitudes, and other topics. *Arithmetic teacher*, 35, 6-16.

Kohlberg, (1984). *The psychology of moral development: The nature and validity of moral stages.* San Francisco: Harper and Row.

Miles, M.B.& Huberman, A.M. (1994). *Qualitative data analysis.* Thousand Oaks, CA: Sage.

Moore, (1989). The learning environment preferences: *Establishing the preliminary reliability and validity for an objective measure of the Perry scheme of intellectual and ethical development.* Unpublished doctoral dissertation, University of Maryland, College Park.

National Council of Teachers of Mathematics. (1991). *Professional Standards For Teaching Mathematics.* Reston, VA.

Pascarella, E. T. & Terenzini, P. T. (1991). *How college affects students.* San Francisco: Jossey-Bass.

Perry, W. G. (1970). *Forms of intellectual and ethical development in the college years.* New York: Holt, Rinehart, and Winston.

Secada W. G. (1991). Agenda Setting, Enlightened Self-Interest, And Equity In Mathematics Education. *Peabody Journal Of Education.* Winter, 22-56.

Secada, W.G. (1994). Equity In Restructured Schools. *NCRMSE Research Review The Teaching And Learning of Mathematics* 3(3), 11-13.

Selman, R. (1980). *The growth of interpersonal understanding: Development and clinical analysis.* New York: Academic Press.

Simon, N.A. and Schifter, D. (1991). Towards A Constructivist Perspective: An Intervention Study Of Mathematics Teacher Development. *Educational Studies In Mathematics*, 22(4), 309-331.

Simmon, M. (1995). Restructuring mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.

Tharp, M. (1992). A Problem-Solving Inquiry-Oriented Approach To Learning Mathematics-Student/Teacher Interactions: A Case Study, paper presented at AERA Annual Meeting, San Francisco, CA. (Eric Document Reproduction Service No. 355 090).

Thompson, A. (1992). (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning.* New York: Macmillan.

EVOLUTION OF A MATHEMATICAL PHILOSOPHY: THE STORY OF ONE SECONDARY MATHEMATICS PRESERVICE TEACHER

Patrick J. Eggleton, Berry College

As mathematics teacher education promotes the mathematics and pedagogical practices of the *Standards*, most mathematics preservice teachers confront philosophies of mathematics significantly different from their personal mathematical philosophy. The espoused constructivist/quasi-empiricist mathematical philosophy promoted by the NCTM *Standards* as opposed to the traditional absolutist philosophy of mathematics as a set of rules and facts characterizes the conflict in philosophies of mathematics. This report shares the findings from an in-depth case study of a preservice secondary mathematics teacher, Ken, by following the subtle evolution of his philosophy of mathematics, as characterized by Ernest's "Mathematics-Related Belief Systems," and the experiences influential to the philosophical evolution over a year of his preservice mathematics teacher education.

Changes in the teaching of mathematics such as those suggested by the NCTM *Standards* will be slow in coming and difficult to achieve because of the basic beliefs teachers hold about the nature of mathematics (Cooney, 1987). Studies in both mathematics education and science education argue that the issue of such change goes further than the beliefs of the teachers, but to the core of those beliefs, the teacher's philosophy (Ernest, 1991a; Schmittau, 1991). The conflict in philosophies of mathematics is characterized by the espoused constructivist/quasi-empiricist mathematical philosophy promoted by the NCTM *Standards* as opposed to the traditional absolutist philosophy of mathematics as a set of rules and facts. In an effort to improve the teaching of secondary school mathematics and secondary mathematics teacher education, the RADIATE[1] research program has focused on the view of mathematics emphasized in the NCTM *Standards* in its experimental secondary mathematics education courses. This research report shares the results of an in-depth case study of Ken, one of the participants of this program, by following the subtle evolution of his philosophy of mathematics, as characterized by Ernest's (1991a) "Mathematics-Related Belief Systems." The documented influences on the adaptations within Ken's philosophy of mathematics provide insight into the psychological aspects of educating teachers.

Methodology

Ken's case study is part of a longitudinal study of preservice teachers that began in April 1994 and will continue into his first few years of teaching. In April 1994 Ken enrolled in the first of a four quarter sequence of secondary mathematics education classes being conducted by the RADIATE staff. A survey and follow up interview during the first two weeks of class elicited initial documentation of Ken's view of mathematics, teaching learning, and becoming a teacher. He then participated in a curriculum which integrated mathematics content and pedagogy, emphasizing reflection of current and past experiences. The present analysis considers the first year of Ken's preservice experience that includes two mathematics methods courses and his student teaching. Field notes record 44 of the 57 class experiences, one full day of his student teaching, and one full week of his teaching a geometry class during his student teaching. Ken participated in eight guided interviews at regular three to four week intervals and 16 weekly informal interviews developed from the ongoing data. Ken also wrote 25 journal entries to add to the artifacts of tests, reports, and other written class work. Ernest's (1991a) five belief clusters concerning mathematics guided the categorization of statements and dialogue from the data. In the clustered data, interpreted themes and changes within a theme characterized Ken's evolving mathematical philosophy. In addition, literature on teacher change and mathematics teacher education informed the analysis of experiences related to evolution in Ken's mathematical philosophy (e.g., Ball, 1990; Cooney, 1994).

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[1] RADIATE (Research and Development Initiatives Applied to Teacher Education) was directed by Dr. Thomas J. Cooney and Dr. Patricia S. Wilson and funded by the National Science Foundation (#DUE 9254475) and the Georgia Research Alliance. Any opinions or conclusions expressed by this report are those of the author and do not necessarily reflect the views of the funding agencies.

Ken's Philosophy of Mathematics

Upon entering the mathematics education program, Ken's view of mathematics as a set of rules and guidelines that he often referred to as "number crunching" grounded his other mathematics-related beliefs. "Applications, that's for someone else. That's for the engineer, that's for the physicist, and the chemist and things like that.....But me personally, I just like number crunching. Bottom line" (1st Interview, 4/7/94). Ken's view of mathematics as "number crunching," of teaching mathematics as "telling" and "lecture," of learning mathematics as "practice, discipline, and memorization," and of assessing mathematics as "objective measures of skill proficiency" reflected what Ken valued from his history of mathematical experiences. Looking for a common thread in Ken's different belief clusters, I interpreted Ken's philosophy of mathematics as "Dualistic Absolutist" (Ernest, 1991b). Ken considered mathematics as certain, made up of absolute truths, and he saw the role of the teacher as an authority on the mathematics for his or her students. Ken's beliefs about mathematics were unquestioned, held without reason or evidence. "Math is what it is." Ken justified his beliefs regarding teaching mathematics, learning mathematics, and assessing mathematics based on the evidence of his experiences in learning mathematics.

Ken's initial philosophy of mathematics is not uncommon among preservice teachers. Like many others, Ken was highly successful with the traditional mathematics he encountered in high school and college. This success with a traditional and dominant absolutist philosophy of mathematics made it difficult for Ken to consider an alternative approach to the mathematics. For instance, when considering approaching basic operations with fractions by using a representation with pattern blocks, Ken commented, "Learning math 'not so good', although I can't say it's bad because this is how I learned it, and I thought it went pretty good, but learning 'not so good' is to say 'Well if you have two-thirds times one-fourth, what's the answer?'" (Interview, 10/14/94) As illustrated in this excerpt, even when Ken tried to be critical of his algorithmic learning of mathematics he could not discredit the success he enjoyed from learning mathematics from that perspective.

During the 1994 spring quarter, the methods course provided mathematical experiences in a unit on functions that allowed students to explore functions and abstract from the activities patterns or regularities regarding the content as well as the methods. Ken initially responded negatively to the experiences, often rejecting activities shared in the class as unrealistic. Along with the class activities, Ken participated in a high school classroom one hour a week. By the end of the quarter Ken acknowledged a difference between "merely memorizing formulas and definitions and learning algebraic manipulations" and "a quality, in-depth understanding of math concepts" (Report, 5/12/94). An assignment related to reading a section of the NCTM (1989) *Curriculum and Evaluation Standards for School Mathematics* at the end of the quarter provided Ken an opportunity to reflect over his experiences in relation to a culturally accepted alternative. This assignment provided a stimulus for emerging changes in several of Ken's mathematics-related beliefs.

Classes in the fall quarter included four areas of focus: mathematics and culture, geometric constructions, assessment, and transformational geometry. Fall quarter classes also presented a process view of mathematics while attending to more specific pedagogical issues such as equity, group work, assessment, and lesson preparation. Ken worked in a family group with three other preservice teachers to develop lessons and teach on three occasions in the local high school. Of all of the fall quarter experiences, Ken most strongly responded to the assessment unit that emphasized creating problems that tested a deeper understanding of the content.

Ken entered his winter quarter of student teaching with a view of a "deeper understanding" in mathematics that focused on problem solving, reasoning, and meaning to append to his original "Theory of Mathematics." Likewise Ken appended other key components of his "Mathematics-Related Belief System" (Ernest, 1991a) so that (1) learning mathematics now included self-discovery and connecting knowledge, (2) teaching mathematics required investigations for self-discovery in addition to lecture and telling, and (3) assessing mathematics added a need to explain meaning and became a regular part of instruction. Tymoczko's (1986) description of a quasi-empiricist philosophy of mathematics related well to Ken's evolved philosophy of mathematics. Ken's "Theory of Mathematics" reflected a strongly held belief in the absolute existence of an objective mathematics, while his evolved view of a "deeper understanding" of mathematics, which connected to his evolved theories of teaching, learning, and assessing mathematics, focused on the meaning of mathematics developed by mathematical processes of reasoning and problem solving. Without denying a reality in mathematics, Ken began to value the processes of mathematics such as the reasoning and problem solving.

Experiences Related to Evolution in Ken's Philosophy of Mathematics

Green (1971) suggested that the difference between instruction and indoctrination was the provision of opportunities for the individual to radically examine his/her belief systems. This idea of self-examination or reflection became a consistent theme in the experiences related to evolution in Ken's philosophy of mathematics. It was important for Ken to pause in his current experiences to reflect back on both his immediate and past experiences. In so doing, he communicated his examination of his beliefs and the influence his experiences had on those beliefs. Looking at those experiences that were most influential to an evolution in Ken's philosophy of mathematics as communicated in his reflections, two issues arose: the context of the reflection and the effect of the reflection.

Cooney (1994) suggested five contexts that should be a part of mathematics teacher education programs. Of those contexts, four described influential experiences to Ken's evolved mathematical philosophy. First, the influence of Ken's involvement in the mathematical investigations in the spring quarter became a stimulus for Ken's view of a "deeper understanding" of mathematics. These experiences not only provided Ken with a context for learning mathematics from a constructivist perspective, but it also provided him with an occasion to reflect on his experience as a learner of mathematics, the second important context. Although Ken began to consider the importance of a "deeper understanding" of mathematics, it is important to realize that Ken viewed the "deeper understanding" as an addition to the essential rules and procedures of mathematics. Ken's response to learning mathematics from a constructivist perspective is in contrast to what Ball (1990) described in her work with elementary preservice teachers. Ball's preservice teachers often approached mathematics with anxiety and feelings of incompetency. Their experiences with mathematics in the methods class provoked some to reinterpret their past mathematical experiences, gaining new lenses, new assumptions, and new ideas to pursue in mathematics. Ken's self-perceived confidence with mathematics, even calling himself a "math god," inhibited him from considering a reinterpretation of his understanding of mathematics, insisting rather on viewing the "deeper understanding" of mathematics as an addition to traditional mathematics.

In contrast to his reflections of the mathematical experiences throughout the 1994 spring quarter, Ken's reflections stimulated by a reading assignment in the NCTM *Standards* at the end of that quarter provided him with a new lens to view his recent experiences that reflected views accepted by many professional mathematics educators. This new lens allowed Ken to see that there were viable approaches to mathematics that differed from what he understood as the only appropriate approach to mathematics. Although he did not reinterpret grade school experiences with mathematics as Ball's (1990) preservice teachers had, Ken reinterpreted his experiences with mathematics in both the methods class and the high school class in which he worked weekly. Using the context of the *Standards* (NCTM, 1989) to reflect on his recent experiences with a constructivist approach to mathematics in the methods class and the traditional approach to mathematics in the local high school provided Ken with a perturbation or dissatisfaction in his beliefs regarding teaching and learning mathematics. As documented in other research studies (e.g., Wilson, 1994; Zilliox 1990), the effect of perturbation or dissatisfaction with current beliefs became an important stimulus for change in Ken's philosophy of mathematics.

In addition to (1) the context that allowed a development of a knowledge of mathematics that permits the teaching of mathematics from a constructivist perspective, and (2) the context of reflecting on his experiences as a learner of mathematics, experiences related to evolution in Ken's mathematical philosophy included (1) contexts of gaining experience in assessing students' understandings of mathematics, and (2) contexts that allowed him to translate his knowledge and beliefs about mathematics into viable teaching strategies. From his experiences in the local high schools, his group work within the courses, and the explicit attention given to culture in the fall quarter, Ken began to acknowledge that different learners will learn and understand mathematics in different ways. This acknowledgment was an evolution from his initial view that all students learned the same mathematics through practice, discipline, and memorization. Ken also found technology and assessment as viable teaching strategies that allowed him to bring his evolving beliefs in exploration and deeper meaning in mathematics into action.

Conclusions

The context of Ken's teacher education program combined with the extra opportunities to reflect on the program through the frequent interviews provided Ken with a very conducive environment for change in his mathematical philosophy. Nonetheless, the evolution noted often seemed insignificant. When asked in a later interview if his student teaching would have been different if he had not had the mathematics education classes he said that it would not have been very different. The evolution in his philosophy had been subtle and Ken did not recognize that there were alternative philosophical views of mathematics from which he could choose. Cobb (1994) states that, "the teacher's role is characterized as that of mediating between students' personal meanings and culturally established mathematical meanings of wider society" (p. 15). Ken developed personal meanings for a "deeper understanding" of mathematics through his experiences and he benefited from the culturally established mathematical meanings presented in the *Standards* (NCTM, 1989), yet he possibly could have benefited from more explicit attention to other culturally established choices for viewing mathematics and its teaching. Such explicit attention would have provided what Ernest (1991a) referred to as "higher levels of reflection and self-awareness" (p. 61) as Ken compared alternative philosophical positions with his beliefs. Neither acknowledging choices among culturally established pedagogical perspectives nor experiencing alternative pedagogical practices is enough for providing a rich context that allows preservice teachers to examine their mathematical philosophies. Rather, the mediation of both the culturally established choices in mathematics education and the

personal meanings attributed to alternative perspectives through experience provides a powerful context for instructing preservice teachers.

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: the role of a preservice methods course. *For the Learning of Mathematics, 10*(2), 10-16.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13-20.

Cooney, T.J. (1987). The issue of reform: What have we learned from yesteryear? In Mathematical Sciences Education Board, *The teacher of mathematics: Issues for today and tomorrow* (pp. 17-35). Washington, DC: National Academy Press.

Cooney, T.J. (1994). Teacher education as an exercise in adaptation. In D. B. Aichele & A. F. Coxford's (Eds.), *Professional development for teachers of mathematics* (pp. 9-22). Reston, VA: National Council of Teachers of Mathematics.

Ernest, P. (1991a). Mathematics teacher education and quality. *Assessment and evaluation in higher education, 16*(1), 56-65.

Ernest, P. (1991b). *The philosophy of mathematics education.* Bristol, PA: Falmer Press.

Green, T. (1971). *The activities of teaching.* New York: McGraw Hill.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Schmittau, J. (1991). Mathematics education in the 1990s: Can it afford to ignore its historical and philosophical foundations? *Educational theory, 41*(2), 121-133.

Tymoczko, T. (Ed.). (1986). *New directions in the philosophy of mathematics: An anthology.* Boston: Birkhäuser.

Wilson, M.R. (1994). One preservice secondary teacher's understanding of function: The impact of a course integrating mathematical content and pedagogy. *Journal for Research in Mathematics Education, 25,* 346-370.

Zilliox, J.T. (1990). *Teacher participation in a mathematics curriculum development project.* Unpublished doctoral dissertation, University of Georgia, Athens.

AN INVESTIGATION OF TEACHER REFLECTION: FOUR MATHEMATICS TEACHERS REFLECT ON EDUCATION

Sacra N. Nicholas, University of Oklahoma

What is the nature and role of reflection for teachers? How is teacher reflection influenced by environmental/situational factors? Examination of stated thoughts and observed practices of four 7th and 8th grade mathematics classroom teachers were used as evidential sources to help answer these questions in this interpretative investigation. Multiple data sources revealed classroom settings elicited mostly spontaneous, technically oriented reflections, while reflections in the interview setting were more personally focused. An analysis of the teachers' metaphorical language indicated that reflections could be increased through the influences of a conflicted educational context and inhibited by deep level belief structures that limited the scope of reflections.

How can mathematics teachers promote professional growth through educational experiences and at the same time address a changing world? Wheatley (1992) suggests learning organizations that respond effectively to changing conditions are comprised of members who constantly process new information "with high levels of self-awareness, plentiful sensing devices, and a strong capacity for reflection" (p. 91). For schools to be true learning organizations, they not only need access to new information but they also need to be comprised of individuals with a particular propensity toward shared reflection based on action and a dynamic view of learning.

The investigation reported in this article was conducted during the 1993-1995 school years. The focus of the study was the nature, role, and relationship to external contexts of teacher reflection. Reflection was defined to be a self-informative analytic process which involves active, persistent and careful consideration of beliefs or 'knowledge.' Through reflection, one holds images and ideas in conscious awareness so they can be interconnected or transformed. Not only does reflection make possible the biological survival of mankind by making possible the adaptation to external changes but it aids in the construction of ideas out of an unlimited supply of potentials.

The importance of teacher reflection is not a new concept in education. Dewey, in 1933, wrote that teachers must avoid acting purely according to impulse, tradition, and authority by becoming reflective inquirers. In the past 10 to 15 years, however, the number of research projects concerning teachers' reflections has steadily increased. This may be partly due to the acceptance of interpretative research as a legitimate research approach thus making the study of complex phenomena, such as reflection, possible. Other factors contributing to the increased interest in teacher reflection include the emphasis placed on the role of reflection according to constructivist theory.

Observations of and interviews with four mathematics teachers (Rose, Belle, Christy, and Joan*) were the primary sources used in the data analysis. These

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* denotes pseudonym

teachers taught at Central City Junior High,* a traditional midwestern school with many common constraints on reflection including isolationism and limited built-in time for teacher reflection.

Theoretical Framework

The theoretical framework used to guide this exploration was based on the work of numerous constructivist theorists, primarily Piaget (1971), O'Loughlin (1992), and Prawat (1993) and the work of philosophers Lakoff and Johnson (1980), Johnson (1987,1993), and Habermas (1971). Analyses of the role and nature of teacher reflection relied heavily on Habermas' three fundamental human interests: technical, practical, and emancipatory. Habermas believed that individuals relate to the world from one of these interests and that much of their thoughts and actions are directed from one of these underlying world views. Teacher participants' statements and actions were considered with respect to their focus on control (central to the technical interest), clarification and understanding (practical interest), or challenging the assumptions of existing systems and the status quo (emancipatory interest).

Research Questions

The guiding questions in this investigation were: (1) What is the nature of the teacher participants' reflections? (2) What role does reflection play for the teachers as they deal with the complexity of teaching?, and (3) What are the environmental/situational factors which influence the teachers' stated reflections, and observed decisions and practices?

Procedures

Multiple data sources were acquired through five months of bi-weekly classroom observations and a weekly individual interview. Five reflective strategies: researcher observation feedback, oral autobiography, personal journals, audio tapes, and video tapes were used during this time as tools to aid teacher reflection. Approximately one year after the study was completed, the researcher conducted a follow-up telephone interview with the four teacher participants to ascertain which educational experiences following the study had provided additional opportunities for reflection.

Applying a constant comparative method (Glaser & Strauss, 1967), categories from the data were developed with respect to the research question concerning the role of teacher reflections. Three other analyses aimed at clarifying the nature of teacher reflections were made. Researcher tools were developed for these analyses based on the work of Louden (1992), Van Manen (1991) and a synthesis of metaphorical research based primarily on work reported by Lakoff and Johnson (1980) and Johnson (1987, 1993).

Conclusions and implications

The Nature of Teacher Reflections

Multiple data sources revealed classroom settings elicited primarily spontaneous, technically oriented reflections. Teachers used reflection in three ways: to make instructional adjustments, conduct on-the-spot assessments of their instruction, and make classroom management decisions.

Teacher reflections in the interview setting were more recollective and personally focused. They helped the teachers analyze themselves, make assessments, process their pasts and anticipate the future.

Contrary to beliefs held by Van Manen and Habermas, it appeared that teacher interests (philosophical orientations) were not solely internal phenomenon but were interactive, dynamic, and sensitive to external circumstances. For example, Joan's intention to represent mathematics not only as a set of rules but also as a discipline that dealt with non-routine, open-ended problems called for both technical and practical interests to be expressed at various times. Belle also expressed a combination of interests in the classroom. At the beginning of her classes she would often display a practical interest as she chatted with her students about their personal concerns, however, when she instructed the students on mathematical concepts and procedures a more technical interest was evident.

Critical theorists have expressed concern that teachers' intentional reflections are too often focused on issues that relate to the immediate demands of their classrooms rather than to external contexts or visions of possibilities of educational alternatives (Grundy, 1987, Smyth, 1992; Zeichner, 1993). These concerns seemed to be legitimized by findings in this study which revealed the teacher participants were too close to their students and instructional situations to be able to be actively introspective.

The Role of Teacher Reflections

The role reflection played for the participants was found to be strongly affected by individual reflection tendencies. Personal philosophy, goals, interests, beliefs, concerns, personal teaching style and environmental influences were some of the factors that influenced these individual differences.

The role of reflection in the interview sessions functioned differently than ongoing reflection in the classroom. In the interview setting teachers used reflections to help them deal with more internal issues. For instance, it was not uncommon for the teachers to discuss conflicted feelings they had about the lack of support from parents and school board members. Opportunities for inward focused reflections were rare in the classroom because the participants had not structured their classes in ways that would allow them to stand outside the action.

Environmental/Situational Factors

The metaphorical analysis of interview data provided insights with respect to focus and increased activity of reflections. The metaphorical language used by the teachers not only demonstrated their concerns but magnified issues that for them were emotionally charged. For example, the emotional reactions to the conflicts experienced within the Central City community over OBE resulted in increased teacher reflective activity and was revealed through the war metaphors they used. The teachers made comments like: "We're fighting this survival battle." "I was just being shot down all over the place." "Sound practices are being challenged."

An additional benefit gained from the metaphorical analysis was an indication of teacher beliefs that might suggest "blind spots" to teacher reflection. Teaching metaphors like: "mathematics is a building-type skill" that has to be "gotten across" to students who may turn into "spoon-fed robots" helped to illustrate how the scope of teacher reflections can become limited due to deep-level belief structures.

Summary

Reflective practice is not a panacea that will solve all the problems of education. However, it is key to most processes and programs designed to meet educational challenges and change. The systematic use of reflection helps teachers combine and integrate their past experiences with current knowledge and information enabling them to respond more effectively to current educational demands. By making explicit those personal theories of knowledge and actions that have been implicit, teachers may confront practice on their own terms and confirm strengths and weaknesses. Appreciating the complexity of teaching and gaining a clearer understanding of their own framework through reflective thought may enable teachers to discuss with others what they believe and explain why they are doing what they are doing in the classroom (Cornet, et al., 1992).

Bureau (1993) argues that for a radical reform of contexts and school structures three issues must be addressed: time, social structures, and social contexts. However, Grant and Zeichner's (1984) findings suggest setting aside time for teachers to reflect does not guarantee productive reflective activity will occur. Those interested in supporting teacher growth through reflective means should be aware that the outcomes of reflective activity may not always be perceived as positive. Teachers may become dissatisfied with current teaching conditions, challenge traditional structures, seek more power and demand a voice in important issues. What this investigation does not intend to suggest is that reflective teachers working alone can tackle the complexities of teaching single-handedly.

Reflection appears to be a key process in promoting the kinds of change in mathematics teaching being promoted by organizations like the National Council of Teachers of Mathematics. What has been learned from this investigation is that any effort to promote reflection needs to (a) be non-coercive, (b) offer a variety of opportunities for reflection, (c) involve social interactions, and (d) be responsive and flexible. By helping mathematics teachers stay open to new information and, at times, off balance (disequilibrium), and by encouraging them to reflect on new information, positive change in mathematics instruction becomes possible. With growth comes the increased likelihood that teachers will be able to think and teach in ways that appropriately meet the needs of a changing world (Wheatley, 1992). Future research goals include exploring some of the more exciting potentials offered by various forms of reflective communities.

References

Bureau, W. E. (1993, April). Seeing supervision differently: The processes of facilitating change in a veteran teacher's beliefs. Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA.

Cornett, J. W., Chase, K. S., Miller, P., Schrock, D., Bennett, B.J., Goins, A., & Hammond, C. (1992). Insights from the analysis of our own theorizing: The viewpoints of seven teachers. In E. W. Ross, J. W. Cornett & G. McCutcheon (Eds.), *Teacher personal theorizing: Connecting curriculum practice, theory and research* (pp. 35-49). Albany, NY: SUNY Press.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process.* Chicago, IL: D.C. Heath.

Glaser, B. & Strauss, A. (1967). *The discovery of grounded theory.* Chicago, IL: Aldine.

Grant, C. A. & Zeichner, K. M. (1984). The teacher. In C. A. Grant (Ed.), *Preparing for Reflective Teaching* (pp. 1-18). Newton, MA: Allyn and Bacon, Inc.

Grundy, S. (1987). *Curriculum: Product or praxis?* New York, NY: The Falmer Press.

Habermas, J. (1971). *Knowledge and interest* (J. L. Shapiro, Trans.). Boston, MA: Beacon Press.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination and reason.* Chicago, IL: The University of Chicago Press.

Johnson, M. (1993). *The psychology of the implications of cognitive Science for ethics.* Chicago, IL: The University of Chicago Press.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by.* Chicago, IL: University of Chicago Press.

Louden, W. (1992). Understanding reflection through collaborative research. In A. Hargreaves & M. G. Fullan (Eds.), *Understanding teacher development* (pp. 178-215). New York, NY: Teacher College Press.

O'Loughlin, M. (1992). Engaging teachers in emancipatory knowledge construction. *Journal of Teacher Education, 43 (5)*, 336-346.

Piaget, J. (1971). *Biology and knowledge: An essay on the relations between organic regulations and cognitive processes.* Chicago, IL: The University of Chicago Press.

Prawat, R. S. (1993). The value of ideas: Problems versus possibilities in learning. *Educational Researcher, 22* (6), 5-16.

Smyth, J. (1992). Teachers' work and the politics of reflection. *American Educational Research Journal, 29* (2), 267-300.

Van Manen, M. (1991). Reflectivity and the pedagogical moment: The normativity of pedagogical thinking and acting. *Journal of Curriculum Studies, 23*(6), 507-536.

Wheatley, M. J. (1992). *Leadership and the new science: Learning about organization from an orderly universe.* San Francisco, CA: Berrett-Koehler Publishers, Inc.

Zeichner, K. M. (1993, April). Reflections of a teacher educator working for social change. Paper presented at the annual meeting of the American Educational Research Association, Atlanta, GA.

ON THE NOTION OF SECONDARY PRESERVICE TEACHERS' WAYS OF KNOWING MATHEMATICS

Thomas J. Cooney, University of Georgia
Patricia S. Wilson, University of Georgia

This study investigated the beliefs about mathematics held by two secondary preservice teachers as they participated in a teacher education program that promoted the NCTM Standards and the use of technology. Of particular interest was what the teachers believed and how those beliefs were structured. Theoretical perspectives developed by Green (1971), Perry (1970), and Belenky, Clinchy, Goldberger, and Tarule (1986) were particularly helpful in this analysis. Analyses of data taken over a 15 month period of time indicated that both the teachers' beliefs and the structures of their beliefs differed. Recognition of these various structures is of considerable importance when developing teacher education programs that promote reflection and adaptive teaching.

This study focuses on prospective secondary teachers' abilities and confidence to do mathematics and the beliefs they express about mathematics as they progress through a four quarter sequence in mathematics education. The sequence consists of two courses in mathematics education, followed by a quarter of student teaching, and concluding with a post student teaching seminar. This study was conducted in the context of the NSF supported project Research and Development Initiatives Applied to Teacher Education (RADIATE).[1] We will explicate three different aspects of knowing 1) what the teachers seem to be able to do mathematically, 2) what beliefs they seem to hold about mathematics and how those beliefs are structured, and 3) the implications of their knowledge and beliefs about mathematics for the teaching of mathematics. To illustrate these different aspects of knowing, we will concentrate on two informants, Harriet and Kyle, who were two of the students who participated in the teacher education program. Data for the study consisted of an initial survey that included mathematical tasks, questions about the teaching of mathematics, and the selection of similes that reflected their views about mathematics and its teaching; eight interviews including a card sort of participant-identified statements from previous interviews; four tests administered during the first two courses; numerous journal entrees in which the informants responded to specific questions related to course activities; and observations of their field experiences including student teaching. The teacher education program placed considerable emphasis on alternate teaching methods, including an extensive use of technology, and daily opportunities for the teachers to engage in various reflective activities.

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[1] Project RADIATE is directed by Thomas J. Cooney and Patricia S. Wilson and is funded by the National Science Foundation (DUE9254475) and the Georgia Research Alliance. Any opinions or conclusions expressed herein are those of the authors and do not necessarily represent the views of the funding agencies.

Theoretical Perspectives

It is our intent to study what the preservice teachers knew and thought about mathematics and also to consider various theoretical perspectives for describing the ways in which the teachers held their knowledge and beliefs. To guide our analysis, a variety of theoretical perspectives were used. Ernest (1991) identified five different belief systems that included theories of mathematics, learning mathematics, teaching mathematics, assessment in mathematics, and aims of mathematics education. We were primarily interested in our informants' theories of mathematics and the teaching of mathematics as we found the line between these theories to be quite blurred in the reality of studying their beliefs. We considered schemes created by Perry (1970) and Belenky, Clinchy, Goldberger, and Tarule's (1986) for considering the way that the teachers' knowledge was held. In particular we were interested in the question of whether the teachers see mathematics from a dualistic perspective where there is some sort of absolute system and the learner is either right or wrong, or whether they see mathematics from a more relativistic perspective where mathematics is dynamic and the learner is influenced by personal as well as community constructions (Cobb, 1994). To provide further dimensionality to the understanding of teachers' beliefs, we considered Green's (1971) metaphorical analysis of the structure of beliefs. In particular, we were interested in the intensity or centrality of the teachers' beliefs (Green's notion of psychologically central beliefs versus peripheral beliefs), possible logical connections among beliefs (Green's notion of primary beliefs versus derivative beliefs), and finally how various beliefs are clustered, noting in particular which beliefs seem isolated from others. By considering the structure of beliefs as well as the substance of beliefs, we can better understand possible entree points for influencing the teachers' beliefs and how those beliefs potentially influence classroom practice. Such an understanding is fundamental to developing teacher education programs that enable teachers to realize constructivist orientations which serve as foundations for in the NCTM *Standards*.

The Substance of Harriet and Kyle's Beliefs and Knowledge of Mathematics

Harriet and Kyle entered their mathematics education sequence with considerable similarity in their mathematical backgrounds as determined by courses taken and grades earned. They both maintained B+ averages in collegiate work. Although they differed in gender and ethnicity, they were both from middle class families and each had a mother who was a school teacher. Harriet frequently spoke with authority in class offering her perspective, occasionally challenging other view points, and sharing her experiences as an African American female. Kyle rarely initiated responses in class but spoke freely when asked for a response or when he worked in his family group. In one interview he commented on his privileged position as a White student.

On the surface, both students seemed to have comparable ideas about mathematics. They each spoke of learning mathematical concepts and the importance of establishing mathematical relationships. However further analysis of interviews, surveys, and class products suggested that Harriet and Kyle held different views about the nature of mathematics and understood mathematics differently. Kyle valued mathematics that could be applied. "I think real world is very important because it's hard to learn something that you can't apply. ... it's got to seem useful in order to learn it." He could easily give examples of applications of mathematics. He enjoyed problem solving and working challenging problems, often drawing from a variety of applications including woodworking, sports, and physics. Harriet also spoke of the importance of applications but when asked for examples she repeatedly referred to the same examples of using knowledge about percents to calculate the price of an item on sale or a gratuity. Harriet explained that she enjoyed mathematics because it was easy and did not involve reading. She spoke of mathematics as being right or wrong and argued that it was important for the teacher to tell students if they are correct or incorrect. She chose an assembly line metaphor for mathematics (written survey, 3/29/94) explaining that, "You start out with a problem, certain parts of your brain perform certain functions and you produce a product of 'answer'." Kyle saw learning mathematics as building a house, where "everything must be thought in advance or else you may have to build and tear down over and over. Same is true for math, one needs to learn it well the first time." Harriet seemed to restrict her view of mathematics to mathematics that she understood and considered appropriate content for high school students. Harriet explained that she wanted "to make sure that I help those [students] and teach as much as I can correctly to young adolescents." Kyle seemed to have two distinct kinds of mathematics. Like Harriet, Kyle believed school mathematics should be primarily rules, formulas, abstractions, and well-defined concepts. However, Kyle also enjoyed a rich mathematics outside of school that helped him solve problems from a variety of perspectives. He struggled with these two different types of mathematics in the classroom. "I think that applications and real world will be real good in high school, but in this sense, they were real good and helped me to learn a lot, but they weren't preparatory for those higher abstract levels of math."

Harriet and Kyle seemed to have different competencies in mathematics. Harriet responded to mathematical questions with vague language, sometimes misinterpreting the question or providing an unusual or incorrect response. Kyle used standard mathematical vocabulary and often provided specific examples to support his answers. The initial survey (3-29-94) posed a question about how to respond to a student who claimed that the area of a rectangle increases as the perimeter increases and provided examples to support this contention. Harriet accepted the student's generalization as correct, but objected to the student's use of a square as an example of a rectangle . Kyle offered a counterexample that disproved the student's generalization. In a later survey (11-4-94) we again see marked differences in how Harriet and Kyle discuss mathematics. In response to the question, "When someone says 'Geometry' what comes to your mind?" Harriet replied by listing names of geometric figures whereas Kyle talked about "finding surface area, volume, and perimeter of shapes", noting that "geometry is extremely useful in the real world." When asked, on another survey (10-26-94), what "transformation" means and what, if any, experiences contributed to your understanding of "transformation", Harriet's answer was brief: "The word transformation means change to me. Vocabulary in my high school English classes contributed to my understanding of transformations." Kyle's responses, however, were more detailed as he talked explicitly about mathematical reflections and rotations, and briefly connected these notions to an experience in his calculus class involving vectors. This pattern of Harriet being vague and general when given the opportunity to talk about mathematics, and Kyle being quite specific and contextual when provided such opportunities prevailed throughout the interviews.

Harriet and Kyle both agreed that a good mathematics teacher should be attentive to the needs of all students. Harriet emphasized "adapting pace of skills to student ability", and Kyle emphasized "teaching a class to *understand* [emphasis his] math, not just memorize ideas" (initial survey, 3-29-94). Harriet seemed to be much more confident in her ability to teach mathematics than was Kyle. Despite the fact that Harriet did not exhibit a strong knowledge of mathematics on tests, projects, interviews, or in her journal discussions, she expressed confidence in her ability to teach it. She was confident that she could relate to students and understand their needs. She was confident that she had command of the mathematics she anticipated teaching. Kyle was concerned about his mathematical content knowledge and how it might affect his teaching of mathematics. He frequently referred to the importance of developing a mathematical foundation which seemed to consist of important rules and facts. Since he could not recall all the rules and facts, he anticipated problems in his teaching. Kyle was confident in his problem-solving ability, but he seemed unsure about how this would help him teach.

Considering the Structure of Harriet and Kyle's Beliefs and Knowledge

We can see similarities in what beliefs Harriet and Kyle share about mathematics and its teaching (e.g., placing an emphasis on relating mathematics to the real world) and also how their beliefs differ (e.g., the way that mathematics should be related to the real world). We can gain further insight into their beliefs about mathematics and mathematics teaching by examining the structure of their beliefs. Of the two, Harriet held a more dualistic orientation. Initially she was critical of opportunities to engage in reflective activities. While she later expressed the view that she enjoyed sharing ideas with others, her orientation was more multiplistic than relativistic. In the main, she relished certainty. Her mother was perhaps the most significant factor in influencing her beliefs about teaching. Indeed, across more than six hours of interviews, Harriet only identified four statements (during the card sort interview) that she thought represented what she believed to be particularly important; two of these involved testimonies about her mother. In some sense her beliefs were held non-evidentially in that she tended to accept as evidence those things that conformed to her perceptions about how mathematics should be taught. This circularity tends to preclude reflection. A possible counterpoint to this orientation was Harriet's perceptions about technology. Initially, she posited the view that it was foolish to spend time using the computers given that computers generally were not available for classroom use. But toward the end of her program, she took quite a different perspective, claiming that technology was going to fundamentally change the teaching of mathematics. Indeed, in her card sort interview, she rejected the notion that students couldn't learn mathematics if they hadn't learned the basic skills because of the availability of technology. This shift may suggest the beginnings of a more relativistic orientation, but one has to wonder about the psychological strength of this belief given that it appears isolated from her other belief structures.

Kyle seemed to appreciate contextuality as suggested by the following statements he identified during the card sort interview: "I would use cooperative learning, but not all the time. Don't make all your examples from the book. You can tell a lot about way they know about a problem by the mathematical terminology they're using. The stuff that I thought was important I would stress and the stuff that I thought was unimportant, would kinda go through quickly." In a summary statement during the card sort interview, he emphasized the importance of learning in groups, receiving ideas from class, and criticism. Kyle's experiences were "connected" in that they fit, generally speaking, into a core belief that mathematics should be made interesting for students by enabling them to see connections between mathematics and the real world. While Kyle felt some tension between the importance he placed on basic skills and his orientation toward real world connections, the fact remains that these two views were not totally isolated. This sense of connectedness suggests a relativistic orientation that may account for why Kyle seemed to prosper from the reflective class activities in a way that Harriet did not.

Since it appears that Kyle's beliefs were held evidentially, there is reason to believe that his beliefs will likely be modified over time. The fact that he was concerned about "holes" in his mathematical background was actually predicated on a more pluralistic or relativistic perspective. That is, he saw mathematics from a broader perspective than did Harriet and consequently was more keenly aware of what he didn't know. The fact that he was better able to integrate various voices (e.g., his two different views of mathematics) about mathematics provides a context in which he both enjoyed and profited from the reflective journal entrees he was asked to write. While Harriet may not have been a received knower, neither was she an integrated knower. Her "filtering system" for what she accepted as evidence for believing what she did was much less permeable than Kyle's. Her apparent confidence in teaching mathematics was likely predicated on her understanding of students rather than her formal mathematics as she tended to shy away from unfamiliar mathematics. Thus her "glitches in mathematical knowledge" did not seem to concern her for she did not see them as impeding her ability to teach the mathematics she anticipated teaching. This isolation helps explain her resistance to engage in reflective activities involving her understanding of mathematics.

We can see that Harriet and Kyle not only hold different beliefs about mathematics and vary in their ability to do mathematics, but the structure of their beliefs varies as well. This difference in structure has considerable implications for their ability to realize the NCTM *Standards*. Harriet's isolation of beliefs, her reliance on authority (e.g., her mother's voice), and her non-relativistic conception of mathematics tend to isolate her from the reflection needed for an adaptive means of teaching. Our evidence indicates that she may be moving toward a more relativistic perspective, as suggested by her views on technology. For Kyle, who believes as does Harriet in the importance of emphasizing basic skills, we see a teacher who appreciates contextuality, thereby suggesting a potential exists for changing and reforming his teaching over time.

A Concluding Remark

By considering both the substance and structure of teachers' beliefs, we provide a certain dimensionality that captures the intensity and interconnections among beliefs. While it is well established that teachers' beliefs influence practice, it may be even more important to consider the means by which those beliefs are structured. Recognition of the way beliefs are structured provides us with the potential for seeing how isolated beliefs can be related to beliefs more strongly held—thus ensuring their endurance when buffeted by the usual obstacles teachers face. By considering both the substance and structure of beliefs, we have the potential for eliminating the random effectiveness often associated with our attempt to reform the teaching of mathematics.

References

Belenky, M.F., Clinchy, B.N., Goldberger, N.R., & Tarule, J.N. (1986). *Women's ways of knowing: The development of self, voice, and mind.* New York: Basic Books.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, 23(7), 13-20.

Ernest, P. (1991). *The philosophy of mathematics education.* London: Falmer Press.

Green, T. (1971). *The activity of teaching.* New York: McGraw-Hill.

Perry, W. (1970). *Forms of intellectual and ethical development in the college years: A scheme.* New York: Holt, Rinehart, & Wilson.

SCIENCE AS BACKDROP FOR REFLECTING ON MATHEMATICS WITH PRESERVICE TEACHERS

Rebecca B. Corwin, TERC and Lesley College
Susan J. Doubler, TERC and Lesley College

This study identifies some of the questions mathematics educators must address in designing methods courses for preservice graduate students. The questions emerge from a pilot study of preservice early childhood graduate students' beliefs about the nature of mathematics and science. In a combined mathematics and science methods course students used each subject as a backdrop for considering the nature of the other.

Mathematics and science are the focus of much new curriculum and course development in preservice education. Combining these with other subjects is not generally new practice in preservice teacher education programs in early childhood, yet few course instructors have developed purposeful strategies for designing and facilitating learning experiences to deepen preservice students' knowledge and understanding of content as well as methods in both science and mathematics. It is important that preservice teachers at all levels develop a personal understanding of the nature of both mathematics and science in order to represent these disciplines with integrity (Steen, 1991).

Background

While co-teaching a mathematics and science methods class for graduate school preservice teachers in autumn, 1994, the authors were surprised to find what appeared to be deeper resistance to mathematics than to science. Subsequently, we studied those differences and their implications for preservice students' teaching and for ours. In the spring science and mathematics methods course we did not want to teach the subjects separately, nor could we justify complete integration of the two. Because of the dual focus we felt the course provided a rich opportunity for reflection on the nature of mathematics and science.

Graduate students in preservice courses bring with them already-established perspectives on content areas, and their stances toward mathematics and/or science often vary. We set out to understand the range of those perspectives, to characterize them, and to develop some notions of how those perspectives might inform the design and implementation of a methods course that dealt with both mathematics and science. We were also concerned to determine the relative flexibility of those perspectives so that we could think realistically about what is possible in early childhood methods courses combining mathematics and science.

It has long been held that what teachers believe about learning is influenced in substantial ways by experiences long before they begin to teach, and these beliefs may not change without some significant intervention. Cooney suggests (1993) that it is crucial to "develop a way of thinking about how teachers orient themselves to their students, to the mathematics they are teaching and to the way that they see themselves teaching."

These three elements—orientation to students, view of mathematics and science, and their vision of themselves as teachers—underlay the design of our combined mathematics and science methods course for early childhood students.

> **Orientation to students**: We required graduate students to observe and interview children four times to analyze understanding of scientific and mathematical concepts (Harlen, in press). Students also viewed, discussed, and analyzed selected videotapes of children engaged in mathematics or science.
>
> **View of mathematics and science**: Central to the course were scientific and mathematical investigations designed to enhance graduate students' content knowledge and their views of how science and mathematics are conducted (Shulman, 1990). We believe our students must themselves participate in such investigations, with all the false starts, uneven results, and excitement of real mathematics and science.
>
> **Vision of self as teacher**: The learning environment in the course emphasized discourse, question-asking, reflection, and listening.

The Study

In this study, we characterize preservice teachers' beliefs and sketch some apparent changes in beliefs about mathematics and science. The questions that guided our research were:

- How do preservice students understand the nature of mathematics and science and the connection between mathematics and science?
- How do graduate students orient themselves to young children's understanding?
- How do they see themselves as teachers of young children?

For this presentation, we emphasize the first question: the nature of mathematics and science as fields of knowledge. In this report we will focus in depth on three graduate students (a reasonably adequate representative sample of the eleven-woman class). These three were interviewed following the first class session and again following the last class session by an interviewer who did not teach the course. Interviews and writing were primary data sources. Written materials included mathematics and science autobiographies, course assignments, responses to course readings, reflections on their own learning, and structured reflections on science and mathematics.

Profiles

These short profiles present three students' contrasting stances toward science and mathematics. Each of the three is a woman between 24 and 34; each intends to teach children who range in age from 3 to 9 years. Their positions and beliefs are different, yet some of the themes are troubling to us. After the profiles are presented, we discuss some of those questions.

Profile #1: Diane: I was always in that "other group"

> My math and science education from Nursery School through Third Grade is very faint to me with a few exceptions. My first recollections begin in Fourth Grade when I remember being in the "other" math group, which continued right through High School.

When first interviewed, Diane sees mathematics as quite different from science. She describes science as philosophical—although there may be established theories, those are debatable and allow choices of what to believe. In contrast, mathematics is quite different.

> I see math as concrete because I see math as adding and subtracting and multiplying and there being one definite answer; whereas in science, theories are always being re-tested and re-tested and sometimes re-evaluated and new answers.

Diane feels mathematics is more bounded; it is computational, focused on one right answer. Mathematics exists as an external body of knowledge, necessary to master by hard work, because "those basic skills will be needed to survive."

Diane's views of science are set in the context of the natural world. In elementary school she stored "precious [rock] samples in my very own egg carton" in fourth grade, and her sixth grade teacher's enthusiasm about birds dominates her memory of that year.

Finding contexts for mathematics is more difficult; Diane's context for mathematics is computational—balancing checkbooks and cooking—"those sort of basic, everyday, need-to-know, help you get through the day kinds of things". Other aspects of mathematics she labels "abstract".

Diane's view of learning seems to focus on hard work, discipline, and control. The teacher needs to be in charge so that knowledge can be passed on to the children. During the course this appears to soften somewhat, as this reflection hints:

> Obviously, the basic mathematics—addition, subtraction, multiplication, and division—those are obviously things that can be applied to everyday life. But I think sort of the process of discovering and of understanding them and asking questions and getting wrong answers—and this goes for both math and science—testing them out and learning through mistakes and trying and trying again until you figure it out really to understand it. I think those are really important skills that come from learning math and science, that can be applied to lots of other things.

Profile #2: Sophia: "Counting spots on lions, tigers, and giraffes"

> I remember as a kid just sitting there and going through the multiplication tables across the classroom or doing "one plus one is two" and I mean *without* music behind it. Not having to do with art. It was really boring. It was the class time that I dreaded most. It was cold. I mean, there's *nothing* creative about it, nothing fun about it. Tedious work.

This somewhat dismal view of her childhood mathematics is Sophia's first statement about the nature of mathematics. She goes on to say,

> You don't study math, you just get at different answers ... There's not different ways to go about it. Like [in] a science project you can go about different ways to get to the same answer. You can answer your own different questions. But in math, it seems like there's one correct way to do it and that's it.

For Sophia, science asks questions, that embraces curiosity and creativity, "the study of ways, things, life, people, bugs." In her first interview, Sophia sees mathematics as separated from the world:

> But I think a lot of students end up learning by rote and end up just hating math because of that. Because there's no actual tie-in to everyday life. I mean, there's nothing fun about it.

This separation between mathematics and life is not evident at the end of the course, when she says, "all real things have to do with math and science" and talks about the mathematical opportunities she finds in her garden. Newly aware of connections among science, math, and art, she thinks about how observations made in her garden and her art work include mathematics—proportions of paint colors, percent yield from seeds planted—and says, "Math is also really intricately involved with science...it's not just balancing a checkbook".

Sophia appropriates mathematical ideas *because* she can think of them in familiar artistic contexts. There is evidence that in her own classroom she will bring mathematics and science to her students in informal, everyday contexts.

> Just so it makes sense. And it's not this weird idea out there that you have to memorize. But it actually does make sense. And it has to do with everyday life.

Profile #3: Sandi: No, no, no! Give me the answer, because I have to have the answer!!!

Until the age of eight Sandi describes herself as fascinated by insects and intrigued by life in the woods on her way to school. She says she always wanted to find out more. Sandi considers science a process of investigation:

> Science is finding out about the things around the world. It's also increasing knowledge. It's finding out why something works the way it works. Or does something the way it does.

She describes mathematics as other course members do:

> Mathematics to me would be all about numbers. Givens. About something that is arguable. Some memorization. Hard. Difficult. Can I say boring?

During the semester, though, there is evidence that she wonders about her view of mathematics. Sandi seems to wrestle with her own ideas. It is as if she argues with herself all semester about mathematics. It is in the context of its relationship to science that she questions whether mathematics can really be that cut and dried. She explains that during the course,

> I started to think of math more, instead of automatically writing down a problem. I could sort of round it off in my head and not worry that it needs to be exact.

Later she mentions what she wants children to gain from doing mathematics:

> Seeing relationships, I guess. Like the example with the baby chick and the Unifix cubes. Seeing how two different things can weigh the same. Or figuring out that a certain number of cubes would increase with the weight increase of a baby chick. It's like that relationship and understanding it as a whole thing.

She seems to move away from a static view of mathematics as finding the right answer:

> I started to think about the whole learning process. ... There's no challenge in the answer. There's challenge in the investigations and the inquiries into how things happen or why.

Puzzling out what she sees as a contradiction—her belief that science is never-ending yet mathematics is limited—Sandi takes Fibonacci and his number sequence as evidence that new knowledge may be possible in mathematics:

> For me, science is never ending. There's always something else to learn. For math it seems like once you know all there is to know about math, that that's it. I mean, even though there are other things like the snail and [the Fibonacci series]...How things have a sort of a pattern. And he came up with that and saw that in things. But then again I guess that's not—you could always maybe find something new. Or a new pattern. Or new sets of numbers that might equal something that people may have never even thought of before, or seen.

Troubling questions

As we focused closely on each of the graduate students' views of the nature of mathematics and science, their views about children, and their views about teaching and learning, the patterns we noticed have left us with questions.

1) *What mathematical contexts are most appropriate?*

Although students talked about science they placed their talk in familiar contexts, their comments about mathematics were almost entirely context-free. In the few instances where students set mathematics into contexts associated with their own lives, it seemed to have more meaning. This was illustrated as Sophia began to see mathematics in the art and gardening she loved.

How do we help our graduate students begin to look at their world with a mathematical eye? How can we contextualize the mathematics to support students' developing awareness of a mathematical perspective?

2) *How can we present mathematics as a growing, changing field that, like science, invents new knowledge?*

Popular American culture incorporates scientific questions. These may be presented in newspaper reports and on television, and lay people are even encouraged to develop explanations, conjectures and theories. By contrast, there is little evidence of current advances in mathematics. Our students have little sense that mathematics has its own compelling questions. In what may be characterized as a deeply anti-mathematical culture in the United States, how can we make mathematics more visible?

3) *How can we make it possible for students to learn to ask their own mathematical questions and pursue their own mathematical investigations?*

All three students we reported on held rigid and limited notions of mathematics. They felt that they had access to some computational algorithms, but little more. This made it nearly impossible for them to ask their own interesting mathematical questions, since they had no context for understanding the nature of mathematical questions.

How can we learn to parallel science education's ability to engage students by helping students pose their own mathematical questions and develop their own mathematical investigations designed to answer those questions? In short, how can we help students to think and act mathematically?

Recommendations: These questions leave us with the recommendation that mathematics educators must acknowledge the narrow, decontextualized view our preservice methods students often hold about the nature of mathematics, and develop some thoughtful responses to begin to counteract it. If we do not, we will not be able to move the field forward as quickly as we would like.

We also have a renewed confidence in the promise of combined mathematics and science methods courses for providing new models to students whose mathematics experiences have been limited. Some aspects of science (investigations, developing questions) seem to help students continue to wonder about the nature of mathematics. We suspect it works both ways.

References

Cooney, T. (1993). On the notion of authority applied to teacher education. In J.R. Becker and B. Pence. *Proceedings of the fifteenth annual meeting of the North American chapter of the international group for the Psychology of Mathematics Education.* Center for Mathematics and Computer Science: San Jose, CA.

Harlen, W. (in press). *Diagnostic Assessment and Learning in Science.* Edinburgh: The Scottish Council for Research in Education.

Shulman, L. (1990). Pedagogical ways of knowing. *International Council on Education for Teaching World Assembly.* Keynote Address. 27-31 July. Singapore: ICET.

Steen, L. (1991). Integrating School Science and Mathematics. *A Network for Integrating Science and Mathematics Teaching and Learning.* Wingspread Conference, April 25-28.

A STUDY OF PRESERVICE TEACHERS' METAPHORS FOR THE DIFFERENT ROLES OF THE MATHEMATICS TEACHER

M. Jayne Fleener, University of Oklahoma
Roland G. Pourdavood, University of Oklahoma
Pamela G. Fry, University of Oklahoma

This study examines preservice teachers' metaphors for describing roles of the mathematics teacher. Previous research (Tobin, 1990; Tobin & LaMaster, 1992) suggested metaphors for teaching typically describe three distinct roles of the teacher: teaching, assessing, and classroom management, and consistency of metaphor is important for classroom effectiveness. The findings of this study reveal student metaphors were not systematic among the three roles. Actualizing visions of mathematics learning consistent with constructivist pedagogy will require teachers and pre-teachers to reconcile beliefs with personal interactions and roles in the classroom by engaging in critical reflection of teacher roles.

From a feminist epistemologic perspective, there is an inherent conflict between the hermeneutic idealism and moral imperative of preservice elementary education majors, (a largely female population), and the policies and techniques of public schools, with an emphasis on "functional efficiency and social control" (Goodman, 1992, 181). If we adopt Dewey's vision of teacher preparation (Dewey, 1933), focusing on reflective practice, an important goal for teacher education programs is to encourage critical, deliberative, reflection-in-action on what being a teacher means.

Tobin's work with practicing science teachers suggests the power of having teachers construct and explore their own metaphors for the various roles of the teacher (Tobin, 1990; Tobin & LaMaster, 1992). Reflecting on her metaphors describing the roles of the teacher for managing, assessing, and facilitating learning of science students, one novice teacher was able to discern inconsistencies in her metaphors and classroom practices and modify her vision of classroom interactions to fundamentally change her approach to teaching (Tobin & LaMaster, 1992). What happens when preservice teachers construct and examine their own metaphors for the various roles of the mathematics teacher? Can exploring the systematicity of metaphor help preservice teachers gain a clearer picture of what it means to be a teacher?

The current study adopts the view that metaphorical language orders individual personal realities (Lakoff & Johnson, 1980) and can be examined to reveal philosophical orientations to knowing (Fleener & Fry, 1994). The role of metaphor for organizing and communicating thoughts about one's personal reality is central to a constructivist approach to language which views individual constructions of personalized realities as limited by individual knowledge and language.

Fundamental human interests are reflected by systematic orientations to problem situations (Habermas, 1971). These fundamental interests intercede and condition human experience and action. Fundamental human interests are affected by cognitive as well as experiential characteristics of the individual. Three fundamental human interests (technical, practical, emancipatory) are described by

Habermas and consistent with theoretical orientations toward knowing (empirical-analytic, historical-hermeneutic, and critical) (Grundy, 1987). In this study, analysis and interpretation of student metaphors use Habermasian categories to orient student perspectives.

Guiding Questions

This investigation considers the systematicity of beliefs about mathematics teaching revealed through metaphor analysis of journal entries of preservice teachers.

Guiding questions for this study are:

1. Within a Habermasian framework, what is the relationship among students' conceptualizations of different roles of teaching as expressed through metaphorical language?
2. What does students' examining the systematicity of their own metaphors reveal about their understanding of the complexity of the classroom and what it means to be a teacher?

Procedures

Participants

Sixty-five preservice elementary education majors in two sections of an intermediate-middle school mathematics methods class participated in this study. All students were in their last semester of coursework before student teaching. All but four were women; 80% were under the age of 25; and 90% were Caucasian.

Data Sources

The primary data source was student journals which included reflections on student logs, class discussions, and written assignments. Previous research (Tobin, 1990; Tobin & LaMaster, 1992) suggested metaphors for teaching typically describe three distinct roles of the teacher: teaching, assessing, and classroom management. Because 'classroom management' is itself a dead metaphor (Fry & Fleener, under review), we asked students to construct metaphors describing the role of the mathematics teacher for a) enhancing learning (teaching), b) assessing learning, and c) performing other duties (which might include but need not be limited to classroom management). Students shared their writings in class and discussed the metaphors for teaching expressed by their writings in small groups. Further assigned journal entries during midterm and finals summarized group discussions and assessed individual changes in metaphors for the roles of the teacher.

Analyses

A matrix of metaphors for the three roles of mathematics teaching was constructed from student journal reflections three times throughout the semester. Metaphors were independently grouped according to Habermasian interest categories of CONTROL and HERMENEUTIC/EMANCIPATORY by each of the investigators. Because the hermeneutic and emancipatory categories seem to overlap (see Fleener & Fry, 1994; Fry, in press), these categories were collapsed for all analyses. Any disparities in categorization were discussed among the researchers until a consensus was reached.

To determine whether Habermasian interests as expressed through metaphor were systematic, patterns of responses across teaching roles were examined. Three separate Chi-square tests of pair-by-pair comparisons of metaphors for the three roles of the teacher were performed to examine whether student metaphors across roles were systematic. A model of expected frequencies for metaphors in each pair-by-pair comparison was determined by prior research (Fleener & Fry, 1994). In order to address the second question, student interpretations of the consistency of their own metaphors and reflections on how their metaphors changed were examined. Students' beliefs about the systematicity of their own metaphors to describe the various roles of the mathematics teacher were examined to assess understanding of the complexity of the mathematics classroom and determine what students believed about what it means to be a teacher.

Findings

Student metaphors for each of the three roles of the mathematics teacher were categorized according to Habermasian interest expressed by the metaphor. For example, the metaphor "Teacher as Guide" was categorized as expressing an interest in hermeneutics if the student description included a focus on understanding and/or consensual agreement of class content. The same metaphor, however, was categorized as expressing controlling interest if the student described a pre-existing path and/or direction down which the teacher guided and the students followed.

A Chi-square test of group differences was performed on categorical data to determine whether student responses across teaching roles were consistent, from a Habermasian perspective. Three analyses were performed, comparing metaphors for enhancing and assessing learning (TEACH x ASSESS), enhancing learning and performing other duties (TEACH x MANAGE), and assessing learning and performing other duties (ASSESS x MANAGE). Significant differences between categorization of metaphors for the various roles of teaching were found by all three analyses: TEACH x ASSESS ($X2 = 5.34, p<.05$), TEACH x MANAGE ($X2 = 9.55, p<.01$), and ASSESS x MANAGE ($X2 = 13.14, p<.01$), suggesting, as a group, these students did not use metaphors for describing the various roles of the mathematics teacher that were consistent across roles from a Habermasian framework.

When asked to evaluate their metaphors for consistency, students provided three specific explanations of how their metaphors for the roles of the mathematics teacher were related. Twenty-five of the sixty-four students felt their metaphors were consistent even though, they admitted, the metaphors were quite different. Eighteen students felt their metaphors were consistent and similar; thirteen expressed the opinion their metaphors were consistent and expressing authority or control relationships; and eight felt their metaphors were inconsistent but offered no clear explanation of how their metaphors were inconsistent.

Even though 56 students felt their metaphors were consistent, only 20 were able to explain the coherence among the roles of the mathematics teachers by expressing a root metaphor connecting the three metaphors for those roles. Those who did indicate a systematicity of metaphor often were not able to articulate the relationship but were able to describe how the metaphors were related. For example, one student implied the root metaphor "Teacher as Role Model" to systematize her metaphors "Teacher as Mentor," "Teacher as Manager," and "Teacher as Mother." Another student described how her metaphors expressed aspects of professionalism. The root metaphor "Teacher as Professional" seemed to tie for her the roles of the teacher as Magician, Electrician, and City Planner.

Eight of the thirteen students who expressed the belief their metaphors were consistent based on the authority relationship the teacher had with the students, were able to provide root metaphors. For example, one student reasoned the metaphors were consistent because they all put the teacher in control, "like a Queen Bee." For her, the root metaphor Teacher as Queen Bee systematized her metaphors Teacher as Guide, Teacher as Coach, and Teacher as Telephone Operator and, from a Habermasian perspective, these metaphors did all reveal an interest in control. Another student explained the common feature of her three metaphors was encompassed by the metaphor Teacher as Authority. Her metaphors were Teacher as Coach, Teacher as Principal, and Teacher as Parent.

Discussion and Implications

By generating metaphors for the various roles of teaching and examining their metaphors for consistency, preservice teachers engaged in opportunities for critical personal reflection and individual meaning-making. This study examined their beliefs and the consistency of those beliefs about the various roles of the teacher as expressed through metaphor.

The findings of this study suggest, from a Habermasian perspective, student metaphors were inconsistent across teaching roles. When faced with the realities of teaching, the idealism and theory of the methods class may be called into question as students' hermeneutical tendencies are conflicted with controlling paradigms for teaching.

Because only one in three students was able to critically examine and assess personal metaphors, providing a root metaphor to explain "what teaching is all about" and systematize the metaphors for teaching, it does not appear opportunities for critical personal reflection are sufficient for students to become aware of their own systematicity of thought, a prerequisite for critical consciousness which may lead to emancipatory transformation. Furthermore, that almost half of the students who were able to express the roles of the mathematics teacher using a root metaphor implying an authority or controlling relationship is distressing since it conveys the Factory model of education (Fry, in press). The Factory model of education, with the implicit role of students as raw products to be molded by the teachers, is precisely the technocratic model most teacher preparation programs are trying to eliminate. As an essential component of praxis, methods class reflections need an emancipatory stance most of these students could not adopt on their own. In order for experiences in methods classes to have meaning for preservice teachers, that is, in order for students to be able to critically examine their own thinking and the function or purpose of schooling, methods instructors must provide more opportunities for examining ideas about teaching from a critical perspective. Actualizing visions of mathematics teaching and learning consistent with constructivist pedagogy will require teachers and pre-teachers to reconcile beliefs with regard to what teaching is about.

References

Dewey, J. (1933). How we think: A restatement of the relation of reflective thinking to the educative process. Chicago: Henry Regnery.

Fleener, M. J., & Fry, P. G. (1994). Analyzing metaphors for mathematics teaching. National Forum of Teacher Education Journal, 4(1), 55-63.

Fry, P. G. (in press). Creation of curriculum reality: Metaphors in education. Journal of Curriculum Theorizing.

Fry, P. G. & Fleener, M. J. (under review). An analysis of teacher-class interaction metaphors.

Goodman, J. (1992). Feminist pedagogy as a foundation for reflective teacher education programs. In L. Valli (Ed.), Reflective teacher education: Cases and critiques, 174-186, Albany, NY: SUNY Press.

Grundy, S. (1987). Curriculum: Product or praxis? New York: Falmer Press.

Habermas, J. (1971). Towards a rational society. London: Heinemann Press.

Lakoff, G. & Johnson, M. (1980). Metaphors we live by. Chicago: University of Chicago Press.

Tobin, K. (1990). Changing metaphors and beliefs: A master switch for teaching? Theory into Practice, 29(2), pp. 122-127.

Tobin, K. & LaMaster, S. U. (1992). An interpretation of high school science teaching based on metaphors and beliefs for specific roles. In E. W. Ross, J.W. Cornett & G. McCutcheon, (Eds.), Teacher personal theorizing: Connecting curriculum practice, theory, and research, 115-136, New York: State University of New York Press.

TEACHERS' THINKING AND RETHINKING ASSESSMENT PRACTICES

Thomas J. Cooney, University of Georgia
Barry E. Shealy, University at Buffalo, SUNY

Five secondary school teachers participated in a three-year project to support the rethinking of their assessment practices. The teachers met regularly to share ideas, submitted assessment tasks and their assessments of students, and received feedback. All of the teachers professed significant changes in their understanding of assessment and four changed their teaching significantly. The teachers' written contributions to the project, interviews with the teachers and their students, and observations of the teachers' teaching were analyzed. Key factors from the nature of the teachers' beliefs and their social situations were identified that facilitated and/or inhibited change.

The mathematics education community currently reflects a significant consensus on broadening student outcomes. We have moved from sole emphasis on computational skills to including problem solving, communication, and reasoning. Teachers tend to agree that this shift is important and many are now able to find curricular materials that reflect the current emphases, if not becoming more adept at developing their own student activities. Teachers are, however, much less comfortable with assessing broader outcomes in student performance (Romberg, 1992), seeing these assessments as less objective than those calling for production of single number answers. This discomfort is troubling as assessment plays an integral role in shaping students' expectations and determining what gets valued and learned (Crooks, 1988). Based on these concerns, it seems important to better understand where teachers are in their capability to utilize a broader view and practice of assessment. A teacher education issue is, then, to understand the process by which teachers grow in confidence and ability to make assessments reflecting broader student outcomes, a process deeply connected to the nature of beliefs and social contexts.

Theoretical Considerations

The belief systems of teachers and the social contexts they find themselves in are crucial interacting agents within a teacher's growth process (Brown & Borko, 1992; Thompson, 1992). Several key ideas assisted us in thinking about the teachers' thinking and practice, particularly changes in thinking and practice. First, beliefs that are central (Green, 1971) to one's belief system, that is, strongly connected to other beliefs, would be more difficult to change than those held peripherally in belief systems and also have greater impact on one's practice. Second, someone who is open to the incorporation of new ideas, who's belief structure is permeable (Kelly, 1955), is also more likely to change his or her beliefs and practice. Furthermore, a person's expressed beliefs and practices will be affected by social supports and constraints, perceived or actual (Brown & Borko, 1992).

Partnership with Five Teachers

A group of grade 7-12 mathematics teachers participated in a study of their evaluation practices (Cooney, Badger, & Wilson, 1993). Following this study, five of the teachers volunteered to join a project to receive support for rethinking their assessment practices over three years. The project included quarterly workshops to help the teachers formulate plans for assessment and to share materials and experiences from their classrooms. The study that follows describes their process of growth over the three years. Data included written responses to surveys, individual and group interviews, and workshop field notes. The teachers regularly submitted copies of their assessment tasks, sample student work with the teachers' assessments, and written rationales for their practices. Each teacher's students responded to a survey and participated in group interviews. Project staff also made classroom observations early in the project. We looked for evidence of the teachers' understanding of mathematics and assessment and the practice of teaching mathematics. We focused on the nature of these beliefs in terms of Green's (1971) central-peripheral organization and the permeability (Kelly, 1955) of the teachers' belief systems. We also looked for social aspects of the teachers' situations which supported or inhibited change.

The Case Studies

Carol

Carol has taught nine years in a middle school where students have not experienced a great deal of academic success. Recognized as an outstanding teacher, she is deeply concerned about her students and, as a consequence, is quite demanding of them. Her principal was very supportive of any innovation she wished to try. Early in the project, Carol's belief that "mathematics consists of computations, concepts, problems, and skills" was central (Green, 1971) to her understanding what students were to learn. Thus, her tests were almost exclusively computational in nature. In interviews, her students also reflected this view of mathematics, no doubt a reflection of their mathematical experiences.

Eventually, Carol modified her orientation toward assessment. While clinging to the notion that learning mathematics consisted primarily of acquiring basic skills, she allowed that the acquisition of basic skills could include experiences with more open-ended items. Consequently, her tests became less computational as she began to pose such questions as, "Terry thinks that 24.36 - 3.6 = 24.00. Where did Terry make his mistake?" Carol also began engaging the students in projects with real world data and requiring her students to keep daily journals. Her students indicated she always asked them to explain their work. Recognizing her earlier computational orientation, she attributed her change to seeing other teachers in the project successfully try new practices and incorporating ideas from the project into her teaching. While she continued to hold a perhaps more computational orientation than other teachers on the project, she reflected progress and change.

David

David had seven years of experience teaching in a private school in which parental expectations were high and the teachers were encouraged to be innovative. He had a keen interest in mathematics and used the following adjectives to describe mathematics: puzzle, game, challenging, logical, and analytical. His stated characteristics of good teaching (being energetic, knowing the content, challenging students, being adaptive, and helping students think analytically) characterized his teaching as well. His emphasis on being adaptive reflects his value of self-improvement and is indicative of a permeable belief system (Kelly, 1955).

It was readily apparent that David concentrated on process with his students and not just procedures. He saw "alternate assessment as a means of [getting] a better understanding of how my students think." What changed over the course of the project was his ability to create open-ended items and an acquired expertise in analyzing students' responses to more process-oriented questions. According to one of his students, "His questions are a lot different than some of my other math teachers. It makes you think." As strong a teacher as David is, however, a caveat is in order. His last few tests regressed slightly in that they did not have the creative items evident on his earlier tests. When questioned about this, David indicated that this was his first year of teaching trigonometry and he was a little anxious about the material. Thus, lack of comfort with the content had a greater effect on his teaching than his commitment to alternate assessment.

Karen

Karen has taught middle and high school mathematics and is currently in a urban/ suburban public school district. Early in the project, like Carol, Karen exhibited a computational view of mathematics and her tests tended to be computational in nature. At one point she said, "I finally decided that testing for deep and thorough understanding was going to be [the students'] total undoing." Later, after giving a test with more open-ended questions, she said "although I liked the results, it took me more than three times as long to grade [the tests] than usual." The issues of what the students could handle and the time required to develop and grade tests were significant concerns for Karen.

Eventually, Karen became more comfortable with creating and using open-ended items and was even an outspoken proponent of the techniques, sharing with her colleagues and other teachers. Karen's instructional practice, however, was slower to change. Several of her colleagues, including her department chair became interested in the project and began to try new approaches to assessment in their own classrooms. This opportunity for support in changing instruction and sharing efforts was important for Karen's development.

Esther

Esther is a high school teacher in an urban school district and has taught for twelve years. Like Carol and Karen, Esther expressed computational and procedural views of mathematics early in the project. She said she enjoyed the project because, in teaching, "you seldom get to interact with other teachers. . . . you are sort of on your own." She appreciated the challenge to think about her tests and the "nudge to do more problem solving activities." Over the first two years of the project, Esther added more open-ended items to her primarily computational tests and used activities she received through the project as "warm-ups" in her class.

Later in the project, Esther showed little evidence of trying alternate assessment activities with her students. For example, one idea emphasized in the project was valuing multiple solutions of problems. When asked about alternate ways of doing a problem a student said, "She says 'yeah, you could do it that way, but this is the way we want you to do it because this is the way we teach you.'" A change in teaching assignments may have precluded much progress in using alternate assessment practices. Esther's concern for losing control of the classroom when using more open activities—losing control both of classroom discipline and of the direction of the mathematics—seems to have been a more significant factor. This concern was evident in observations and interviews with Esther and explicitly expressed by her to the group.

Linda

Linda is a middle school teacher in the same private school as David. She has taught for ten years. When she describes mathematics she uses words like "real life," patterns, and colorful. She regularly involves her students in open-ended projects. In education, she tends to focus on a holistic view of the child, emphasizing life skills and conceptual understanding. She sees assessment as determining "what goes on in [the students'] heads. From the beginning of the project, her written tests could range from 10% to 70% open-ended in nature with such questions as, "Write a problem where the commutative property can be used to make it easier to solve and explain why the commutative property is helpful in your problem." From the student interviews, it is clear these types of question are representative of the tasks she provides for her students.

While Linda's tests did not change significantly over the course of the project, she was more open to trying new practices than the other teachers. Linda was the first of the teachers to use portfolios, student interviews, and student-generated tasks in her assessments. Clearly, the support of a colleague, David, and the supportive administration contributed to Linda's freedom to innovate. The project provided Linda with ideas and people to "bounce ideas off of."

Reflecting on the Case Studies

Several clear struggles arose as the teachers tried to use new assessment techniques. They found that having a process-oriented classroom is a challenge—for both students and teachers. It takes more time, seems to makes life in the classroom more complicated, and surely less certain. Teachers have to revise the nature of their roles as teachers and students have to assume greater responsibility for their own learning. The project teachers held these concerns, but interestingly, these teachers saw the rewards as clearly worth the extra effort. They could not envision returning to teaching in which assessment consisted solely of students performing algorithmic tasks.

Key aspects of the project and the teachers' own circumstances contributed to their growth or lack of growth. The initial support of the project staff, the collegial support of the teachers and later peers in their own schools were important elements for change for the teachers. The on-going encouragement over an extended length of time was another aspect of the project the teachers felt contributed to their development. Linda, David, and Carol had interested and supportive principals who provided freedom and encouragement for innovation. Esther, in contrast, was in a situation where her teaching was strictly prescribed inhibiting, if not ruling out, trying new practices. Thus, local and external support and perceived freedom within the teaching situation are important change agents.

Investigating the teachers' belief systems provides further insight into the differing nature of the changes and struggles they experienced. One difference among the group of teachers is the central beliefs (Green, 1971) relating to the nature of mathematics. Carol's view of mathematics as a set of rules by being central is less amenable to change. This impermeability may explain her initial reluctance to try new practices and her reticence to share within the group; although by the end of the project she was involving students in extended open-ended projects. Karen's view of mathematics, while similar to Carol's, was more open to broader process-oriented student outcomes. Thus, she was able to assimilate many ideas of alternate assessment and be an outspoken advocate while initially changing little in her classroom. Over the course of the project she was able to accommodate alternate assessment into her instructional program—if for no other reason than to train the kids to do open-ended items. Thus, we see a case where peripheral beliefs about classroom practice change allowing alternate assessment in without changing her basic notion of the nature of mathematics. For David, however, using such practices as open-ended tasks is at the core of what he believes, but his peripheral fear of teaching new content temporarily requires him to back off his commitment to alternate assessment.

The foregoing analysis raises a "chicken-and-egg" question. The teachers who were most innovative had the most supportive teaching contexts, appeared to have the most permeable belief systems, and held central beliefs about mathematics most in line with current reform ideas. It would be interesting to see the result of Linda and David changing places with Carol, Karen, and Esther. Would permeable belief systems and open views of mathematics and teaching withstand restrictive, unsupportive teaching environments? The nature of belief systems—central beliefs about mathematics and teaching and the openness to new ideas—and the social situation of teaching—support of peers and administrators, time for interaction with peers, supportive curricular materials, and freedom to innovate—have critical effects on the ability of teachers to rethink and change their practices.

References

Brown, C. A., & Borko, H. (1992). On becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, (pp. 209-242). New York: Macmillan.

Cooney, T., Badger, E., & Wilson, M. (1993). Assessment, understanding mathematics, and distinguishing visions from mirages. In N. L. Webb (Ed.), *Assessment in the mathematics classroom* (pp. 239-247). Reston, VA: NCTM.

Crooks, T. (1988). The impact of classroom evaluation practices on students. *Review of Educational Research, 58*, 438-481.

Green, T. (1971). *The activities of teaching*. New York: McGraw-Hill.

Kelly, G. A. (1955). *The psychology of personal constructs* (vol. 1). New York: Norton.

Romberg, T. A. (1992). Evaluation: A coat of many colors. In T. A. Romberg (Ed.), *Mathematics assessment and evaluation* (pp. 10-36). Albany, NY: SUNY Press.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, (pp. 127-146). New York: Macmillan.

TEACHING WITH TECHNOLOGY: TWO PRESERVICE TEACHERS' BELIEFS[1]

Pamela Turner, The University of Georgia
Jennifer Chauvot, The University of Georgia

The goal of this longitudinal study was to conceptualize the belief structures of preservice teachers with regard to technology. We were concerned with what beliefs were held, how those beliefs were held, and to what extent those beliefs influenced the teacher's use of technology. We followed two preservice teachers through four quarters of a secondary mathematics education sequence. We analyzed their beliefs and found that prerequisite mathematical knowledge and the role of the teacher played a major part in the structure of their beliefs toward technology. We plan to continue working with these teachers as they move into their first year of teaching to see if these belief structures change.

Much research has highlighted the importance of beliefs of preservice mathematics teachers and the way beliefs affect the teaching of mathematics (Cooney, 1994; Thompson, 1992). In terms of the use of technology, we need to be aware that the purposes for which the computers are used, the software used, the ratio of students to computers, the location of the computers, available time, and the curriculum are all likely to influence these beliefs (Kaput, 1992; Schofield & Verban, 1988). Our longitudinal study investigated the beliefs that preservice secondary mathematics teachers held toward technology and its use in the secondary mathematics classroom. Accordingly, we focused on the following questions:

- What beliefs did the preservice teachers hold about the use of technology?
- How are the preservice teachers' beliefs about technology structured?
- How did their beliefs seem to promote or impede their use of technology?

We plan to continue working with these teachers into their first year of teaching.

One way of conceptualizing belief structures is through the work of Green (1971). Green's theory on beliefs allowed us to consider not only what beliefs teachers held, but also the way in which those beliefs were structured. We were concerned with many dimensions of the belief structure. How the beliefs were held could be discussed through the use of quasi-logically held primary and derivative beliefs. The strength at which the beliefs were held could be considered through the use of psychologically central and peripheral beliefs. The reasons for holding these beliefs could be determined by considering whether the beliefs were held evidentially or nonevidentially.

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[1] RADIATE (Research and Development Initiatives Applied to Teacher Education) is directed by Thomas J. Cooney and Patricia S. Wilson and is funded by the National Science Foundation (DUE9254475) and the Georgia Research Alliance. Any opinions or conclusions expressed herein are those of the author and do not necessarily represent positions of the funding agencies.

Primary beliefs form the basis for derivative beliefs. There is a quasi-logical relation between primary and derivative beliefs, though there may be no empirical basis for the primary belief itself. In the absence of empirical evidence, the primary belief may be based on proclamations from authority. For example, if a person held a primary belief that he would use technology in his teaching, perhaps only because the NCTM Standards promote this, then a derivative belief would be that he would be willing to use graphing calculators, though the commitment to use such calculators may be questionable. Psychologically central beliefs are those which are held very strongly. These beliefs are less open to rational criticism or change compared to psychologically peripheral beliefs which are more open to examination and possible change. Evidentially held beliefs are those that are held with regard to evidence. They are beliefs that may change in light of further evidence. Nonevidentially held beliefs cannot be changed by the introduction of evidence. They are those beliefs which when challenged cause a person to respond, "Don't bother me with facts, I have made up my mind" (Green, 1971, p. 48).

Methods

Over the course of a four quarter sequence in secondary mathematics education we studied the beliefs of Christine and Liz, two of 15 participants in the Research and Development Initiatives Applied to Teacher Education (RADIATE) project. The four quarter sequence consisted of two courses in mathematics education, student teaching, and a post student teaching seminar. All 15 of the participants were followed, but for the purpose of this study we chose to focus on Christine and Liz. They were chosen because of their willingness to share their thoughts and ideas.

Data for this study consisted of an initial survey that involved mathematical tasks and questions about the teaching and learning of mathematics; three interviews during the first quarter, two interviews during the second quarter, one formal observation and interview during student teaching, and three interviews, one of which was a card sort interview, during the post student teaching seminar (all of the interviews ranged from 45 to 90 minutes); four exams administered during the first two quarters; weekly journals in which the participants were asked to respond to questions related to course activities; and observations of their work on campus as well as their field experiences.

Analysis of Beliefs

Through experiences in the mathematics education courses both Christine and Liz were exposed to situations where graphing calculators and computers were used regularly as investigative tools. These experiences involved the use of technology as an integrated approach to learning mathematics. During the first mathematics education course the students were able to spend time in a computer lab.

One activity involved the use of Algebra Xpresser, a graphing program. Liz explained.

> We graphed $y = x^2$, then we graphed $y = ax^2$. Then we compared the graphs. It took a lot of time, but it helped me to see the effects of each part of the function. (Journal, 5/12/94)

The second mathematics education course was taught in an enhanced classroom which contained 17 Power Macintosh computers. During the course of the activities the students could turn their chairs so they would be at a computer. It was a powerful situation especially when the students could move between using technology and using a paper and pencil. We hoped that we were creating an atmosphere where technology was a tool to be used in an interactive process of learning mathematics.

As we began to analyze the beliefs of Christine and Liz, it was clear that as they became proficient in and confident toward their use of technology, they were forming similar beliefs. One of these beliefs was that their success with technology resulted from the fact that they already knew the mathematics involved in the activity. Thus it was their mathematical knowledge that helped them understand the use of technology, hence the technology was simply "icing on the cake." Both Christine and Liz held this belief as primary in the sense that it drove other beliefs, and further, this belief was psychologically peripheral. It was also evidentially held and was open to rational criticism and possibly change in light of new evidence. For example, during a unit on transformational geometry, Liz had an experience which provided an opportunity for her to examine her belief.

> Throughout the week, I learned to use my available materials (computers, MIRAs, paper, etc.) and try to visualize the transformations of a figure. . . I never knew that formulas stood behind each of these transformations. I think it helped to first work on the computer and experience the transformations, and then discover how the computer followed our commands. (Journal, 11/4/94)

Her belief that success with technology comes after the mathematical knowledge was acquired was peripheral and amenable to change. She was willing to consider a new belief which was contradictory to her primary belief. Though this episode challenged her belief, the evidence was not significant enough to cause her belief to change. Perhaps more such challenging episodes would promote a change.

Both Christine and Liz extended this primary belief of a prerequisite mathematical knowledge to many derivative beliefs. One of these derivative beliefs was that once the mathematical knowledge, "paper and pencil skill", as they both called it, was obtained, then and only then could technology be used for further mathematical investigation. The graphing calculator or computer could be used, as Liz explains, to "speed up the busy work" so the teacher "can get to the real gut of the lesson" (Interview, 5/24/94). Christine added,

> so, if let's say I had an algebra 3 trig class, and I knew that they had learned or were supposed to have gotten something and we reviewed, then we could go on into the computer looking at graphs and trig functions and things like that. (Interview, 5/17/94)

Christine extended this belief about a prerequisite knowledge of mathematics to include other derivative beliefs. For example, she held a belief that technology is to be used only in the upper level classes.

> If you have a class that has number one a problem with getting it on paper....you might run into some problems....If I was trying to teach functions [in an algebra 3 or trigonometry class] I would love to have a computer in my classroom. But if I was teaching general math and....I was teaching basic skills and what you're going to need to graduate....I would probably do things that were outside....[like] balancing a checkbook. (Interview, 5/17/95)

Liz extended this belief of a prerequisite knowledge of mathematics to derivative beliefs which were different from those of Christine. Liz was concerned with her students using the calculator or computer as a crutch.

> [I would use] computers maybe once every two weeks....I don't think I would rely on it and I don't think I would want the kids to rely on it because....I would want them to understand it. (Interview, 5/24/94)

Later she stated that the students:

> Trust the calculator way more than their confidence....cause they haven't been taught it [mathematics] without the calculator. (Interview, 5/30/95)

In addition to sharing the primary belief of prerequisite mathematical knowledge, both Christine and Liz shared a belief that they would use technology in their teaching. This belief tended toward psychologically central. It was a derivative belief which was based on their primary beliefs of teaching. Throughout their field experience it became apparent that this belief had an entirely different meaning for each of them.

Christine saw technology as an alternative method of teaching. This stemmed from Christine's primary and psychologically central belief that she wanted to reach every student and, in order to do this, alternative methods of teaching needed to be used.

> It [technology] is a different way to teach. It's a different way that a student might understand something. Somebody might not get it looking at it on the overhead....but if they were put in

> front of a computer maybe they'd have a whole lot to say. (Card Sort Interview, 5/30/95)

Christine's field experience in the second mathematics education course offered an opportunity to use technology with an algebra 3 class that was currently working on the law of sines. At first she chose not to use technology.

> I felt like we had so much we needed to cover that any structured use of technology would be a hindrance. (Journal, 11/11/94)

Once she was "coerced to use something" she realized that:

> Not only were our students given an opportunity through us, but they also benefited from the exploration of right triangles on GSP. (Journal, 11/11/94)

With such positive experiences such as this, it seemed that Christine would use technology in her student teaching.

Her student teaching took place in a small city high school with a traditional teacher who, in Christine's words,

> Doesn't have a clue about technology, but she's really excited about using it. (Interview, 11/29/94)

Christine had possession of a powerbook and an overhead projection panel, but she rarely used them. Most of her classes were introductory geometry with the exception of one algebra 2 class. On the surface it seemed as though her belief in using technology was in conflict with the fact that she did not use technology in her student teaching. In fact, a deeper analysis revealed that her beliefs were not in conflict. She believed technology to be an alternative method of teaching, so she replaced it with other alternative methods such as group work, manipulatives, and peer teaching.

Christine also believed that using technology in her classroom would give students much needed skills since "our world is becoming more of a technological focus in general" (Card Sort Interview, 5/30/95). It is interesting to note that this belief was contradictory to her belief that technology was to be used only in the upper level classes. One might wonder, if our world is becoming technological and it is important to teach these skills, then why should we only use technology in the upper level courses? For Christine, the answer to this question may be rooted in the belief she held about a prerequisite mathematical knowledge. Perhaps the lower level students have not yet acquired that knowledge and therefore using technology would not be worthwhile.

Liz, on the other hand, was primarily concerned with using technology as a demonstrative tool. Liz's primary belief was that the role of the teacher was that of an authority figure. She believed that her students should be exploring mathematics with technology, but at the same time she saw her role as controlling the direction and substance of the activity. Her teaching was basically teacher-centered in nature. She was not ready to give up or even share the authority in the classroom.

Liz's student teaching took place in a large suburban high school with a teacher who was well versed in computers and computer software. The classroom itself had only one computer with a television monitor but there was a lab directly across the hallway that contained 15 Macintosh and 15 IBM computers. Also, the students were familiar with the TI graphing calculators since there was a classroom set that was available daily. Throughout her student teaching Liz constantly used technology in a very structured, demonstrative way. When she was encouraged by her cooperating teacher to let the students explore on their own, she was frustrated. She wanted to be the one to give them guidance. Nevertheless, she heeded the advice of her cooperating teacher, even if reluctantly. At the end of the lesson she felt that it had been unsuccessful and next time she would be sure to give the students more direction (Interview, 4/20/95). In another student teaching episode, a student who was absent the previous day was asking about the assignment. Liz responded, "It's real easy, the calculator will do it all for you." Liz then handed a piece of paper to the student and stated, that "[the handout] is to tell you all that you need to put in the calculator" (Student Teaching Observation, 2/23/95). It was a summary sheet of all the keystrokes. It is interesting to note that even though Liz had access to computers and graphing calculators on a daily basis, she used them only after the students had learned the mathematics and even then she would tell the students what they needed to be doing. On a positive note, she did use them!

Conclusion

Green's (1971) theory on beliefs has given us a framework to examine the belief structures of these preservice teachers. Analyzing their belief structures has helped us to determine how and when these teachers would use technology in their classrooms. For example, Christine's belief that she would only use technology in the upper level classes was derived from the primary belief that success in technology resulted from a prerequisite knowledge of mathematics. In order to challenge Christine's belief that technology was to be used only in the upper level classes, we could challenge her belief of a prerequisite mathematical knowledge. Since this belief was peripheral, it would be open to examination. This in turn may cause her to examine her belief of technology in the upper level classes. With Liz, her belief that she needed to provide direction in the use of technology was derived from two primary beliefs-success in technology resulted from a prerequisite knowledge of mathematics and the role of the teacher is one of authority. To challenge Liz's belief that she needed to provide direction, it might be beneficial to challenge her belief that success in technology resulted from a prerequisite knowledge of mathematics. Her belief that the teacher's role is one of authority is centrally held and is therefore less open to rational criticism than her belief of a prerequisite mathematics.

Awareness of these preservice teacher's belief structures has given us insight into possible changes in our preservice secondary mathematics education program. As teacher educators, we need to be aware of our preservice teacher's beliefs and we need to offer opportunities to challenge those beliefs. We intend to continue following both Christine and Liz into their first year of teaching to determine if their present belief structures about technology change and if so, what caused the change. Through continuation with the RADIATE project and others like it, we hope to continue to gain an understanding of the beliefs of preservice teachers as they continue through the mathematics education program and into their first year of teaching.

References

Cooney, T. (1994). Research and teacher education: In search of common ground. *Journal for Research in Mathematics Education*, 25, 608-636.

Green, T. (1971). *The activities of teaching*. New York: McGraw-Hill Book Company.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan Publishing Company.

Schofield, J. W., & Verban, D. (1988). Computer usage in the teaching of mathematics: Issues that need answers. In D. A. Grouws, & T. J. Cooney (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 169-193). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Thompson, A. G. (1992). Teacher's beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan Publishing Company.

TWO PROSPECTIVE TEACHERS STRUGGLE WITH THE TEACHING-IS-NOT-TELLING DILEMMA

Randolph A. Philipp, San Diego State University

This study is one of a set designed to investigate how preservice teachers' understanding of mathematics and views of teaching are affected by working with children. In the study, two preservice secondary school teachers tutored an eleventh grade student over a period of two sessions. The student teachers were surprised that the student was not considering fractions as involving equal areas, and they spent 45 minutes unsuccessfully attempting to induce disequilibrium in the student. The author viewed the videotape with the student teachers, during which time it became clear that the student teachers were struggling with the notion that teaching means not telling students anything. Implications of this view are discussed.

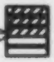
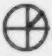
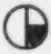
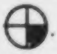
Lampert (1985) proposed a provocative view of teaching when she suggested that the teacher manages dilemmas. These dilemmas differ from problems because they do not lend themselves to solutions, and therefore, rather than being viewed as obstacles to be eliminated, dilemmas instead should be viewed as endemic conflicts that teachers learn to work with and even find useful (Lampert, 1985). Four examples of dilemmas faced by teachers attempting to transform the way they teach mathematics include *"telling" students vs. students constructing knowledge; fostering unconventional and meaningful strategies vs. being socialized into the broader mathematical community; achieving immediate success vs. long term development of ideas; and fostering diversity vs. having convergence of ideas as a goal* (Harel et al., 1995). This paper will address the first dilemma listed above, which Romagnano (1994) referred to as the *ask them or tell them* dilemma. Romagnano (1994) described an incident whereby a lesson he taught to a ninth grade general mathematics class did not go as well as the same lesson taught by a first-year teacher using a more direct teaching approach. Romagnano wondered where one could draw the line with respect to the *ask them or tell them* dilemma:

> Perhaps the most subtle and important aspect of this black-and-white dilemma is the apparent absence of any shades of gray. Can you tell the students some things so you can move on to the more important goals of the lesson? Is it possible to wean students away from being told what to do all the time by telling them less and less and asking more and more as the school year progresses? Or does any telling to students of this age reinforce their expectation that they will be told what to do? (Romagnano, 1994, p. 101)

This paper will highlight this dilemma by sharing the difficulty experienced by two student teachers who had adopted the view that teaching means not telling.

Method

Louis and Ethan were two prospective secondary school mathematics teachers who undertook an assignment that called for them to meet with a student at least two times and work with the student on a mathematical topic of their choosing. They were to assess the student and then, based on the assessment, plan a follow-up lesson that they might teach to the student. The sessions were videotaped. Louis and Ethan worked with an 11th grade student named Donald, an average student in Ethan's geometry class. Louis and Ethan chose to assess Donald's understanding of division of fractions, because it was only in the previous semester's methods course that they themselves had come to conceptually understand this idea.

Following the sessions with Donald, Louis and Ethan entered a student-teaching seminar excited and eager to share with their peers what they had learned. During their interview with Donald on the first day, Louis and Ethan planned on developing the idea of fraction division by working up to asking how this picture  could be used to show why $1 \div 3/5 = 5/3$. To develop this idea, Louis and Ethan first asked Donald to draw a picture of fifths, which Donald drew: . Donald explained that he could draw fifths, but not equal fifths. Louis and Ethan were quite surprised by Donald's response, and they began their second session by again asking Donald to draw fifths, which Donald drew the same way he had the previous day. Louis and Ethan had expected that, and they had altered their plan so that they might work with thirds, which they expected Donald to be able to construct. When asked to draw one-third, Donald drew: . The student teachers spent the duration of the tutoring session, over 45 minutes, trying to induce disequilibrium in the student so that he would come to recognize that the thirds must be equal. At one point they asked the student to draw one-fourth, and he drew: . When they asked Donald how was it that the shaded region in the first circle was one-third whereas the same sized shaded region in the second circle was one-fourth, he responded that in the first case the region was one out of three whereas in the second case it was one out of four. Ethan and Louis expressed surprise that an average eleventh grade student would think in this manner. They also explained that they found the experience to be difficult because they did not want to "tell" Donald. Intrigued by what Louis and Ethan seemed to have learned from their experience, I invited them to view their videotape with me. Prior to the meeting, I arranged to have the videotape transcribed and I sent Louis and Ethan each one copy of the transcription. I then met with Louis and Ethan to discuss the videotape. This session was also videotaped, and the videotape was transcribed and served as the primary source of data for this report.

Results

During the session it became clear that not only had the student teachers read the 26-page transcript of their sessions with Donald, but they talked to each other on the telephone about it. Their session with me began with me asking the student teachers to comment on the transcript. They explained their surprise that Donald did not draw equal fifths or thirds. They said that although they knew where they wanted to go with Donald, they were not sure how to get him there. Louis said, "There was still this mystical magic land we wanted him to get to — you know, the equal parts."

After Louis and Ethan shared their thoughts about the transcript, I asked them whether there were portions of the tape they wanted to watch. Louis responded, "No, not really, (but) there are portions I'd rather skip." He and Ethan described how they were disappointed with how much they thought they had led Donald. As Ethan and Louis reflected upon the "telling" they did with Ethan, they wondered how much telling they did in their own classrooms:

E: I just wonder how much I do that in my own class.

L: And I do it in my class, too. Now since doing this.

R: Is this bad to do?

E: I think sometimes it is. And sometimes.

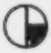
R: What are the implications for doing it, and what are the implications for not doing it?

E: The implications for doing it is that you're telling the student maybe what to think. And you're perhaps telling them, maybe not literally, but figuratively telling them that their thinking should be the same as yours.

L: I think maybe you're also telling them they don't have to think.

E: Yeah. You're also telling them, "Well, you're going to give us the answer anyway."

Ethan and Louis spent 40 minutes trying to induce disequilibrium in Donald. Following is a portion of their attempt, transcribed from their interview with Donald. It begins with them asking Donald to draw one-third.

Donald draws: .

E: That's 1/3? Are all those pieces equal?

D: No.

E: So is that a third?

D: I guess. (Inaudible)

L: You guess? ... Don't be nervous; we're trying to figure out how you're thinking. So you're saying a third because it's one of three pieces? (Donald agrees) This third right there—if I take a third of something, it doesn't matter what it is, whether it's a pie or something else, it's just one piece out of three?

D: Yeah.

L: Do the pieces have to be equal sizes?

D: Yes. No.

L: What happens if we have three 3rds?

D: All three of them make 1.

L: Okay so, they don't actually have to be exactly the same size as long as three of them add up to the whole.

(Pause)

E: When you were dividing it (the circle) into half, did you have two equal halves?

D: When I divided it in half?

E: Yeah. When you divided it in half the first time—those two pieces. So each of those pieces represented what?

D: One half of the whole.

E: So both of them were equal?

D: Yeah.

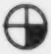
E: Okay. What about here (Donald's representation of 1/3)? You divided it into 3 parts. Are the 3 parts equal?

D: No.

E: So is that one-third of the whole?

D: I guess not.

L: Can you show me one-fourth?

Donald adds a line to his previous drawing: 

L: **So now the same thing you called a third is a fourth now.** Does that make sense to you?

D: I added another piece.

L: You added another piece.

Even though Ethan and Louis understood that Donald was seeing one-third as one-out-of-three, they continued to struggle with why their attempts at inducing disequilibrium failed. I finally suggested we role play, with one of them playing the role of Donald:

R: Okay, let's do it again. "So both of them were equal; they were halves." Okay? "You divided the two pieces in a whole. Were they both equal?"

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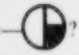
E: (as Donald) Yes.

R: What about here—? You divided into three parts. Are the three parts equal? How much is this?"

E: (as Donald) One third.

R: Why did you answer one third? Now we're popping out [of the roll playing]. Why do you think that he would have answered that that's one third?

E: Because it's one out of three pieces. I see what you're talking about.

L: I see what you're saying now. Yeah. To us there is a connection between three equal parts and a third, but **not to Donald!**

E: To him... We have to get inside his brain, so to speak, and try to question it so that we don't give away that maybe he's wrong or he's right....

L: That's interesting. Even talking as much as we did, we didn't even come close to seeing that one.

Ethan and Louis talked about how they felt they had been going around in circles. I suggested that at times the best thing to do is to back off:

R: Sometimes the best thing to do is just to back up ten yards and punt when you're in a position like this, because you're really sort of stuck. You don't really know where to go with it.

E: And I really didn't know where I was going with this.

L: We're not very adept at punting yet.

R: I realize that.

L: Punting is a hard thing to do...

E: We go for it on fourth down still.

L: Fourth and ten, we're running—up the middle!

E: Up the middle (laughing).

R: That's good. That's funny.

L: No, it's not.

E: No, it's not funny.

R: Well, look. You know, one of the points of being here is to sit back and reflect upon stuff that you probably don't often have a chance to reflect upon.

E: And this is only five minutes into the tape. And our frustration level.... If you could have been in that room—and you know Louis and I and our mannerisms—you could tell just by looking at us that we were frustrated.

I asked them to describe the source of their frustration. Ethan responded, "The frustration was because we didn't know what to do next. We didn't know what to do with his responses. Later during the discussion they again came back to the difficulty they had helping Donald see that thirds were supposed to be equal.

L: Uggh! It's frustrating because *I* see this, and I want *him* to see it.

R: What would you like to tell him right now?

L: I don't know.

R: Forget about all this constructivist stuff, and my methods class. Forget about all the stuff that's in your head. What would you like to tell him right now?

L: **Damn you, child. Thirds are equal!**

R: So what would happen if you told him that?

L: I don't know.

E: I don't think he would necessarily take that to heart.

L: I don't know.

R: Pat and I talked about this quite a bit. And he suggested that what you're trying to do is have him discover a convention.

L: **Ouch!** I didn't think of that.

E: That's exactly what it is. Ahh, man. That's exactly what it is.

L: Do you want to slap me now or later?

E: Slap *me*, man! I said they had to be equal. That's the convention, right there. His thirds don't have to be equal.

L: Ohhhh, man! But *why* are thirds equal? Why did we decide that they *should* be equal? Aah. Okay. Excuse me for a moment.

E: Can we hide our faces?

Note how fragile Louis and Ethan's understanding was. They were grasping for straws, ready to jump on any suggestions I made. They were stuck, with no idea how to proceed. Albert Einstein once said, "The world we have made as a result of the level of thinking we have done thus far creates problems we can not solve at the same level as that which they were created at." Ethan and Louis chose not to tell Donald, but in so choosing, they were left with a problem they could not solve. How could they facilitate Donald seeing what they wanted him to see?

Discussion

Was the previous situation an artificial result of not wanting to "tell" in order to be "constructivists"? Is the condition of equal area a convention, in the sense of an arbitrarily socially-agreed-upon condition? Should Louis and Ethan have simply told Donald that the fractional parts must have the same area?

My answer to all these questions is yes and no. On the one hand, yes, Louis and Ethan felt that telling Donald how to split the circle into equal areas would be inconsistent with a constructivist approach, a thesis that can arguably be considered naive and impractical; but, on the other hand, by restraining themselves from explaining how-to-do-it they created an opportunity to be surprised by how Donald conceptualized the idea of "thirds." This was not only a source of insights for Donald's view of fractional splits, but also for Louis' and Ethan's own questioning of the nature of fractional quantities.

It was important to the student teachers that they not tell Donald the "solution," and yet, at certain points, it might have been useful to talk with Donald about the "convention." Donald's ideas could have provided a background of relevance to the "convention." That is, by noticing how the convention is different from what he had done, the whole conversation might have made the issue more salient for Donald.

Is the condition of equal area a convention, in the sense of an arbitrarily socially-agreed-upon condition? Yes and no. Taken in isolation it can be thought of as a convention, but from a broader perspective it is not just a convention. For example, 1/2+1/2 would not necessarily be equal to one if each 1/2 could be a "different" half. So, should we tell the student that this is a convention? Well, sometimes. Telling Donald the condition of equal areas might have been useful to start him on a shared-upon activity. But there is something shallow in just thinking of it as an arbitrary norm.

Whether something is a convention or not is context dependent. For someone viewing fractions as ratios, equal areas may appear to be a convention. However, for Louis and Ethan, the condition that equal fractions must involve equal areas was an unquestioned principle of mathematics. Which raises another dilemma: Just when and how do conventions become principles, and for whom? At what point should teachers tell students conventions? There are no general rules for this. Furthermore, if Louis and Ethan decide when it would have been appropriate to tell Donald, that does not prescribe when "to tell" another student.

The current reform movement calls for teachers to move from the role of the "sage on the stage" to the "guide on the side." Romagnano (1994) suggests that the dilemma introduced here is black-or-white, with no shades of gray. For some, this black-or-white view is reflected in the notion that "constructivism means never having to tell anyone anything." The study reported in this paper suggests that there are shades of gray.

In order to help teachers work with these shades of gray, we must provide them ways to think about such questions as: Are there some things teachers can tell students? Are there some things teachers must tell students? What are the implications for either telling or not telling in each of these cases? What is the difference between the roles played by convention and principle in mathematics, and how do each of these play out in a classroom committed to constructivist principles? If it is true that there is some knowledge students can not be expected to construct (Williams & Baxter, 1994), then which knowledge is constructible by whom, and how do teachers provide supportive learning environments?

References

Harel, G., Kaput, J., Lesh, D., Nemirovsky, R., Philipp, R., & Thompson, P. (1995). *Mathematics Curricula based in Conceptual Experience* [Unpublished manuscript].

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review*, *55*(2), 178-194.

Romagnano, L. (1994). *Wrestling with change: The dilemmas of teaching real mathematics*. Portsmouth, NH: Heinemann.

Williams, S. R., & Baxter, J. A. (1994). *Reconstructing constructivism: The dilemmas of discourse oriented teaching*. Manuscript submitted for publication.

THE FOCUS OF PRESERVICE SECONDARY SCHOOL MATHEMATICS TEACHERS' OBSERVATIONS OF CLASSROOM MATHEMATICS INSTRUCTION

Laura R. Van Zoest, Western Michigan University

As part of their teacher education program, prospective secondary school mathematics teachers spend a significant amount of time observing classroom mathematics instruction. The purpose of this study was to investigate the focus of these observations.

The participants in this study were three volunteer preservice secondary school mathematics teachers. They were enrolled in courses towards the beginning of their teacher education program, prior to any subject-specific methods course. The time the participants spent in this study was credited towards the observational clock-hours required by their courses and the state board of certification.

The researcher accompanied each preservice teacher to observations of two secondary school mathematics classes. Immediately after each session each participant was interviewed for approximately fifty-five minutes about what he or she had noticed while observing. During the interviews the participants were encouraged to "think out loud" about their observations. Questions building on the participant's previous comments and of an open-ended nature were asked only when a preservice teacher seemed to have run out of things to talk about. Transcripts of the tape-recorded interviews were analyzed using grounded theory procedures and techniques.

The preservice teachers' observations and suggestions for improvement were filtered through their sometimes contradictory beliefs about mathematics teaching and learning. These beliefs, in turn, were affected by their experiences as both students and teachers. The experiences could be vicarious as well as actual and included such things as being an undergraduate teaching assistant, tutoring, imaging their future, and empathy with other teachers. The focus of the preservice teachers' observations fell into five major categories. In decreasing order of time and attention that the preservice teachers allocated to them, these categories were: 1) classroom management strategies, 2) instructional strategies, 3) teaching style, 4) student behaviors, and 5) mathematics content.

The fact that the preservice teachers filtered their observations through their past experiences and beliefs suggests that observations may reinforce the status quo. In a time of major reform in mathematics education, this raises questions about the effectiveness of the observation component of early field experience. The preservice teachers' failure to focus on the mathematics is especially troubling as the reforms require teachers to have a flexible and far-reaching grasp of the mathematics they teach. Further research is needed to assess various means of increasing preservice teachers' concentration on the mathematics in the mathematics lessons.

PERSONALITY TYPE AS A FACTOR IN THE MATHEMATICS ANXIETY OF PRESERVICE ELEMENTARY TEACHERS

Oakley D. Hadfield, New Mexico State University

Numerous research studies have indicated that a large percentage of elementary school teachers possess a level of mathematics anxiety that interferes with their teaching effectiveness. Many mathematics educators suggest that uncomfortable and unsuccessful experiences with mathematics during the teachers' previous schooling is the chief cause. Others suggest that the majority of elementary teachers are simply not cognitively inclined towards analytical thinking. Their cognitive styles (or even personality types) may consist of factors that contribute more interest and skill in global thinking and interpersonal relations. The purpose of this study was to investigate the possible relationship of personality type, as measured by the *Myers-Briggs Type Indicator* [MBTI], to mathematics anxiety among preservice elementary teachers.

The MBTI is comprised of four pairs of dichotomous personality categories: introversion-extroversion, sensing-intuition, thinking-feeling, and judgment-perception. The combined scores of the four subscales result in the delineation of 16 specific personality types. Each of the four subscales was tested for possible relationship to scores on the *Revised Mathematics Anxiety Rating Scale*. Main effects were found on the "thinking-feeling" scale only, with those scoring at or near the "feeling" end of the continuum registering significantly higher levels of mathematics anxiety. This implies that those elementary teachers who are more concerned with the human as opposed to the technical aspects of problems (as is the case with two-thirds of the sample in this study) will be more predisposed to mathematics anxiety.

Two first-order interactions were also indicated. Those subjects who relied more on their senses to establish meaning in the world ("sensing types") tended to be more affected by the "thinking-feeling" variable than those who relied more on meanings and insights ("intuitive types"). Also, those scoring higher on the "extrovert" subscale tended to be more influenced by the "thinking-feeling" variable than were the "introverts". These interactions indicate that if a "feeling" personality type were to be an "extrovert" or "sensing" type (or both), it could perhaps compound any mathematics anxiety problem that may be present. These results agree with the additional finding that there was a prevalence of high levels of mathematics anxiety among 4 of the specific 16 personality types within the sample of 206 preservice elementary teachers. The most math anxious groups were primarily combinations of "feeling", "extrovert", and "sensing" types.

The results suggest that perhaps educators should accept the theory that a majority of elementary teachers are of the "feeling" personality type. Consequently, they are not interested in focusing on the underlying concepts involved in teaching mathematics from a constructivist approach. The most popular solution to this problem would be to continue to improve the quantity and effectiveness of professional development opportunities for elementary teachers in the area of mathematics. An alternative and probably more effective solution would be to employ and rely upon mathematics specialists in the elementary grades to administer the bulk of the conceptual learning experiences of children.

PRE-SERVICE TEACHERS' ATTITUDES TOWARD FEMALES AND MALES, AND THEIR BELIEFS ABOUT MATHEMATICS

Victoria Boller LaBerge, Northern Illinois University

Compared with males, females have experienced lower levels of achievement and participation in mathematics and related fields. Among the factors associated with differences in students' performances in mathematics are teachers' attitudes and beliefs. This study examined pre-service teachers' [PST] attitudes with regard to gender roles and mathematics.

Data for this study were collected using a five-part questionnaire. Ninety-two PST, representing certification levels from elementary to secondary, participated in the study. In the first section of the questionnaire, the PST were given a list of adjectives and were asked to indicate whether each adjective was associated with males, females, or both. The PST were given the same list of adjectives in the third section of the questionnaire; this time they were asked to indicate whether or not they associated the adjectives with mathematicians.

For the second portion of the questionnaire the PST used a five part Likert-type scale to respond to a series of statements relating to the roles of females and males. Responses for each item were scored according to the idea expressed in the statement. A higher score on this portion of the questionnaire indicated a more egalitarian attitude concerning acceptable behaviors for females and males. The fourth section of the questionnaire was similar in construction and scoring method to the second, except that the items included were related to mathematics and not gender roles. A more positive view of mathematics was reflected by a higher score on this section. The final section of the survey instrument collected demographic data from the PST.

Analysis of the data indicated the PST held generally egalitarian attitudes towards roles of females and males (mean = 4.2900), and generally positive (mean = 3.5931) views of mathematics. "Male" adjectives, i.e., those associated with males more often than with females, tended to be associated with mathematicians to a greater extent than "female" adjectives. No "female" adjectives were included among those most frequently selected by the PST as associated with mathematicians. The correlation between the PST's mean scores on the second and fourth sections of the questionnaire, $r = 0.2673$, was significant ($t = 2.587$, $a = 0.05$); PST with more egalitarian attitudes toward gender roles also tended to hold more positive views of mathematics.

Responses to specific items from the questionnaire will be presented and the implications of these results relative to teacher development and students' mathematical understandings will be discussed.

TRANSITION FROM TRADITIONAL HIGH SCHOOL MATHEMATICS TEACHERS TO INNOVATIVE MULTI-DISCIPLINARY TEACHERS

Kenneth L. Shaw, Florida State University
David Ashley, Florida State University

Purpose

The purpose of the paper is to share some of the psychological components of how two self-proclaimed, traditional high school mathematics teachers have modified their beliefs and practices as a result of teaming with two science teachers, two language arts teachers, and a technology/vocational teacher. Our aim was to explore and understand the internal struggles the mathematics teachers were experiencing.

Setting

The team of seven teachers has established a school within a school, called the Master Academy, where 9th and 10th graders form one group and 11th and 12th graders form another. The purpose of the Master Academy is assisting the "average" students to better understand the world around them by having them experience a dynamic interdisciplinary approach to learning.

Methods

In-depth interviews with the mathematics teachers over the course of 1.5 years accompanied with classroom observations were analyzed to determine what factors sustained or constrained their daily routines. Extensive observations and interviews during a one-week planning session in the summer of 1995 were triangulated with the preliminary findings of classroom observations and teacher interviews.

Summary Of Results

Both teachers strongly believe that students learn best when they are "doing" and that by integrating the curriculum, they enrich the students' learning. The internal struggles have come from many sources including: (a) their colleagues in the mathematics department who feel they are not teaching to the state standards; (b) standardized test scores which they felt would be low, and thus used to evaluate them negatively; (c) their lack of knowledge about mathematics applications; and (d) their customary routine of teaching by lecturing. These struggles were somewhat offset by the county's enthusiasm for their project, and their learning as a team of interdisciplinary teachers. A more detailed paper will be shared after the presentation.

DIFFERENCES IN THE CALCULATOR TASKS ASSIGNED IN A VOCATIONAL PREPARATORY AND IN A COLLEGE PREPARATORY CULTURE

Todd M. Johnson, Illinois State University

In this study, differences in the calculator tasks that a teacher assigned in Technical Mathematics and Precalculus were investigated. Differences were investigated in relation to the teacher's conceptions of his roles in each course. His roles were described in terms of his goals for himself in each course, his conceptions of the expectations of individuals and institutions he identified as important, and his conceptions of the situations in which he performed his roles. Symbolic interactionism was used as the conceptual framework to integrate key concepts and data gathering procedures used in this study.

Method

Interviewing the teacher, collecting artifacts, and observing classes were the primary methods used to collect data. Interviews were conducted to: (a) collect the teacher's descriptive accounts concerning how and why he used calculators in his classes and (b) confirm conceptual categories generated from data collected during the study. Artifacts collected during the study included calculator tasks assigned by the teacher and documents that represented the expectations of individuals and institutions the teacher identified as important. Classroom observations focused on calculator tasks assigned by the teacher. Data were analyzed throughout the study. The analyses involved the classification and organization of data to identify categories to describe the teacher's conceptions of his roles and calculator tasks he assigned, descriptors for these categories, examples of elements of these categories, and relationships among categories. After a category was identified, existing data were re-examined and additional data were collected that might indicate the usefulness of the category.

Results

The calculator tasks assigned in Technical Mathematics and Precalculus differed in terms of: the purposes of using calculators, the types of calculators available, how often calculator tasks were assigned, the physical locations students were to work on tasks, the grouping of students during tasks, and the mathematical topics of tasks. These differences were linked to the social norm that the teacher should prepare students for post-secondary education and the different expectations of a university and a technical college.

THE INFLUENCE OF A PROBLEM-SOLVING APPROACH TO TEACHING MATHEMATICS ON PRESERVICE TEACHERS' MATHEMATICAL BELIEFS

Charles E. Emenaker, University Of Cincinnati

This study examines the impact that T104, a problem-solving based mathematics content course for preservice elementary education teachers (PSTs), has on challenging the beliefs PSTs hold with respect to mathematics and themselves as doers of mathematics. The study also addresses the influence the course has on challenging the mathematical beliefs of PSTs by achievement level. T104 employs cooperative learning, alternative assessment, and reflective writing to help the PSTs develop a conceptually based understanding of mathematics.

The research literature indicates that the beliefs teachers hold with respect to mathematics have a major influence on their mathematical performance and their presentation of mathematics in the classroom. These beliefs can be passed on to students thereby limiting the students' mathematical abilities as well. Improving student performance requires, in part, improving their mathematical beliefs which in turn requires helping teachers develop positive mathematical beliefs.

Five specific mathematical beliefs were considered in this study. The beliefs are step-by-step procedures and are needed to do mathematics (STEP), memorization is essential to success in mathematics (MEMORY), only very intelligent people are able to understand mathematical concepts (UNDERSTAND), there is only one way to correctly solve any mathematics problem (SEVERAL), and problems taking more than five to ten minutes to solve are impossible (TIME). The impact of the course on these beliefs was studied using both surveys and PST interviews. The surveys were used to provided insight into the degree to which these beliefs were influenced while enrolled in T104. The interviews focused on verifying changes that were identified by the surveys and identifying specific aspects of T104 that were instrumental in producing these changes.

Statistically significant ($p<.01$) positive changes were observed for STEP, MEMORY, UNDERSTAND, and SEVERAL (N = 137). When changes in beliefs were studied by achievement level, students with a final grades of A in T104 showed statistically significant changes ($p<.01$) in STEP, MEMORY, UNDERSTAND, and SEVERAL. Students with a final grade of B showed statistically significant changes ($p<.01$) in STEP, MEMORY, and UNDERSTAND. No statistically significant changes in beliefs were observed in those groups of students scoring a C, D, or F for the course. PSTs who were interviewed consistently reported increased confidence in their mathematical abilities as a result of T104. Most of these PSTs also reported an increased likelihood of using the innovative instructional approaches from T104 in their own classrooms.

THE INTERACTION OF PRESERVICE TEACHERS' MATHEMATICAL BELIEFS AND AN EARLY FIELD EXPERIENCE

Ronald M. Benbow, Taylor University

Recent research on teachers' instructional practices and on the process of learning to teach has sometimes focused on the issue of teacher beliefs and conceptions. Research is still needed to help us better understand how and under what circumstances beliefs change and for whom or in what situations mathematical beliefs will influence behavior. The research in this presentation explores the possibilities of challenging preservice teachers' mathematical beliefs through a specifically designed early field experience intended to encourage both reflective analysis and instructional skill acquisition. The subjects in this 1-semester study were 25 preservice elementary teachers in a liberal arts university teacher education program, who were enrolled in an early field experience connected to a Mathematics for Teachers course. This experience presented the first opportunity for the PSTs to plan weekly mathematics lessons, teach in elementary school classrooms, and evaluate their own efforts in a systematic manner. The study, utilizing both quantitative and qualitative methods, involved collecting data by administering 3 mathematical beliefs inventories, observing classroom teaching episodes, analyzing students' written lesson plans and reflective evaluation reports, and conducting interviews with 8 selected PSTs and their respective supervising classroom teachers.

The theoretical framework of this research was based on the premise that beliefs and practices are not linear-causal but are more interactive in nature. Therefore, subjects' beliefs about 1) the nature of mathematics, 2) the learning and teaching of mathematics, and 3) self as a learner/teacher of mathematics, were investigated in terms of their influence in shaping classroom instructional practices but also, how classroom experiences may help strengthen or modify those beliefs.

Results indicate that *key* beliefs do influence choices of content and methods in instructional settings and that particular aspects of a first teaching experience have differential effects on beliefs relating to personal teaching efficacy, curriculum content, roles of teacher and student, and appropriate learning tasks. Conclusions from this research include a discussion of various factors relating mathematical beliefs to classroom practices and suggested guidelines for the structure of early field experiences in teacher preparation programs.

INVESTIGATING UNIVERSITY FACULTY MEMBERS' BELIEFS ABOUT INTEGRATED MATHEMATICS AND SCIENCE IN THE MARYLAND COLLABORATIVE FOR TEACHER PREPARATION

Mary Ann Huntley, University of Maryland at College Park
Tad Watanabe, Towson State College (Towson, Maryland)
J. Randy McGinnis, University of Maryland at College Park

The Maryland Collaborative for Teacher Preparation (MCTP) is a statewide NSF-funded innovative interdisciplinary program to prepare teachers who can provide exemplary instruction in upper elementary and middle school grades in Maryland. These teacher candidates have special course and field experiences in mathematics and science content and teaching methods with the goal of making them exceptionally qualified for teaching mathematics and science in ways that emphasize the connections between these disciplines.

One facet of research on the MCTP project involves an investigation of university faculty members' beliefs about mathematics, science, and the connections between these disciplines, and how these beliefs are played out in their instructional practice. This research is consistent with current research on teaching, which indicates that teachers' beliefs about the discipline(s) they teach affect *what* and *how* they teach (Thompson, 1992; Tobin, Tippins, & Gallard, 1994). This research also investigates university faculty members' perceptions of the factors that have facilitated and the factors that have been barriers in their teaching mathematics and science in an integrated fashion to the teacher candidates. The term *integrated mathematics and science* is ill-defined (Berlin, 1991). To what does *integration* refer–an infusion of content, or an infusion of methods from one discipline into the other? Furthermore, does teaching integrated mathematics and science necessitate a particular model of instruction? Data has been collected from participating faculty members across the state in the form of interviews, classroom observations, and faculty journals. This poster presentation will present preliminary analyses of this data.

References

Berlin, D. F. (1991). *Integrating science and mathematics in teaching and learning: A bibliography*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). Reston, VA: National Council of Teachers of Mathematics.

Tobin, K., Tippins, D. J., & Gallard, A. J. (1994). Research on instructional strategies for teaching science. In D. L. Gabel (Ed.), *Handbook of research on science teaching and learning* (pp. 45-93). New York: Macmillan.

The preparation of this material was supported by a grant from the National Science Foundation (Cooperative Agreement No. DUE 9255745).

PRESERVICE ELEMENTARY TEACHERS' BELIEFS: A ROLLER COASTER RIDE

Annette Ricks Leitze, Ball State University

This longitudinal study examined preservice elementary teachers' beliefs about mathematics and mathematics teaching and learning as they progressed through three required mathematics courses at a mid-sized public university. The five-point Likert type beliefs inventory used in the study included the following eight scales:

- I can solve time-consuming mathematics problems. (Difficult problems)
- There are word problems that cannot be solved with simple, step-by-step procedures. (Steps)
- Understanding concepts is important in mathematics. (Understanding)
- Effort can increase mathematical ability. (Effort)
- Mathematics, in general, is useful. (Useful)
- The mathematics I'm studying this semester is useful to me. (Useful course)
- Mathematics is enjoyable to me. (Enjoy)
- I am confident that I can teach elementary school mathematics. (Teaching confidence)

The beliefs inventory was administered at the beginning and end of each of the three mathematics courses. Paired sample *t*-tests were used to check for changes in beliefs over the duration of each course: (1) a manipulative-based content course focusing largely on number concepts, (2) a manipulative-based content course focusing primarily on geometry/measurement concepts, and (3) a mathematics methods course including a field experience.

The preservice teachers' beliefs exhibited a roller-coaster-like behavior—some beliefs oscillating from significant positive changes to significant negative changes and back to significant positive changes. Beliefs represented by the Useful scale, for instance, were enhanced during the first content course, deteriorated during the second content course, and were enhanced again during the methods course ultimately ending up at a level significantly higher than that at which they began. Beliefs represented by the Enjoy scale showed a similar roller coaster pattern.

The research reported here was supported in part by a grant from Ball State University. Any opinions expressed herein are those of the author and do not necessarily reflect the views of Ball State University.

It is particularly worthwhile to mention that significant changes during the mathematics methods course were found on five of the scales: Useful ($p<.01$), Teaching confidence ($p<.001$), Useful course ($p<.01$), Steps, ($p<.001$), and Understanding ($p<.01$). Each of these changes was in a direction indicating enhanced beliefs, reversing in part the trend of deteriorating beliefs that began during the second mathematics content course. While it is comforting to know that the preservice teachers' beliefs are, overall, enhanced by the time they finish their three-course sequence, it remains disturbing that their beliefs significantly deteriorate during the geometry and measurement course.

Teacher Change

CONTENT KNOWLEDGE, BELIEFS, AND PRACTICES: A COMPARISON AMONG SIX PRESERVICE TEACHERS

Cheryl A. Lubinski, Illinois State University
Albert D. Otto, Illinois State University
Beverly S. Rich, Illinois State University
Patricia A. Jaberg, Illinois State University

This paper compares the beliefs and practices of six elementary education majors prior to and during their senior year clinical and student teaching experiences. Three of these preservice teachers had the minimum requirement of two mathematics content courses. The other three were mathematics specialists and had at least six additional courses in mathematics. Comparisons are made from data, pre-intervention baseline to post-student teaching, on their teaching practices and on their reported beliefs. Although results from this comparison suggest that the change profiles of these two groups are different, at the end of their student teaching experience these two groups are not significantly different in their teaching practices and in their reported beliefs.

Typically, elementary education majors believe mathematics is a set of rules and procedures (Ball, 1990; Ball & Wilcox, 1989). The image they have of what teaching is, influences what they do with their student teaching experience (Calderhead, 1988). Their methodology involves a "show and tell" approach and they believe listening to children is an important factor in the learning environment (Feiman-Nemser, McDiarmid, Melnick, & Parker, 1989). Often, they tend to teach as they were taught (Ball, 1990; Lappan & Even, 1989). Since teachers' understanding of mathematics is an integral component of their knowledge base for teaching (Ball, 1991), it could be assumed that an increase in mathematics understanding would have a positive effect on teaching practices. The purpose of this study was to provide insights into the effect of an increase in mathematical knowledge on teaching practices by comparing the reported beliefs and observed practices of elementary K-9 mathematics specialists and non-specialists.

The Project

This study is part of a five-year National Science Foundation grant that is designed to prepare teachers to base their instructional decisions on research findings about children's mathematical thinking. During the initial phase of the project, 25 experienced K-6 teachers developed learning environments to reflect their perception of teaching mathematics for understanding. This was done in collabora-

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The preparation of this paper was supported in part by a grant from the National Science Foundation (Grant No. DUE-9250044) on the "Influences on Preservice Teachers' Instructional Decision Making. Any opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation. We acknowledge the assistance of Deborah Carter and Rosanna Siongco in the data analysis component of this paper.

tion with university researchers and involved a four week summer seminar, classroom visits, and a bi-weekly meeting for one school year. During the second phase of the project twenty-five K-9 preservice teachers joined the project. In the summer preceding their senior year, a two week summer seminar was held in which experienced and preservice teachers collaborated on mathematical tasks. In the fall semester the preservice teachers were enrolled in the same mathematics methods class where they had access to the same information and research that the experienced teachers had had during the previous year. Further, as part of their clinical experiences the preservice teachers were assigned to observe and teach for six hours per week in the classrooms of the experienced teachers. For the student teaching experience during the spring semester, the preservice teacher was assigned to a different grade level with another experienced teacher. During the fall and spring semesters, both the experienced and preservice teachers attended bi-weekly meetings conducted by the university staff.

Data Collection

Data collected on the preservice teachers included information from the Perry Scale, a written belief survey, a belief interview, and pre- and post-intervention video tapes of model mathematical lessons followed by stimulated recall interviews.

Subjects

Three mathematics specialists Barbara, Quincy, and Faith and three non-specialists Evelyn, Nancy, and Wanda (pseudonyms) were selected for more in-depth analysis. Barbara and Nancy student taught in a second and first grade classroom, respectively. Evelyn is a non-traditional student having an undergraduate degree in psychology. She and Quincy were both placed in fifth grade classrooms. Wanda experiences great degrees of mathematics anxiety. She and Faith were at the same school teaching sixth and fifth grade, respectively. Both were in departmentalized situations and taught mathematics throughout the day. Selection of these six was based on commonalties in their grade-levels taught for both clinical and student teaching experiences, of their Perry scale ratings, of their belief survey results, and among their cooperating teachers.

Findings

The analysis of all data focused on the observed and reported changes of the preservice teachers with respect to their beliefs, practices, and decision-making processes before, during, and after the intervention. This is approximately a period of one year.

Written Beliefs Survey. The written belief survey was adapted from the Cognitively Guided Instruction project (Peterson, Fennema, Carpenter & Loef, 1989). A high score indicates a reported belief that learning and teaching decisions need to be based on the consideration of developing students' understandings. Prior to any intervention, the range of belief scores for all mathematics specialists (n=6) went from 3.08 to 4.5. Non-specialists (n=19) ranged from 3.04 to 4.0. After intervention, the range for the mathematics specialists was 3.6 to 4.8 and for the non-specialists from 3.56 to 4.8. We conclude there was no significant difference between the two ranges at the end of the intervention.

Stimulated recall interviews. A videotape of a 15 minute model lesson was made prior to any intervention. Three other model lessons were also taped, one during the clinical experience and two during the student teaching experience. A stimulated recall interview was conducted after each of the last three taped lessons. Comparisons were made of the students in areas identified as relevant (Lubinski, 1990) and related to lesson planning: objectives, content, materials, task selection, consideration of students, and role of assessment.

Objectives. Nancy and Evelyn both shifted from focusing on review and practice of appropriate operations for their grade level to developing strategies for use in word problems which involved all four operations and allowed for multiple solution strategies. Wanda initially focused on increasing student knowledge through the use of definitions and formulas and developed her style to focus on finding solutions to problems that she felt were the types to which her students could make connections to real life situations.

Barbara and Quincy moved in different directions. Barbara's focus went from an open-ended perspective to one being linked to a review or task in which she was only minimally focused on the mathematics involved. Quincy shifted from evaluating equations to teaching more open-ended types of problems that allowed for multiple strategies. Throughout her involvement in the project, Faith had the objective of teaching mathematics by placing her problems in a real-world setting.

Content and Task selection. For ease of comparison, all subjects were instructed to focus on whole number operations during the videotaping of any model lesson. All subjects except Barbara developed teaching styles that incorporated problem situations reflecting students experiences and interests. Both Evelyn and Faith began with problem situations and continued with this style throughout their teaching experiences.

Nancy selected tasks that were fun, informal, and non-threatening in her initial lesson, posing word problems with sums less than 20. These problems appeared contrived rather than based on students' interests or experiences. One year later her problems included not only all four operations and whole numbers, but also fractions. She now selects activities that are integrated with literature, makes connections to previous mathematics lessons, and focuses on problem solving. Her lessons appeared to be influenced by tasks presented in the methods course and at project meetings.

The content of Evelyn's lessons were influenced by her belief that students need challenge. Evelyn consistently used small groups and developed her own activities that she believed would relate to real-life problem solving, for example, a game of Jeopardy. She went initially from focusing on one operation problems to more open-ended problems.

At first, Wanda focused on formulas and definitions and tasks that focused on computational proficiency. Later she developed a style using more relevant problems that involved interpreting and solving written word problems. Wanda also used group work but struggled to have her students complete tasks in the assigned time and was aware that her limited mathematics background contributed to her inability to implement lessons that flowed smoothly. The implementation of her lessons appeared to be influenced by her cooperating teacher's style.

Barbara initially used problems with multiple answers which incorporated all four operations. In her last three lessons she used problems which did not readily allow for multiple answers or alternative strategies. Her additional mathematics background experiences were not readily apparent in their selection of tasks. Barbara's task selection during her student teaching experience appeared strongly influenced by her cooperating teacher, who interrupted her several times during the videotapings.

During initial and clinical videotapings in grade one settings, Quincy's content went from join change unknown problems using numbers under 20 to making connections between repeated addition and multiplication. During student teaching, Quincy selected tasks to develop understandings related to curriculum guidelines and to which students could relate. However, during this time, his groups were structured in order to keep students on task and control behavior, not to develop understanding. During his final videotape, he appeared anxious and focused on controlling behavior more than usual. He relied heavily on his own perception of how mathematics should be taught and struggled to establish a working relationship with his cooperating teacher.

Faith was consistent with using topics to which the children could relate. In the model lesson during her first grade clinical experience she selected to teach comparison problems. That is, she did not focus on developing strategies, but rather on teaching problem types. By the end of her student teaching experience she was still using interesting problems, but incorporating techniques to develop students' thinking. Her content selection was often influenced by the experiences she had within the mathematics department.

Materials. All used a variety of materials but for different reasons. Nancy initially selected materials that were "fun for the students". As they developed their teaching style, both she and Evelyn realized that some materials and manipulatives can detract from the learning. Nancy's choice of materials became dependent upon student behaviors and she resorted to paper and pencil during her final two lessons. Evelyn often selected materials based on assumptions she made about how the students would use them to develop their thinking. Wanda used materials that suggested strategies for solving the problems posed.

Both Barbara and Quincy initially used only paper and pencil, but progressively used more materials; however, their rationales for doing so differed. Barbara's materials were related to projects needing to be completed before the mathematics could be introduced. Quincy introduced materials and manipulatives to develop diverse thinking strategies. Faith used a variety of visual aids and manipulatives throughout.

Consideration of students and the role of assessment. Initially, Nancy based her decision on her conceptions of what students can do at a particular grade level and on what they like. She was aware that the level of difficulty of the problems may have been inappropriate because she asked the students at the end of the lesson if the problems were too easy. She developed an awareness of her students' understandings and interests, tried to incorporate assessment as part of instruction, and began to make connections among the strategies collected. In her final lesson she was aware of different students' abilities based on the kinds of strategies they used and considered this in planning. During instruction at the end of her student teaching experience, she solicited students' thinking but did not alter plans or pose follow-up questions based on their responses.

Evelyn assessed continually as she taught. In her initial first grade situation, the second sentence she said was, "How high can you count?" She became increasingly aware of her students' strategies and modified her lessons accordingly. She had the students verbalize their thought processes to not only develop their understandings, but to collect information about their thinking. She progressed from simply collecting strategies to comparing the strategies suggested. This indicates a more developed level of pedagogy than exhibited by most of the other student teachers.

Wanda had preset assumptions of what the students knew. Initially, she made little effort to solicit students' thinking. During her clinical experience, she focused more on her own thinking even though she solicited students' ideas about their thinking. She developed her questions to better determine how students were thinking about solving a problem by stating, "Could you walk me through it" or "How do you know?" Consideration of students' experiences was not apparent until the final lesson in which problems were taken from real-life situations but there was little evidence of assessment in lesson planning.

Barbara considers students' background throughout her teaching experiences, however assessment was often based on assumptions. There was little evidence of using students' thinking to formulate questions during instruction or to plan for further instruction. Emphasis was more often on a procedure, not on understanding the concepts. She frequently referred to "doing procedures correctly" as providing evidence of understanding. Her emphasis was on obtaining the right answer.

Quincy initially based his decisions on what he perceived interesting to the students, considering their skill level. At the end of student teaching he exhibited an increased awareness of and ability to illicit students' thinking and multiple problem solving strategies. He professed to address a variety of cognitive styles he believes his students possess. Quincy's assessment progressed from walking around the class watching students work to adjusting problems to individual's abilities based on their responses. Faith considered students' interests, experiences, and needs throughout. She focused on their strategies, maintained a flowing dialogue, and used both written and verbal feedback for assessment.

Discussion

It was hoped that an increase in the amount of mathematics coursework where instructors modeled reform-based pedagogy along with a change in reported beliefs would affect teaching practices. Our results suggest that it is not clear what effect more experiences with mathematics being taught for understanding has on teaching practices specifically during student teaching. Other factors may be of greater influence. The cooperating teacher and classroom environment were major influences on some specialists and non-specialists alike, but for others there was little evidence of this. One mathematics specialist was influenced by her mathematics project and non-project experiences, while another appeared to be influenced only minimally by the mathematics coursework. Overall confidence allowed two of the mathematics specialists and one of the non-specialists to be less concerned about control within the learning environment. We attribute this in part to their mathematics backgrounds, level of maturity, or level of cognitive development.

If teachers tend to teach as they were taught, the question becomes "Taught when?" Further, the "show and tell" approach discussed in the literature extended with our preservice teachers to collecting strategies without making connections among them. Our conclusions at this time suggest that there are many factors that affect teaching practices. Rich descriptions are still needed of the relationships among preservice teachers' beliefs, knowledge, and practices. Further, it is important to follow these new teachers as they begin to teach in order to determine if our findings are consistent over time.

References

Ball, D.L. (1990). The mathematical understandings that prospective teachers bring to teacher education. The Elementary School Journal, 90(4), 449-466.

Ball, D.L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (Ed.) Advances in Research on Teaching, Vol. 2 (pp. 1-48). Greenwich, CT: JAI Press, Inc.

Ball, D.L. & Wilcox, S. K. (1989). Inservice teacher education in mathematics: Examining the interaction of context and content (NCRTE Report No. 90-12). East Lansing, MI: NCRTE.

Calderhead, J. (1988). The development of knowledge structures in learning to teach. In J. Calderhead (Ed.), Teachers Professional Learning (pp. 51-64). New York: Falmer.

Feiman-Nemser, S., McDiarmid, G.W., Melnick, S.L. & Parker, M. (1989). Changing beginning teachers conceptions: A description of an introductory teacher education course (NCRTE Report No. 90-12). East Lansing, MI: NCRTE.

Lappan, G., & Even, R. (1989). Learning to teach: Constructing meaningful understanding of mathematical content (NCRTE Report No. 90-12). East Lansing, MI: NCRTE.

Lubinski, C.A. (1990). Cognitively-guided mathematics instruction and teachers' decision making (Doctoral dissertation, University of Wisconsin-Madison, 1989). Dissertation Abstracts International, 50, 09A, p. 2773. (University Microfilms No. DA 8917661).

Peterson, P.L., Fennema, E., Carpenter, T.P., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. Cognition and Instruction. 6 (1), 1-40.

COOPERATIVE LEARNING IN RESPONSE TO AN INNOVATIVE CURRICULUM AS A MANIFESTATION OF CHANGE IN TEACHING PRACTICE

Thomas G. Edwards, Wayne State University

This paper traces the development of cooperative learning strategies in the practices of two middle school mathematics teachers who implemented an innovative mathematics curriculum with their 7th and 8th grade students. The evidence suggests that these teachers increased their use of cooperative learning strategies, and this increase was related to the nature of the curriculum materials, as well as to their views of the nature of mathematics and its teaching and learning. Moreover, both teachers appear to share a similar view of the social context of the classroom.

A critical problem facing mathematics education reform is the translation of a vision of mathematics teaching and learning contained in the National Council of Teachers of Mathematics (NCTM) *Standards* documents (NCTM, 1989, 1991) into actual practice in classrooms. This vision suggests learning environments that require substantive changes in current norms of teaching practice (Brown & Borko, 1992; Schifter & Fosnot, 1992; Simon, 1994; Simon & Schifter, 1993; Thompson, 1992). Among questions associated with the current reform, Nelson (1993) asks about the role of innovative curricula. Speaking at the Research Presession of the 73rd Annual Meeting of NCTM in Boston, Susan Jo Russell, in what she terms a hopeful observation, suggests two ways in which curriculum materials can be powerful tools for reform:

- Innovative curricula allow teachers to focus on the particularity of their own classrooms, and
- Innovative curricula support teachers' efforts to establish mathematical communities (Lindquist, et al, 1995).

This research report will trace the development of cooperative learning strategies in the practices of two middle school mathematics teachers who implemented an innovative mathematics curriculum.

Methodology

In the fall of 1992, a large, urban school district in the northeast began a district-wide implementation of an innovative mathematics curriculum in grades 8-10. The materials implemented were the University of Chicago School Mathematics Project (UCSMP) *Transition Mathematics* (grade 8), *Algebra* (grade 9), and *Geometry* (grade 10). Teachers involved in the UCSMP implementation received some staff development. This took the form of a series of eight Saturday morning paid inservice meetings spaced at 4 or 5 week intervals. The content of these inservice meetings typically involved teachers presenting their reactions to the UCSMP materials, as well as training in instructional strategies supportive of the UCSMP materials. Cooperative learning strategies were among those in which such training was provided.

In the fall of 1992, a case study of an eighth-grade teacher who was implementing the UCSMP *Transition Mathematics* materials for the first time was begun. This case study was developed over the course of two school years, and in the fall of 1993, a second case study was opened. The second teacher was part of a pilot project testing the efficacy of teaching the UCSMP *Transition Mathematics* and *Algebra* courses over three years: grades 7, 8, and 9. She was also using the UCSMP materials for the first time and implemented about 60% of the *Transition Mathematics* materials in her seventh-grade classes.

These two case studies were developed using qualitative data. The data were triangulated across time and across three primary sources:

- Fieldnotes taken from observations of the teachers using the UCSMP materials.
- Interviews of the teachers.
- Reflective journals kept by the teachers.

Diane, the subject of the two-year case study, was observed 13 times while Gina, who was studied for a single year, was observed 8 times. The observations were spaced at approximately 4-5 week intervals. Each teacher was also interviewed on the day of the observations. The interviews were semi-structured using an interview guide (Patton, 1990) and were typically 45 minutes to an hour in duration. Both teachers also kept journals of their reflections regarding the UCSMP implementation. These journals were interactive with the researcher.

The data were analyzed using a constant comparative approach (Glazer and Strauss, 1967). Of particular interest was the question of what instructional changes, if any, these teachers would make in response to their implementation of the UCSMP materials. An attempt was made to ground results in the ongoing interpretations of the researcher (Strauss, 1987). The data were coded and sorted in a manner suggested by Jorgensen (1989). Conceptual categories were derived from the research questions, and themes that emerged from the data themselves provided keywords. Matrix arrays of the data were also produced to aid in understanding relationships. As theories emerged from the analysis, they were tested against the data and further refined.

Findings

As the studies unfolded and patterns began to emerge from the data, it became apparent that Diane and Gina were changing their instructional practices in quite similar ways. One of the most striking of these changes was the increased use, albeit for somewhat different reasons, of cooperative learning strategies by both of them.

The Case of Diane

Diane teaches 7th- and 8th-grade mathematics in a K-8 school and is the only teacher certified in secondary mathematics in that building. She was selected for the study in part because her responses to a survey questionnaire were fairly representative of a large group of teachers who would be using the UCSMP materials for the first time.

At the close of the school year prior to her first year of UCSMP implementation, in completing the survey questionnaire, Diane had indicated disagreement with the statement: Students working in cooperative groups can learn just as well as from whole class instruction. During our first interview at the beginning of the next school year, I asked Diane to elaborate her response:

> I guess in my experience, with these students, at least, I haven't come up with a way yet to get them to work in small groups that's productive.

When I probed further, Diane indicated that she would find cooperative learning groups acceptable, if she could be shown ways to bring such strategies to fruition in her classroom.

This interaction apparently kindled, or perhaps rekindled, Diane's interest in cooperative learning strategies, for in a November journal entry, she wrote:

> I've been reading some books on cooperative learning teams, and I've signed up for a cooperative learning inservice. I plan to really work at using these strategies in my classroom.

Diane also attended the cooperative learning workshop that was part of the series of UCSMP inservices mentioned earlier. Moreover, her regular use of the Teacher's Edition of the UCSMP textbook brought her into almost daily contact with subtle hints and suggestions that are provided in the Teaching Notes that accompany each lesson in the text. These notes cover a variety of topics, including small group work.

The fieldnotes of my observations of Diane's practice confirm that she attempted to use cooperative learning strategies, with some success, through the end of the second year of the study. During an interview near the end of the first year, I asked Diane if working with the UCSMP materials had influenced her instructional practice in any ways.

> I think that through the UCSMP, I started to use cooperative learning more. And that was one thing that I had wanted to do.

The Case of Gina

Gina teaches 7th-grade mathematics in a K-8 school that has a large proportion of students for whom English is a second language (ESL). She was selected for the study due to the possible tension between the large number of ESL students in her classes and the reading requirements of the UCSMP materials.

During our first interview, prior to her use of the UCSMP materials, I asked with what importance she viewed cooperative learning strategies:

> In the past I haven't used it, definitely not most of the time, and probably not half of the time. I don't think I'm going to have much choice with this program. I'm just going to have to have them work in groups, just to get as much experience as possible ... because of the pace of the course; because of the whole setup of everything.

In this instance, I interpreted Gina's remarks to indicate that she viewed the UCSMP materials, with their heavy reliance on student reading, as problematic due to the large number of ESL students in her classes. She then decided to use cooperative learning groups, which provide a format for students to use each other as resources, as a response to this problematic situation.

Indeed, the fieldnotes of my observations of Gina's work confirm her use of such learning contexts. Her use of cooperative strategies was often limited to pairs and triads, but for activities such as Think-Pair-Share (Davidson, 1990) and semantic mapping (Carrell, Pharis, & Liberto, 1989), larger groups were utilized.

Toward the end of the school year, I asked Gina to tell me about any changes in instructional practice that she might have made that were related to her use of UCSMP materials. Without hesitation, at the top of her list was, *I've used more group work.*

Teacher Beliefs

Thompson (1992) acknowledges the relationship between teachers' beliefs about the nature of mathematics and their instructional practices that is suggested by a number of studies. Ernest (1989) proposes three main categories of views of mathematics, which he characterizes as Platonist, instrumentalist, and problem solving. Ernest also notes that mathematics curriculum reform efforts are often based on mathematics perspectives and links the current reform efforts with the problem solving view.

Diane and Gina both selected the problem solving description of mathematics from among Ernest's three views as most closely describing their own view of mathematics. When asked to generate adjectives or adverbs to describe mathematics, both offered "broad;" Gina suggested "art" and "science;" while Diane chose "logical" and "invented." Moreover, in choosing words to describe the processes of learning and teaching mathematics, Gina offered "collaborative," "fun," and "interesting," and Diane "inexact," "challenging," and "never-ending." These selections seem to fit well with both Ernest's problem solving view of mathematics and the underlying philosophy of UCSMP, which embraces the NCTM *Standards* (UCSMP, 1992).

Discussion

Both Diane and Gina attribute their increased use of cooperative learning strategies to their use of the UCSMP textbook. It seems no accident that their espoused views of mathematics are a good fit with those of the NCTM *Standards* authors. In addition, these teachers were observed to make changes in instructional practice that suggest a transition in their conception of the role of the teacher away from one as a "transmitter of knowledge" and toward one as a "facilitator of learning." An increased use of cooperative learning strategies was one embodiment of that transition.

The observation that the teachers' use of the innovative UCSMP materials fostered changes in teaching practice may be interpreted in a manner that is consistent with constructivist thinking. These teachers' daily interactions with the innovative textbook and materials, their students, and their students' reactions to the materials required them to interpret the innovation on a regular basis. This may well have provided a source of continuing perturbation in their understanding of their own practices, and the resolution of any such perturbation may well have resulted in changes in instructional practice.

These results also seem to fit well with several points in Ernest's (1991) analysis of the influence of social context on teaching practice. Among the factors that Ernest includes in the social context are the textbook and the curriculum. If there is indeed a good fit between these teachers' belief structures and the epistemology that underpins the work of UCSMP, it is likely that the use of these materials acted as a powerful catalyst for change in these two teachers' instructional practices.

In his analysis, Ernest (1991) notes that "the socialization effect of the context is sufficiently powerful that despite having differing beliefs about mathematics and its teaching, teachers in the same school are observed to adopt similar classroom practices" (p. 289). However, it is likely that such a constraint did not arise in the case of Diane, because she is the only secondary mathematics teacher in her school, nor in the case of Gina, who has little day-to-day professional contact with the only other secondary mathematics teacher in her school. Moreover, if my interpretation is accurate, both Diane and Gina were changing their conception of their role as teacher, and the nature of that transition toward more of a facilitator's role suggests that they viewed the social context within their own classrooms as more of an opportunity than a constraint to change.

Conclusion

The research herein reported suggests that innovative textbooks and curriculum materials might serve as catalysts for change in instructional practice, even when implemented with minimal support. The teachers in this study appear to embody both of Russell's observations previously noted. In focusing on the particularity of their own classrooms, they seemed to view the development of mathematical learning communities as an advantageous change, while the innovative curriculum materials that they were using supported their efforts to establish such communities.

One extension of this research is to ask the question, "What if an innovative textbook or curriculum is implemented together with a high level of support for teachers' attempts to bring their practices in line with the innovations?" Developers provide some support for innovative materials which are currently available. The role in the process of change of both the innovative materials and the adjoined support efforts, as well as their interaction, deserves our attention.

References

Brown, C. A. & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209-239). New York: Macmillan.

Carrell, P. C., Pharis, B. G., & Liberto, J. C. (1989). Metacognitive strategy training for ESL reading. *TESOL Quarterly, 23*(4), 647-678.

Davidson, N. (1990). Small-group cooperative learning in mathematics. In T. J. Cooney & C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s - 1990 Yearbook* (pp. 52-61). Reston, VA: National Council of Teachers of Mathematics.

Ernest, P. (1989). The knowledge, beliefs, and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching, 15*(1), 13-33.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer.

Glazer, B. C. & Strauss, A. L. (1967). *The discovery of grounded theory*. Chicago: Aldine.

Jorgensen, D. L. (1989). *Participant observation: A methodology for human studies*. Newbury Park, CA: Sage.

Lindquist, M., Ferrini-Mundy, J., Russell, S. J., Tate, W., Usiskin, Z., & Lappan, G. (1995, April). *The NCTM standards six years later: Where are we with mathematics education reform.* A panel discussion conducted as the Plenary Address at the Research Presession of the 73rd Annual Meeting of the National Council of Teachers of Mathematics, Boston, MA.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Nelson, B. S. (1993, April). *Implications of current research on teacher change in mathematics for the professional development of teachers.* Paper presented at the annual meetings of the National Council of Teachers of Mathematics, 1 April 1993, Seattle, WA.

Patton, M. Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage.

Schifter, D. & Fosnot, C. T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform.* New York: Teachers College Press.

Simon, M. A. (1994). Learning mathematics and learning to teach: Learning cycles in mathematics teacher education. *Educational Studies in Mathematics, 26*, 71-94.

Simon, M. A. & Schifter, D. (1993). Toward a constructivist perspective: The impact of a mathematics teacher inservice program on students. *Educational Studies in Mathematics, 25*, 331-340.

Strauss, A. L. (1987). *Qualitative analysis for social scientists.* Cambridge: Cambridge University Press.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

University of Chicago School Mathematics Project. (1992). Professional sourcebook for UCSMP. In Usiskin, Z., Flanders, J., Hynes, C., Polonsky, L., Porter, S., & Viktora, S., *Transition mathematics* (Teacher's Edition) (pp. T19-T52). Glenview, IL: Scott Foresman.

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THE DEVELOPING ROLE OF TEACHER: ONE PRESERVICE SECONDARY MATHEMATICS TEACHER'S BELIEFS AND EXPERIENCES

Jennifer Chauvot, The University of Georgia
Pamela Turner, The University of Georgia

This study followed one preservice teacher, Liz, as she progressed through her senior year of a secondary mathematics education program that illustrated and encouraged views congruous with the NCTM *Standards*. At the start of the program Liz saw her role as one in which it was her responsibility to create a classroom environment defined to be non-intimidating, non-frustrating, interesting, and motivating for her students. Throughout the year, her beliefs defining her role of teacher were strengthened by the program. In addition, a belief in the use of problem-solving activities evolved from her beliefs of her role of teacher. This belief in problem-solving activities paired with her student teaching experience caused Liz to re-examine some of her earlier beliefs. We will follow Liz into her first year of teaching to see how or whether this evolution continues.

Preservice teachers enter mathematics education programs with preconceived notions or ideas about the role of teacher in the classroom. As mathematics education programs continue to implement and encourage the underlying ideas and concepts espoused by the NCTM *Standards* (1991), preservice teachers are encouraged to develop and identify their role as teacher. Their initial notions, often constructed through their own classroom experiences, are the beginning of a more structured development and identification of beliefs about teaching. This study was conducted as part of the Research and Development Initiatives Applied to Teacher Education (RADIATE)[1] project. We followed one preservice teacher, Liz, as she progressed through her senior year of a mathematics education program that illustrated and encouraged views congruous with the NCTM *Standards*. At the start of the program Liz saw her role as one in which it was her responsibility to create a classroom environment defined to be non-intimidating, non-frustrating, interesting, and motivating for her students. Throughout the program her beliefs defining her role of teacher were strengthened. In addition, a belief in the use of problem-solving activities evolved from her beliefs of her role of teacher. This new belief combined with earlier notions and her student teaching experience, caused Liz to re-examine some of her earlier beliefs.

We used Green's (1971) theory of belief systems to help us organize and understand Liz's beliefs and how they were structured. Considering a quasi-logical structure, Green described beliefs as either derivative or primary. A derivative belief is a belief that follows from, or is derived from other beliefs. For example, a teacher may have a belief of frequent use of cooperative learning. The teacher

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[1] RADIATE was directed by Dr. Thomas J. Cooney and Dr. Patricia S. Wilson and funded by the National Science Foundation (grant # DUE9254475) and the Georgia Research Alliance. Any opinions or conclusions expressed by this report are those of the authors and do not necessarily represent an official position of the funding agencies.

may argue that this belief followed from a belief that one needs to be able to function as part of a team to be successful in the real world. If the teacher cannot provide a reason, or argues "it just is", then the belief is described as primary.

Psychologically, beliefs are either central/core or peripheral. Green used concentric circles as a model in which the interior circles represent psychologically central beliefs and the exterior circles represent psychologically peripheral beliefs. The teacher's teamwork belief (to do well one must be part of a team) may be held centrally, or psychologically strong. On the other hand, the belief may be held peripherally to a core belief that the teacher must prepare students to do well in the real world. If it is held peripherally, then it is not held as strongly and is more likely to be examined and perhaps changed.

Green (1971) uses evidentially and nonevidentially held beliefs to describe grounds of beliefs. A belief held nonevidentially is less apt to be modified despite evidence or reasons provided. The teacher may believe that teamwork is the way to go regardless of the success of individual workers. Evidentially held beliefs however are more susceptible to modification. They can be changed through the introduction of more evidence, for example, success of individual workers.

Methodology

Liz, one of fifteen preservice teachers of the RADIATE program, participated in two mathematics education courses, a practicum, one quarter of student teaching, and a post student teaching seminar. She was chosen for this study because of her willingness to participate and share her views on teaching and learning. Data collection came from several sources. At the start of the study, Liz completed an initial survey that asked her to reflect on her views of mathematics and her views of the teaching and learning of mathematics. She submitted journal entries weekly the first, second and fourth quarter. Journal questions focused on reflections of course experiences, how they related to herself and to her teaching. Coursework artifacts (papers, exams) and student teaching artifacts were also collected. Nine audiotaped interviews were conducted throughout the year, one of which was a card-sort interview. For the card sort Liz highlighted passages from her first seven interviews that she felt were important. She defined "important" to be what she thought was valuable. Her choices were placed on cards and she was then asked to sort them into categories which she defined. Lastly, Liz was observed in separate field experiences: team-teaching a technology enhanced lesson, team-teaching a week long unit, and her ten week student teaching assignment.

Liz's Developing Role of Teacher

Liz entered the program with many notions of teaching in mind. She described the characteristics of a good mathematics teacher as follows.

> A good mathematics teacher can explain one problem in several ways, can deviate from the lesson plan to meet the needs of his/her students, can help the students visualize with the use of dia-

> grams or props, can vary teaching to increase interest and motivation, can spend extra time with students, is patient and flexible, verifies comprehension before leaving one topic, is aware of his/her body language and comments toward students in order to not disencourage [sic] students from learning, is comfortable with his/her mathematical knowledge, is always properly prepared to teach. (initial survey, 3/29/94)

In the first interview, Liz reiterated these characteristics and stressed the responsibility of the teacher to behave in the above manner. From this, we began to identify some of Liz's beliefs about teaching, specifically her beliefs about her role of teacher. A core belief seemed to be that it was her responsibility to create a classroom environment that demonstrated the above characteristics. This classroom environment was defined by three other beliefs: students should not be intimidated or embarrassed, students should not be frustrated, students should be kept interested and motivated. These four beliefs and Green's (1971) theory helped us understand Liz's reflections and actions as she shared her perception of her role of teacher.

Liz based her three beliefs defining the environment on her experiences as a student in the classroom. She made references to classes where the instructor had been intimidating, material was not explained clearly (causing frustuation), and to classes that were boring. These served as counterpoints and defining elements of what she considered good teaching to be.

Liz's role of teacher became more and more defined as she expressed how she would fulfill her responsibilities. For her, the creation of a classroom where students did not feel intimidated could be achieved using the following strategy.

> If the teacher, on the first day, randomly picked someone to come up front and do a problem on the board and you know, do this. And if they embarrass themselves, it's no problem. You just keep going and, you know, everyone's going to get the chance to embarrass themselves. And it just becomes that kind of environment ...it encourages questions and someone's more liable to raise their hand and say, "I'm confused." You know? (1st quarter interview 4/7/94)

This strategy remained consistent throughout our year with Liz. This belief that students should not be intimidated seemed centrally held and had a primary structure (Green, 1971). It was not subject to change. To Liz it was common sense to believe that part of an ideal classroom environment would include students who did not feel intimidated or embarrassed to ask questions.

It was also common sense to believe that students should not feel frustrated in a classroom. Liz had two strategies that fulfilled this belief which again helped define her role of teacher. The first was for the teacher to demonstrate flexibility. Flexibility was defined to be an ability to "deviate from the lesson plan to meet the needs of his/her students". This was, of course, contingent upon students' willingness to ask questions. Liz expressed this belief throughout the year and demonstrated it while she was student teaching. (The text in italics represents passages Liz highlighted in the card sort interview.)

> I was gonna go over everything again before I talked about standard error. But umm they, they, it sounded like they knew what they were doing so I went ahead to standard error and we covered everything and they were ready to try out what they knew. And um, so I was, it ended up being a lot more organized 'cause when I started, *the lesson was all dependent on what they remembered from the day before.* (student teaching interview, 2/23/95)

Flexibility had been a part of her plan. She would not have gone on with new material had her students expressed confusion. Her second strategy to reduce frustration of her students was to present and explain material clearly. She demonstrated this consistently throughout the program. Her mathematics education courses included several activities which were open-ended and provided limited direction. In reflecting on such activities and how or if they would be used in her classroom, Liz consistently modified the activities to provide more direction. This belief of providing direction appeared to be peripherally held, contingent on the level of student. In her second interview she mentioned different approaches for different level students. She was asked to elaborate.

> Uhm, for the advanced students, probably more challenging, more individual or group work that doesn't show as much an objective and they figure it out for themselves. The general classes, maybe more give them a lead, give them an objective of what we're working on so that they're going in the right direction, you know, they can still work on the problem solving *but they're at least led in the right direction kind of because I feel like they're more likely to maybe get discouraged and quit rather than the advanced student.* (1st quarter interview, 5/24/94)

However, student teaching data did not illustrate this differential treatment. Although activities in the higher level courses that Liz taught were open-ended explorations, handouts she provided with the activities were very structured and leading. This suggested that perhaps she held her belief of low frustration levels of her students more strongly than we thought, representing a core belief (Green, 1971).

According to Liz, her third belief in defining her ideal classroom environment, students should be kept interested and motivated, could be fulfilled by varying her teaching styles. In the fourth interview, Liz discussed the use of lecture.

> *I think that sometimes that that's the best way depending on your time, the size of your group, and everything, and the material you have to teach, then I think that that could easily be the best way. Also I think it's good to vary the way you teach just the*

> *same.* I mean I think if they get used to activities and how you're gonna test them and everything, then they can easily get into that group and start slacking so you know? *If they never know what they're gonna get when they come into the room, it's probably a little better.* (2nd quarter interview, 10/13/94)

Although Liz's experiences as a student were mostly in classrooms that had been teacher-centered and textbook-based, Liz was very receptive to the multiple teaching styles demonstrated in the program. We suspect her belief of keeping the classroom environment interesting and motivating, coupled with mathematics education course experiences, provided a catalyst for a belief in the use of group work in the classroom.

> It's easy to just get drawn into the normal way of presenting the material, teaching it step by step you know. The normal, what the book says, the books suggestions of teaching it...but I would like to throw in group activities and more exploration on the student's part...*I think it definitely helps their learning a lot so (pause) cause it's helped mine just in our class.* (1st quarter interview 5/24/94)

Related to this belief in group work was a simultaneously evolving belief in problem solving. It began with her view on word problems.

> I really do like word problems, but as far as that being the main point of it, I think that the students, if they learn their math they're going to be able to apply it...I don't think that it has to be a number one stress in the classroom. (1st quarter interview 4/7/94)

At the start of the program, the terms *word problems* and *problem solving* were interchangeable. Her belief in word problems seemed to be evidentially held in that she was successful in her mathematics courses, and word problems did not play a significant role in this success. She was also working under the assumption that her students would learn the same way she did. However, as the program progressed, more experiences forced her to re-examine her beliefs. She began to realize not everyone is the same as she, word problems and problem solving are different, and there is a lot to gain from problem-solving activities.

> I probably will not be teaching many students with my perspective of math [enjoyment, success]. By observing at [a local high school], I am learning the different students' perspectives...I have learned that what worked with me will not necessarily work with everyone. (1st quarter journal 5/10/94)

At the end of the first quarter, she differentiated between the terms "word problems" and "problem solving." Word problems were what she encountered as a high school student - problems with words at the end of each chapter. Those problems were routine once you extracted the information. However,

> [Problem solving is] *more like you have to think of your own method of doing it* [not just follow the directions]...[problem solving] *is really important to just sort of bring everything together for the students to see how it connects and how it's not just being used, you know, in one specific area, that it can be applied to other things and that all the concepts can be put together to solve a large problem, you know, that they work over a period of time...* (1st quarter interview, 6/2/94)

Time constraints of using problem-solving activities and covering school curriculum were a consideration. Her practicum experience, which occurred between the first and second quarter, allowed her to see group work and problem-solving activities in action. "*I was sold 100% because they learned more from that than it looked like they were learning from lecture*" (2nd quarter interview, 10/13/94). This acceptance of the use of problem-solving activities was easily derived from her belief of keeping the classroom interesting and motivating. Not only that, but Liz's concern about time constraints was diminishing by the end of her student teaching experience. Under the guidance of her cooperating teacher, she saw flexibility in the curriculum.

> *We've got two weeks to teach these three main topics that we'll expand into other stuff and overlap through activities. So suddenly it was like I've got more than enough time to do this.* (4th quarter interview, 4/20/95)

Problem solving was the largest of ten categories Liz formed in the card sort interview. Over 20% of her cards were placed in this category. She saw problem solving as mathematics that was not contained in school mathematics, but she would be sure to include it in her teaching. In the last quarter, Liz defined problem solving as the heart of mathematics. She added "if you have good problem-solving skills, then you can tackle a lot of things mathematically as well as in other areas" (4th quarter interview, 5/30/95).

Liz's view of problem solving interacted with her belief in her role of providing direction to her students. In problem-solving activities her role was to "*point them in a direction...but not tell them where to go with it...*" (student teaching interview, 2/23/95). Her belief in *how much* direction to provide was unstable, peripherally held to the core belief of keeping her students from being frustrated. Lack of direction on the teacher's part will cause confusion and frustration in the students. During student teaching, it was brought to her attention that perhaps she was providing too much direction.

> Well I don't realize when I'm doing that, but uh my cooperating teacher kept thinking that I did, you know? *That I was giving them too much direction where they're going, but I guess that's*

> *just getting to know your students and knowing what they're capable of doing, you know?....[I need] to find a balance. Give them just enough direction so that they feel like they have found their answer on their own and they feel confident...But then not giving them too little so they don't get frustrated.* (4th quarter interview, 4/20/95)

Liz was examining her belief in providing direction for her students and how it related to her belief in problem-solving activities. We will follow Liz into her first year of teaching to see how these two beliefs continue to interact.

Final Comments

Green's (1971) theory of belief systems provided a perspective for organizing the structure of Liz's beliefs of her role of teacher in the classroom. This in turn helped us consider experiences that promoted modification of her beliefs. If we are receptive to preservice teachers' beliefs as they enter mathematics education programs, we may provide catalysts that could promote change and growth in beliefs congruous with the NCTM *Standards*.

References

Green, T. F., (1971). The activities of teaching. New York: McGraw-Hill Book Company.

National Council of Teachers of Mathematics, (1991). Professional Standards for Teaching Mathematics. Reston, VA: The Council.

HIGH SCHOOL TEACHERS' EXPERIENCES IN A STUDENT-CENTERED MATHEMATICS CURRICULUM

Melvin R. (Skip) Wilson, University of Michigan
Gwen Lloyd, University of Michigan

Three mathematics teachers and ten of their ninth grade students were observed and interviewed during a six-week period. One teacher claimed that her main challenge implementing a student-centered curriculum was her doubt that students would make the right connections without her explanations. Another teacher struggled with the dynamics of operating both small-group and whole-class discussions and ultimately decided not to hold whole-class discussions. A third teacher achieved a more equal balance between teacher-directed and student-centered activities. All three teachers demonstrated more difficulty than students changing their expectations about appropriate mathematical activity.

It is now well documented that many mathematics teachers communicate a narrow view of mathematics—as a set of fixed procedures to be mastered by students (e.g., Thompson, 1992). In contrast, reform movements, including many specific curriculum development projects in the United States, are guided by a growing consensus that school mathematics should be portrayed as an exciting subject to be understood and explored. This paper describes the experiences and conceptions of three high school teachers attempting to implement a student-centered mathematics curriculum that explicitly supports the assumption that mathematics is a vibrant and useful subject. The paper focuses on two main themes: (a) how teachers (and to a lesser extent, students) made the transition from a teacher-centered to a student-centered classroom, and (b) teachers' and students' beliefs about mathematics, particularly the mathematics suggested by a specific set of curriculum materials.

Because we were interested in describing how mathematics teachers' conceptions were related to their classroom decisions and actions, we considered research related to teachers' conceptions of mathematics and mathematics teaching (e.g., Thompson, 1992). Since we wanted to interpret teachers' and (to a lesser extent) students' conceptions, decisions, and actions in a climate of change, we investigated literature (both general and in mathematics) related to the intellectual growth of adolescents and adults (Belenky et al., 1986; Cooney, 1994; Copes, 1982; King & Kitchener, 1994; Perry, 1970). Expecting students to explore and understand mathematical ideas requires them to accept much of the responsibility or authority for determining, for example, appropriate procedures and methods to solve problems. We were interested particularly in learning about participating teachers' and students' beliefs about pedagogical authority (i.e., where the authority of mathematical correctness and understanding lies—with the teacher, the textbook, or the student) and so we focused on what this literature said about authority.

Design

Curriculum Materials, Participants, and Research Sites

Teachers and students in this study used materials generated by the *Core-Plus Mathematics Project (CPMP)*, a curriculum that encourages and supports teachers in organizing the classroom so students can explore mathematical concepts, work cooperatively, learn from each other, and think about and solve interesting problems (Hirsch, Coxford, Fey, & Schoen, in press). Three teachers (one female, two male) were selected from two high schools in neighboring public school districts located in a small northeast urban community. All three had taught for 10 or more years, but none had previously taught using *CPMP* materials. Ninth grade mathematics classes (one taught by each teacher) were observed daily for 5-6 weeks during November and December of 1994. This study took place while classes participated in a unit called *Patterns of Change*, the second unit in the ninth grade *CPMP* sequence. All three teachers were well-established traditional teachers (by their own admission) and recognized at the outset of the study that change would not be easy. Yet all were extremely enthusiastic about using the *CPMP* materials and incorporating the accompanying suggestions.

Data Collection

Teacher and student data were collected between September 1994 and June 1995 using interviews, observations, and students' and teachers' written work and plans. Two one-hour interviews conducted during September and October of 1994 investigated each teacher's conceptions of mathematical functions and of mathematics and mathematics teaching more generally. During a five- to six-week period in which classes were observed daily, each teacher was interviewed four or five times. These hour-long interviews allowed teachers to comment on recent classroom events. One group of students (3-4 students) was identified by each teacher as a target group. Students in target groups were observed and interviewed periodically during class sessions. After students had completed the observed unit, hour-long interviews with 10 target group students encouraged them to comment on their experiences in the observed classes. An hour-long interview at the end of the observation period (December 1994) assessed each teacher's immediate reflections on his or her experiences teaching the unit. To allow teachers to reflect on the entire year's experience, as well as enable them to comment on their resolutions for the next year, teachers were also interviewed at the end of the academic year (June 1995) in a group setting. Fieldnotes were taken during classroom observations and all observations were video recorded, with a cordless microphone carried by the teacher. Teacher and student interviews were audio recorded and transcribed for ongoing analysis. Photocopies were made of the written artifacts (e.g., student work).

Findings

Student-Centered vs. Teacher-Centered Activities

For all three teachers, a dichotomy existed between "a cooperative learning atmosphere" and "a teacher-centered" or "traditional" one. This dichotomy manifested itself in various ways as teachers contrasted their prior experience to what they were trying to do in the *CPMP* class. Ms. Gifford, like the other teachers, communicated an understanding of the intent of *CPMP* curriculum materials. In an early interview she explained: "In theory the teacher goes from being ruler or dictator who is disseminating all the information, to a facilitator. . . . As kids are doing work, you're there to assist." She further described the students' role in this model classroom: "[the mathematics is] something they can get out of their seats and put their hands on and have some ownership of the data that's being used. Rather than my giving them whatever equation, they come [up with the] relationship." However, Ms. Gifford felt like she did "too much front of the room instruction" in her *CPMP* class. Citing as her biggest challenge the fear that her students would not be able to make appropriate connections on their own, she frequently interrupted or even eliminated small group work. She lamented often feeling that she had to "jump in there and save them" and that she had difficulty "letting students go." Of the three classes we observed, Ms. Gifford's was indeed the most teacher-centered. Additionally, her students were the least inclined to describe their *CPMP* class as being radically different from other mathematics classes they had experienced. However, Ms. Gifford (as well as the other teachers) expects that her transition from "dictator" to "facilitator" will become easier in subsequent years, as she gains experience with the *CPMP* program.

Mr. Allen also communicated concern about student abilities, but in his class we observed a substantially different routine. During almost any given class period, after a brief introductory whole-class discussion, students worked in groups of three or four for the entire time. Mr. Allen circulated among the groups, answering questions and helping individuals and groups to stay "on task," but he rarely called the whole class together for sharing, questioning, or summarizing. One reason for his decision to permit extensive group work was that Mr. Allen wanted his students to learn to rely upon other group members (and themselves) instead of solely on him. However, he admitted that his decision to not interrupt group work was due mainly to his previous unsuccessful attempts to gather students together for whole-class discussions. Although he believed whole-class discussions were important, he felt uncomfortable with the "in and out" movement between group work and teacher-centered instruction and thus chose to explain and share important connections and generalizations with small groups of students. In an effort to further compensate for what he perceived to be inadequate teacher direction, Mr. Allen often supplemented curriculum materials with teacher-constructed review sheets.

Mr. Johnson was more inclined than Mr. Allen to interrupt small-group discussions to discuss important conclusions but he was not nearly as directive as Ms.

Gifford. Mr. Johnson attributed his ability to easily and comfortably lead transitions between student-centered and teacher-directed activities to his experience teaching computer science courses, in which he organized class as a sort of "lab." The following example is somewhat typical. Before the class period, Mr. Johnson made overheads of one group's work. During the class discussion, the students from this group stood in front of the class and pointed out connections between a graph, equation, and table they had generated. Mr. Johnson helped them do this by standing in the back of the room and asking questions, making comments, and providing encouragement, but he was careful to let students explain most of the important connections and conclusions.

Although Mr. Johnson considered himself to be very capable of leading effective whole-class discussions and he believed that such interactions were sometimes necessary (even teacher lectures), he was convinced that to meet the needs of all his students, he needed to let them do most things in groups. During interviews he frequently commented that many students thought that teacher lectures were "boring and confusing" but that group explorations were "refreshing." For example, during one interview he claimed that he lost "half his students when he talk[ed]." On another occasion, he explained that in past years 90% of his students were so "turned off" by his lectures that he could "do anything and they would just sit there." In such situations, Mr. Johnson would plead, "My God, you guys are dead!" and students would respond, "but it's boring." He agreed, describing traditional, teacher-directed classrooms as being very boring and ineffective.

A related issue concerns sources of pedagogical authority. The teachers communicated, both by the ways they taught and by the things they said, that they wanted students to take more responsibility for learning. For example, they insisted that students work cooperatively. Teachers wanted students to become less dependent on them (teachers) and more dependent on each other. During our post-unit interview we asked students to describe how, in this mathematics class, they would typically decide when they had done a problem correctly. Students who did ultimately refer to the teacher as a source of "correctness" (only about half did), did so after first explaining the important role of peers and discussion within groups to determine correctness. This result, together with our observation and teachers' claims that students preferred to work cooperatively on interesting problems (rather than in a teacher-centered environment), contrasts with the image described by Borasi (1990) of the "invisible hand" of students' expectations operating in mathematics classrooms. Borasi claims that students' expectations often encourage the adoption of a traditional, teacher-directed classroom model, despite teachers' efforts to do otherwise. Students in our study *enjoyed* the student-centered, problem-driven activities and had little difficulty adapting to them.

What Constitutes Appropriate Mathematics?

Participating teachers were deeply committed to change. As the previous section illustrates, to varying degrees, all three teachers were successful in changing their practices to incorporate student exploration and cooperation. Not surprisingly, during our end-of-the-year interview, all three teachers were able to identify positive and negative aspects of the *CPMP* and traditional curricula, and all favored the more useful understandings that students acquired in the *CPMP* curriculum. Although they did maintain concerns about public acceptance of the program (e.g., college admission, state testing), in general they were happy with the *CPMP* approach. But despite their claimed and observed shift, these teachers struggled longer than their students in changing their traditional views about what constitutes appropriate mathematics and mathematical activity. Further, the teachers' struggle to change (at least their early struggle) seemed to be masked by reference to the difficulty of getting students to change their expectations. From the outset of our study, teachers maintained that because *CPMP* classes were not like ordinary mathematics classes emphasizing traditional topics such as solving equations, factoring, etc., many students in those classes felt like they might "miss out." Although we observed some dissatisfaction and concern among students at the beginning of our study, during our formal interviews with target students at the end of the *Patterns of Change* unit (December, 1994), few of them commented that they were concerned with the non-traditional focus or content of the class. In fact, students reported *liking* the fact that the mathematics they were studying had meaning and application.

Discussion

Our results about teachers' and students' conceptions of what constitutes appropriate mathematics, as well as student conceptions about where the authority of mathematical correctness lies, point to at least two possible implications for curriculum and teacher development. First, individuals who develop and implement new curricula need to be aware that teachers often perceive student pressure or resistance to be stronger than it really is. Second, teachers who plan to do innovative things should understand that although student resistance is often strong at the outset, it lessens as time goes on.

Students in our study had a less difficult time than teachers adjusting to a student-centered classroom environment, one in which the teacher, to a lesser extent, was the ultimate authority. In one sense, this seems surprising. Adolescents are generally *more* inclined than adults to rely on outside authority for verification of legitimacy or truth. However, it is also the case that adolescents are less reflective than adults (King & Kitchener, 1994). Students did not struggle as much with the adjustment to a student-centered environment because they probably did not think much about it. To them, the authority (teacher) set things up that way so that is the way it was. On the other hand, the teachers were not only attempting to do something with which they were unfamiliar and felt considerable outside pressure to resist, but were actively thinking about the pros and cons of the new setup and whether it actually worked better. It is not surprising then that they struggled longer than students.

All three teachers commented that *CPMP* students were more difficult to bring together for whole-class discussions than students in other, more teacher-centered classes. This difficulty posed such a problem for Mr. Allen that he rarely attempted to gather students together, except at the beginning of class. Some teachers (particularly very traditional ones) may interpret this student tendency as a lack of respect or as evidence of a teacher's inability to appropriately "manage" the classroom (Mr. Johnson would not agree—he claimed that *CPMP* classes were *easier* for him to manage). But perhaps this tendency is simply an indication that students are accepting responsibility for their own learning. When students are given more of the authority or responsibility for learning, it is more difficult to "interrupt" their activities to do activities that are primarily teacher directed. This new classroom dynamic needs to be recognized and addressed by curriculum developers and others interested in reform.

References

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing*. Basic Books.

Borasi, R. (1990). The invisible hand operating in mathematics instruction: Students conceptions and expectations. In T. Cooney (Ed.), *Teaching and learning mathematics in the 1990s* (pp. 174-182). Reston VA: National Council of Teachers of Mathematics.

Cooney, T. J. (1994). Research and teacher education. *Journal for Research in Mathematics Education, 25*, 608-636.

Copes, L. (1982). The Perry development scheme: A metaphor for learning and teaching mathematics. *For the Learning of Mathematics, 3*, 38-44.

Hirsch, C. R, Coxford, A. F., Fey, J. T., & Schoen, H. L. (in press). The Core-Plus Mathematics Project: Teaching sensible mathematics in sense-making ways. *Mathematics Teacher.*

King, P., M, & Kitchener, K. S. (1994). *Developing reflective judgment*. San Francisco, CA: Jossey-Bass.

Perry, W. G. (1970). *Forms of intellectual and ethical development in the college years*. New York: Holt, Rinehart, and Winston.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan Publishing Company.

MATHEMATICS FOR ALL STUDENTS!
MATHEMATICS FOR ALL TEACHERS?

Ronald V. Preston, East Carolina University
Diana V. Lambdin, Indiana University

Mathematics for all students is a goal of the current mathematics education reform movement. But is today's reform also workable for all teachers? We profile two teachers who dropped out of field testing an innovative middle grades mathematics curriculum. The teachers are of interest because (1) their mathematics backgrounds seemed strong, (2) their espoused philosophies seemed compatible with the innovation, and (3) their work environments provided considerable support. We detail their difficulties and offer recommendations for helping teachers succeed with curricular reform.

One of the goals of the current mathematics education reform (NCTM, 1989; 1991) is to help students and the general public understand that all students can learn mathematics. Just as important might be this question—are today's reform recommendations workable for all teachers? Successful traditional teachers are finding much of the mathematics they are confident about teaching de-emphasized in favor of new—often unfamiliar—topics, while their tried and true methods are being challenged as well.

As external evaluators in charge of the nation-wide field testing of materials from a reform curriculum (the NSF-sponsored Connected Mathematics Project—CMP), we are involved in studying teachers' adaptation to change. CMP units of study engage middle school students in learning mathematics through contextualized investigations—and in reflective writing and oral communication about the mathematical concepts they encounter. Our field testing data include a variety of pre-, mid-, and end-of-year questionnaires from teachers; questionnaires three times a year from students; teacher calendars of daily plans; and classroom observations and feedback from individuals hired as site recorders.

Teachers' reactions to CMP have been diverse. Some teachers enthusiastically accept the program, finding a match with their own philosophies. Others experience philosophical differences with the approach or content difficulties with the mathematics. However, most teachers seem to agree—after trying the CMP materials—that the new approach is worthwhile. Though there is much to be learned from these teachers described (we have previously written about them—Lambdin & Preston, 1993; 1994; 1995), stories of teachers who have dropped out of the project can also inform the research community. Two of the nearly 100 teachers involved with CMP for at least one year—whom we will call Hannah and Laura[1]—dropped out during 1993-94. Their cases intrigue us because we initially believed they were strong candidates for field testing CMP. What went wrong?

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[1] Hannah and Laura are pseudonyms for real teachers. All quotes from them are actual quotes and all details are accurate, although—in the interest of maintaining their anonymity—we have omitted mention of certain details that might identify them or their schools.

After combing through our data, we developed a list of questions that we used in interviewing the recorders who had observed first-hand in Hannah's and Laura's classes. We also talked with Hannah and Laura themselves to get their perspectives on their CMP experiences. In this paper, we detail their difficulties and provide some recommendations for helping teachers succeed with curricular reform. (Due to page restrictions we have been reduced to including here only a small fraction of the evidence we have for many of our claims. Additional data will be presented at our PME presentation.)

Hannah's Case: Easier Said than Done?

Hannah came to the project as a teaching veteran of 18 years, with the past seven years being in middle school mathematics. Originally elementary prepared, Hannah earned her mathematics certification six years ago. Her philosophy and style of teaching (as self-described on a pre-project questionnaire) seemed largely in line with CMP:

> My main belief is that every student must be made to feel comfortable with themselves and the subject in order to grow and achieve. No question posed by a student is too trivial to warrant an explanation and can almost always be answered by another student. Understanding what to do is more important than calculations that are performed for tests.

Hannah's school provided plenty of collaborative assistance: regular meetings of all project teachers, a masters-level student/recorder who could help or observe in the classroom four days per week, a site director who offered content and methodological assistance, and a supportive administration. Hannah was (at least at first) a regular and productive contributor to weekly teacher meetings.

Hannah's dissatisfaction with the project came to our attention through strident self-report questionnaire responses at mid year; very different start-of-year and mid-year responses on the Stages of Concern Questionnaire (SoCQ) (Hord, Rutherford, Huling-Austin, & Hall, 1987); undeniably inflammatory comments from her students on mid-year questionnaires; and most of all, from observations of the site recorder. Hannah wrote, "students are uncomfortable with materials—explanations are not clear," and "Get rid of ACE [homework] #5—too difficult," and quizzes are "not clear enough to use." At mid-year, on her concerns questionnaire, Hannah marked "very true" for three items that we had previously identified as possible "red flag" indicators of dissatisfaction with the project:

> At this time I am not interested in learning about this innovation. I would like to know how to supplement, enhance, or replace the innovation. I would like to know how this innovation is better than what we have.

Students in Hannah's class reacted angrily on the mid-year questionnaire. Their responses to "What else would you like to tell the writers of CMP?" included numerous adolescent vulgarities and even one death threat. Fifteen of Hannah's 28 students had more negative comments at mid-year than four months before. The recorder believed Hannah was actually the source of the students' discontent, claiming "the students are confusing their [negative] feelings toward the teacher with their feelings toward the project."

In a tape-recorded interview, the recorder suggested that though Hannah talked quite knowledgeably about mathematics education reform, she actually taught very little mathematics in her CMP class, leaving her students to flounder uncomfortably:

> People coming in might not know it's a math class …she would talk about other things, not the math …students and parents had complaints that they weren't doing any math. She knows about NCTM Standards. She knows everything. You'd get so motivated when you'd talk to her but when you'd go to her class it was like a bombshell. [Site Director] was shocked too….She doesn't believe in telling them [students], she doesn't guide. She'd let them go for weeks not knowing what the real answer is. Some of them that really need to know, they'd leave her class even more confused, because there's never closure…she doesn't pose other questions to focus them.

Shortly after mid-year, it was mutually agreed that Hannah would stop field testing CMP. In our subsequent interview with her, Hannah claimed that the curriculum "was jumping around a lot," that "there was no provision in that book for practicing basics, which is what a 7th grader needs," that "there was a lot of resistance" from students and parents because "parents couldn't help," and that "the book was not math friendly, especially if the student had trouble reading." Though Hannah believed "most of the NCTM standards definitely fit with the [CMP] philosophy," she also believed that the standards confirmed her own teaching philosophy, which she claimed to have espoused for decades: "when I saw it in the NCTM standards, I said, 'hey, I've been doing the right thing.'" Hannah did not note any conflict between these statements and her comment that CMP "was just so radical to the students that I could not overcome the resistance."

Hannah confirmed the recorder's observation that she had provided almost no direction in her CMP class. She also confirmed that her students floundered, learned little, and became increasingly hostile and belligerent. We believe Hannah may have thought that minimal teacher direction was a prerequisite for a student-centered environment such as that espoused by CMP. Perhaps she was confused because she had not attended the week-long summer inservice that provided an introduction to CMP. (Hannah also refused to read the Teacher Edition because she preferred to learn along with the students.)

Hannah talks as if she believes in active, student-centered learning, but she is unable to actually pull it off in her classes. She also seems to suffer somewhat from conflicting beliefs, alternately claiming that rules, computational practice, and algorithms are important and not important. Though she claimed that "understanding what to do is more important than calculations that are performed for tests," she was distressed that "fractions were thrown in almost immediately [in CMP]. I know they should have had fractions, they should be able to remember that, but there was no provision in that book for practicing basics, which is what a seventh grader needs." Upon closer investigation, it became clear to us that pressure from a district-imposed standardized test that emphasizes a very algorithmic, computational approach created some of this conflict for Hannah.

We believe that Hannah is an example (albeit extreme) of a teacher who claims (and probably even believes) that she is a proponent of new ways of teaching mathematics, but who finds it very difficult to actually implement them appropriately in her own teaching. Though Hannah talks convincingly about reform in mathematics education, she shows quite clearly that such changes are often much easier said than done.

Laura's Case: "I Think It's Great, But . . . "

Laura had been teaching for 12 years when she began to field test CMP. Originally an English major, she had returned to college for classes leading to a K-9 license with reading specialization. Then, after one year teaching elementary school, she became a middle school mathematics teacher. She reported 52+ quarter hours of math/math methods (the mathematics classes for her elementary degree "so captured my interest that I continued to take them after earning my teaching certificate"). In recent years, she has been involved in Math in the Mind's Eye, the Middle Grades Mathematics Project, and spent a sabbatical year working on an alternative assessment project.

Laura field tested CMP materials for nearly a year-and-a-half before dropping out. The first year, she taught seventh grade. On an initial questionnaire, Laura wrote:

> Activity based mathematics is my favorite way to teach where kids look at a situation or problem. Kids have access to manipulatives at all times. They're encouraged to use them, make models or draw pictures to help solve problems. I like to relate mathematics to history and current issues or other situations.

Her philosophy seemed to align well with that of CMP. In fact, to talk to Laura, to read her questionnaire responses, and to listen to descriptions from her on-site recorder was to experience a teacher so enthusiastic that she could have been a spokesperson for CMP. For example, in her first year of CMP, Laura declared, "I can't help but feel my kids will be much better prepared for algebra next year than with the [other math] book." On other occasions she volunteered "I love Investigation 6 because of the history tie in," and "I *really* like the "Filling and Wrapping" play dough and rice [activity]. YES IT WORKS!"

On the other hand, Laura sometimes sent other signals. She was torn between using CMP or favorite activities from previous years. When asked if she supplemented CMP she always answered affirmatively (e.g., "*Yes*, almost daily [picture

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of sad face] 'sorry.' I need to work on sticking to the CMP material. It will save time."). Later, she wrote, "I try to use your materials faithfully, but I am readily distracted." She said that she supplemented CMP with "nifty, human interest things that spark excitement and connections." Laura also admitted spending insufficient time preparing, confiding that "my best lessons were ones that I'd thought about for a couple of days, not just in the car on the way to school" and "I need to see how do-able certain questions are for homework *before* I assign them. I've really screwed up on this when I didn't know."

Laura's SoCQ questionnaire revealed high concerns in every category (almost all above the 50th percentile)—in fact, higher than any of the 50+ other teachers for whom we developed concerns profiles. In her first year of CMP, she depended heavily on her recorder (who was also teaching CMP and willing to plan collaboratively). After her first year, when asked to provide advice for new teachers, she wrote "It *really* helped having [name] as our recorder. I was able to go to her with questions and concerns."

In her second year with us, Laura moved from 7th grade to 8th grade—where she taught one CMP class and several traditional algebra classes. Amazingly, by mid-November, Laura had spent only four days using CMP with her CMP class! Her calendar detailed the variety of things she had done instead (e.g., history of the word "algebra," pattern work, logic problems, fraction worksheets, survey project, film on Platonic solids, area bingo, area of silhouettes, and activities from two other sets of materials).

What contributed to this enthusiastic, if somewhat harried, CMP supporter's avoidance of the 8th grade project materials? We have identified several possible factors. The first and perhaps the most important factor is Laura's mathematical background. It appears that the mathematics of the eighth grade curriculum challenged her (see Ball, 1991) to the point where she was uncomfortable and thus avoided it. The eighth grade materials have a strong algebraic emphasis. Laura was experienced at teaching algebra traditionally, with an algorithmic approach, while CMP stresses learning algebraic ideas through investigation, which can be quite challenging. Upon further investigation we determined that Laura's alleged 52+ quarter hours of math/methods had actually focused much more on methods than on content. Her mathematics background was weaker than it looked. This helps explain why she was so often bothered, in both the 7th and 8th grade draft materials, by typographical errors and incomplete solutions. She was apparently quite insecure about teaching from materials that she could not rely upon for answers and explanations.

A second factor involves collaboration. In her second year of trialling CMP materials, Hannah was the only eighth grade teacher using CMP in her school. The recorder was no longer teaching and planning collaboratively with her, as in the previous year. This no doubt contributed to her decision to drop out of the project (see Little, 1987). A third factor relates to planning. By Laura's own admission she often waited until the last minute to put together a lesson, acknowledging that it "is really up to me to find the time."

A fourth factor is the student population. At the private school where Laura teaches, all the eighth grade students had taken algebra the previous two years, with mixed results. (Some ended up repeating the course in 9th grade.) Laura's 8th grade CMP class consisted of students counseled not to take algebra—the bottom 25% of the 8th grade. The recorder opined that Laura "believes that her kids won't 'get it.' She spends time going back over 'rules' for algorithms for rational number operations . . ."

Laura's case provides an example of a teacher who enthusiastically embraces the methods of reform, but whose limited understanding of mathematics, lack of collaborative support, and limited confidence when faced with errors in the materials led to insecurity, and eventually to avoiding the innovation entirely.

Discussion

Though Hannah and Laura had what looked to be above adequate mathematics backgrounds and appeared to agree philosophically with CMP, upon closer examination, we discovered problems in both these areas. Both women had become "mathematics specialists" through course taking and attendance at workshops, but the tenuousness of their mathematical expertise was revealed when they were confronted with unfamiliar mathematical ideas. Both talked enthusiastically and informatively about mathematics education reform, but had difficulty actually implementing their visions in the classroom. Hannah and Laura's cases concern us because indicators of trouble became apparent only in hindsight, when we began to look closely after they dropped out of the project.

Reform curricula seem likely to succeed only to the extent that teachers are helped to become knowledgeable and confident about mathematics content, and well supported in their efforts to use new methods of instruction (e.g., inservice and collaborative assistance). Content knowledge and pedagogical beliefs must be primary considerations for those who design inservice workshops and teacher manuals for innovative materials. It is also apparent that it is not easy to predict success with an innovation by the typical completing of forms and brief interviews. Use of the SoCQ, observations, and other means of identifying concerns and problems are important for identifying areas that can then be addressed (e.g., provide content assistance). Without significant efforts along these lines, it is looking more and more likely that reform success "for all teachers" may be an elusive dream.

References

Ball, D.L. (1991). Teaching mathematics for understanding: What do teachers need to know about subject matter? In M. M. Kennedy (Ed.), *Teaching academic subjects to diverse learners*, (pp. 63-83). New York: Teachers College Press.

Hord, S.M., Rutherford, W.L., Huling-Austin, L., & Hall, G. E. (1987). *Taking charge of change*. Alexandria, VA: Association for Supervision and Curriculum Development.

Lambdin, D.V. & Preston, R.V. (1995). Caricatures in innovation: Teacher adaptation to an investigation-oriented middle school mathematics curriculum. *Journal of Teacher Education, 46*, 130-140.

Lambdin, D.V. & Preston, R.V. (1994). How middle school teachers adjust to change: Classroom testing of materials from an innovative curriculum project. In D. Kirshner, (Ed.), *Proceedings of the Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 2 (pp. 217-233). Baton Rogue, LA: Louisiana State University.

Lambdin, D.V. & Preston, R.V. (1993). Caricatures in innovation: Three "middle school teachers" try teaching an innovative mathematics curriculum. In J. R. Becker & B. J. Pence, (Eds.), *Proceedings of the Fifteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 2 (pp. 138-144). San Jose: San Jose State University.

Little, J. (1987). Teachers as colleagues. In V. Richardson-Koehler (Ed.), *Educators' handbook: A research perspective* (pp. 491-518). New York: Longman.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

ONE TEACHER'S LEARNING: A CASE STUDY OF AN ELEMENTARY TEACHER'S BELIEFS AND PRACTICE

David Feikes, Purdue University North Central

This is a case study of one teacher's beliefs, practice, and learning during his first year of participation in a problem-centered second-grade mathematics project. The teacher in this study, attempted to realize an alternative approach to teaching mathematics that differed dramatically from his former practice. This study looks at his former beliefs and practice, his teaching of mathematics during his participation, and the process by which he learned and consequently changed his beliefs and practice. The teacher in this study learned and consequently changed his beliefs and practice through his actual practice. Paradigm cases often consisted of alternative interpretations of classroom incidents.

Teachers' development of beliefs and knowledge is synonymous with teachers' learning. Teachers learn as they reflect on and reorganize their knowledge, and modify their previously taken-for-granted practices. Teachers, like students, are considered as active reorganizers of their experiences who actively construct knowledge. This social constructivist perspective on teacher development draws on student development (Bauersfeld, 1995; Cobb, 1989) as a source of analogies, a position that ties connected perspectives to a common, consistent, theoretical foundation of how people learn.

Teachers are viewed as learners, not as empty vessels to be filled. Mathematics educators often attempt to fill the vessel by supplying teachers with research knowledge or modeling the direct results. However only opportunities for teachers to learn can be provided. The most productive opportunities for teachers' learning arises in the course of their practice as they interact with students. Consequently, this is where teacher/researcher interaction is most vital. This paper attempts to illustrate how the teacher in this study learned and markedly changed his practice and beliefs.

The Case Study

Carl Willis, the teacher in this study, was interviewed and his classroom teaching was video-taped and analyzed over the course of a school year. At the time of the study Carl was teaching second-grade in an inner-city elementary school where he had taught various grade levels for 27 years. Carl expressed and exuded an exuberance for teaching, especially mathematics.

Carl taught arithmetic through extensive drill and practice with a heavy emphasis on flash cards. He indicated that he might do mathematical activities (e.g., flash cards) three times a day: in the morning, the regular lesson, and at the end of the day.

> Ok, I believe in them, [flash cards] drill over these, drill over these ... I would start in September. Every day, every day the good Lord says so ... Lot's of time, at least some of the time when we line up at the door, I would dismiss them by a fact, you

> know. For them to get out of the classroom, I would ask them [a flash card].
>
> They would really know that [nine plus eight] is 17. They didn't have to take the time... I believe they really understand the basic ones when they come to this [nine plus eight], they didn't have to do all this [counting], they just knew it.

For Carl, if a child knew his facts, that is, he could rapidly answer a series of flash cards correctly, then that child was learning. He believed mathematics was the application of known procedures and basic facts to compute solutions to problems that had already been predetermined.

Carl viewed learning as a process that he could significantly influence. For him, it was important that children were exposed to something if they were going to learn it. For him, learning was like exposing film to the light; an image is left on the film and the more times the film, **or student**, is exposed to the light, the stronger the image becomes. Carl believed that learning consisted of memorizing and, as such, the sequencing and timing of mathematical content was not crucial to students' ability to memorize.

> I always teach my kids that they are the best second graders here at Lincoln School, no matter what class I have [referring to the ability grouping of classes]. To get them to think that. And by me drilling these, these cards every day ... And at the same time I'm telling you [his students], you are the best and they really believe that, so getting them to really think that they are, which they would be ... success, always success, always success you know, in these [the basic facts], every day in math class. I would make them feel like they'd really done something and they really would deserve that and just build that up into them.

Carl made his children feel successful by having them master the facts. More importantly, he believed that his children were successful in mathematics. He sought to instill this belief in his children. Carl attempted to build up the children's self image as a means of motivating the children.

Carl was a caring teacher who helped his students develop proficiency in memorizing the basic facts and through this he also attempted to build his own and children's self image. His focus was on the mathematics, the facts, and through his teaching of the facts he focused on the child. After 27 years of teaching, in which he considered himself to be an excellent mathematics teacher, Carl believed that his former practice was not problematic and that it might only require a slight enhancement. Since he thought of himself as an excellent mathematics teacher, he saw no need to change his practice. He had not volunteered for the project but assumed that he was selected because of his expertise in teaching mathematics.

Carl's Practice

His students had the greatest opportunity to express their mathematical ideas in the initial whole-class activities which were designed to generate a variety of responses and solution methods.

The class was working on Double Ten Frames. Carl had placed four red chips in the left frame and six green chips in the right frame. Carl asked the class what they saw and how many chips there were in all. After they agreed that there were ten in all, the class discussion centered around the idea that one could say four plus six equals ten or six plus four equals ten. Theresa indicated a different way to express the same idea.

Th: You could say. ... it's six on this side [pointing to the right frame] and take one from that side [and] put it on the red side [the left side] ...

T: Listen to her.

Th: And [you] would have five plus five.

T: All right. Do you understand what she [said]? I like that. She said, if we were to take one of these green and put it over here with, with the four [pointing to the four red chips] ... you could say five plus five. That's good! (9/25/89)

Carl had created an atmosphere where Theresa could express her ideas. In general, as long as his students arrived at the correct answer, Carl encouraged their creative thinking. As students expressed their varied mathematical ideas he began to see how the instructional activities encouraged students to develop competency and understanding of the basic facts. Although, Carl was still the authority in the classroom, his students began to express their mathematical ideas. Their thinking became increasingly accepted and valued in his classroom.

Carl also became more knowledgeable of the ways that his students used to solve problems and he began to see that there were several ways to solve particular problems. With this new knowledge, he began to have his students explain their solution methods in more detail and to encourage alternative methods. This in turn gave rise to learning opportunities for his students as they explained their thinking.

For example, Carl began one class with a warm-up activity using balances. He wrote several balance problems on the overhead and asked the students how they solved each problem.

T: Let's put, six and seven. Sheryl? [Carl put a six and seven in two boxes on one side of the balance and a blank box on the other side.]

Sh: Thirteen.

T: How did you know? Oh I like that. I love it ...

Sh: 6, 7, 8, 9, 10, 11, 12, 13.

T: All right would you [to Theresa] like to tell us something different? ...

Th: I had six plus six is twelve and the six on the right, I just added one more to it.

T: I love it. (9/27/89)

Increasingly, Carl asked students to justify their solutions, "How did you know?". He was aware of how a less able student like Cindy might 'count on,' and a more able student like Theresa might use a thinking strategy. What was significant was that he encouraged both students to explain their methods and he was no longer satisfied with just the answer. In Carl's class, mathematical discussion was starting to mean: 'How did you solve the problem, if you have the right answer'.

The following episode contrasts with his prior practice in that he encouraged a student to explain an incorrect solution and he also refrained from directing her to the correct solution. He had written 9 + 10 + 11 on the board and asked the class to solve the problem mentally. In the preceding discussion, two students had given correct answers and their explanations.

T: Someone else who didn't get a chance from last time. Dana.

D: I had 31.

T: How did you get 31? [Carl's tone of voice was much softer than it had been in the past.] (12/15/89)

His actions indicated that he really wanted to know how Dana had solved the problem.

Dana went on to explain that 10 plus nine was 19 and that 19 plus 11 was 31. Carl asked her how she added 19 plus 11. She explained that she had counted. Instead of "straitening" her out he asked the class about her solution. Ralph indicated that 19 plus 12 is 31. However, she still insisted that 19 + 11 was 31.

T: Are you still going to stick with 19 plus 11 would be 31?

D: ... I think that's still 31.

T: If we take, I'm not going to say that! Ha ha. [Carl stopped his question and looked to the back of the room at the project staff member and laughed] (12/15/89)

Carl caught himself in the act of directing a student to the answer. He acknowledged his actions and laughed at his intentions. He had frequently attempted to lead students to a predetermined process, but this was the first time that he stopped to examine this approach. It was as he interacted with his students that the suggestions of the project staff made sense to him. Even though his new practice was induced by the suggestions and comments of the project staff, Carl himself had to do the reflecting and learning.

Carl's Beliefs and Learning

He no longer viewed the correct answer as the most important part of mathematics.

> That's another thing I've learned, too. I was too much hung up on what's right and what's wrong, getting the right answer, and that's not as important as how ... the child or the method they used to get their [solution].

He viewed the processes by which students solved problems as important and, hence, gave students the opportunity to express themselves.

The project staff had noted that one of Carl's slower students could only add by counting on his fingers. In his former practice Carl expected students to memorize the basic facts; now he was being asked to consider how this student could solve problems.

> When she [the project staff member] told me, what really struck me, Travis was only able to count up to five with his [fingers]... When a child, [Travis], will have to say five plus one, he really doesn't know what [he's doing]....He has to say one, two, three, four, five, six....[Drilling with flash cards] he has no real idea what he did....I was denying him the chance....He really didn't understand the relationship.

As this example illustrates, Carl learned and altered his practice as he learned more about how his students solved mathematical problems. His interactions with the project staff influenced him to question his taken-for-granted assumptions about children's learning and his teaching of mathematics. However, his interaction with his students was the primary source of his learning and it was here that the project staff's suggestions began to make sense for him.

Carl indicated that one key aspect of his learning was that he now listened to his students. This enabled him to learn more about how they learned mathematics and, in turn, how to teach mathematics.

> I really didn't listen to the children. I didn't give them an opportunity to express themselves. That's the key thing right there, to be patient and to give them time to express themselves. I think the whole thing is to give…the child an opportunity to…tell how they got their solution to the problem and which I had never really…given a child that [opportunity].

As he learned how his students solved problems, he saw a need for change. Specifically, listening to his students was the basis for much of his learning. This influenced his beliefs about how to teach mathematics.

The project staff attempted to make interventions that might influence Carl to reformulate his beliefs and practice. One means they used was to suggest to him what he might expect from his students. Salient experiences which influenced his learning were specific examples of his students' mathematical activity together with the interpretations offered by the project staff. As previously mentioned, Carl was amazed when he learned that Travis could only add by using his fingers. These became paradigm cases which Carl could verify in his actual practice. Carl had to learn to listen to Travis to verify the staff's assertions. As Carl learned to listen to his students, he became less dependent on the project staff. He learned about his students' mathematical understandings by interacting with them.

Implications

The Curriculum and Evaluation Standards for School Mathematics, (1989); & Professional Standards for Teaching Mathematics, (1991); have attempted "to establish a broad framework to guide reform in school mathematics in the next decade. In particular, they present a vision of what teaching should entail ...". The project in which Carl participated fits with the recommendations of the NCTM Standards. The NCTM Professional Standards for Teaching indicate what teachers should do and know. However, the Standards do not elaborate in detail how to support this change and develop this vision. "These standards focus on what a teacher needs to know about mathematics, mathematics education, and pedagogy to be able to carry out this vision of teaching" (p. 6). Simply providing teachers with the appropriate knowledge will not be sufficient to transform mathematics education as assumed in the Standards. Reform efforts are destined for failure unless teachers are viewed as active learners, are consequently provided with opportunities to learn in the classroom, and are provided with on-going classroom support.

Teachers learn from their actual practice. Paradigm cases, which often consisted of alternative interpretations of classroom incidents, were important in the teachers' learning. Teachers learned as they used alternative perspectives to explain and make sense of classroom events. In this study the project staff offered these interpretations of classroom events and attempted to encourage the teacher to question his taken-for-granted assumptions about teaching mathematics.

References

Bauersfeld, H. (1995). Development and function of mathematizing as a social practice. In L. Steffe and J Gale (Eds.), *Constructivism in Education* (pp. 137-158). Hillsdale, NJ: Lawrence Erlbaum.

Cobb, P. (1989). Experiential, cognitive, and anthropological perspectives in mathematics education. *For the Learning of Mathematics*,9, 32-42.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: NCTM.

ONE TEACHER'S STRUGGLE TO BALANCESTUDENTS' NEEDS FOR CHALLENGE AND SUCCESS

Margaret Schwan Smith, University of Pittsburgh

This study investigates a dilemma faced by an experienced teacher during the early stages of participating in a mathematics instructional reform project. The dilemma arises as the teacher's past practices come into conflict with new forms of instruction. Factors that assisted the teacher in dealing with the dilemma and arriving at a satisfactory resolution are identified.

Dilemmas or conflicts in teaching are situations that arise when a teacher is confronted with two alternatives, which seem to stand in contradiction to each other, neither of which is considered a solution. Lampert (1985) suggests that dilemmas are commonplace in teaching and grow out of the uncertain and unpredictable nature of classroom activity. She recounts a dilemma from her own fifth-grade classroom where her practice of teaching from the "boys' side" of the room, so as to ensure that the boys' behavior did not get in the way of mathematics learning, created a conflict since the girls came to believe that they were being ignored. She was confronted with two alternatives—make the girls feel that they were being unfairly treated or risk misbehavior from the boys—neither of which she saw as a viable option.

Lampert argues that teachers need to be thought of as dilemma managers rather than problem solvers, recognizing that some situations cannot be solved and that facing dilemmas need not result in a forced choice between competing alternatives. According to Romagnano (1994), "the image of teachers as dilemma managers is one that gives teachers themselves opportunities for their own professional growth and development" (p.14). In the case of the "order vs. equal opportunity" dilemma noted above, Lampert made the decision to begin the next unit by reorganizing the class into four small groups (two of each gender) and moving one group of boys to what had been the "girls' side" and one group of girls to what had been the "boys' side". With each group working individually with the teacher or the student teacher, there was less of a chance that the teacher's behavior could be viewed as preferential towards one group. In addition, it allowed the teacher to respond to the students as individuals rather than as members of a gender group. This strategy allowed the teacher to manage the conflict without making a forced choice.

Lampert (1985) suggests that "our understanding of the work of teaching might be enhanced if we explored what teachers do when they choose to endure and make use of conflict" (p. 194). The Second Grade Mathematics Project (Wood, Cobb, & Yackel, 1991) provides a context for exploring Lampert's question. Wood et al. (1991) contend that it is through the resolution of conflicts or dilemmas that learning occurs. The second grade teacher, who was the focus of their research, encountered conflicts which challenged her previous assumptions about her role as the authority in the classroom (e.g., using teacher-given procedures vs. student-

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created procedures). Through refléction on her practice, work with students in her classroom, and support from researchers, she resolved the conflict and came to develop a form of practice that placed an emphasis on students as constructors of mathematical knowledge. Hence, the conflicts with which she dealt provided occasions for new learning and in turn led to the development of an enhanced instructional practice.

Ball (1993) contends that while dilemmas can occur under any conditions, they are even more likely to occur when teachers begin to implement reform. What teachers are trying to do stands in sharp contrast to what teachers have previously done. New ways and old practices that seem contradictory put the teacher in the position of having to accommodate new knowledge and previous beliefs and ways of doing and knowing. Potential conflicts are inherent as teachers make significant shifts in the types of mathematics tasks used, the nature of the classroom discourse, the learning environment, and the analysis of teaching and learning (NCTM, 1991).

If mathematics education reform is going to take hold on a large scale, we need to facilitate the process by "being able to recognize certain familiar dilemmas, crises, or choice points and understand something about the typical range of routes through those points" (Goldsmith & Schifter, 1993, p.12). Understanding the dilemmas faced by teachers in the early stages of reform, the process by which a teacher deals with or "works through" the dilemmas associated with reform, and the factors that contribute to teachers "hanging in there" rather than abandoning reform, have implications for in-service education, especially with respect to the experiences and support needed to make this transition.

The purpose of this study is to investigate an instructional dilemma with which a teacher struggled during the early stages of implementing mathematics education reform in her classroom. Of particular interest is the situation which gave rise to the dilemma, the way the teacher dealt with the dilemma, and the factors that support successful resolution of the dilemma.

Method

Subject

The subject of this study is Ellen Hyde, a veteran teacher with 25 years of experience, who was participating in QUASAR — a national project aimed at improving mathematics instruction for students attending middle schools in economically disadvantaged communities in ways that are consistent with the recommendations of the National Council of Teachers of Mathematics (1989, 1991).

Data

The data, gathered during the first three months of the 1990-91 school year, provide information on classroom instruction, staff development, and personal perspectives of the teacher. Specific data from this period include 37 journal entries written by the teacher; videotapes of three consecutive teaching episodes;

interviews conducted with the teacher before and after the sequence of teaching episodes; and videotapes of two-full day staff development sessions in which the teacher participated along with her mathematics teacher colleagues and teacher educators from a local university (i.e., "Resource Partners") who provided support during project implementation. The resource partners encouraged Ellen and her colleagues to keep journals of reflections on the issues or concerns regarding their practice that were most salient to them.

Analysis

All data were systematically reviewed, and the dilemmas with which the teacher struggled were identified. An area the teacher repeatedly identified as problematic was then traced chronologically through the data sources. The trace involved looking for ways in which the dilemma manifested itself in the teacher's oral and written accounts of her practice and in actual classroom instruction and for factors that appeared to influence the teacher's perspective regarding the dilemma.

Results

One dilemma that emerged early in the first year was the tension between structuring learning opportunities so that students could experience success and facilitating the development of problem-solving skills. While the teacher stated that one of her goals for the year was to help students approach problem solving openly, she also felt that if the students were given tasks which were too hard or too frustrating, students would become defeated by their failures. If that happened, she felt that she would not have fulfilled her obligation as a teacher. Thus, she structured her lessons so that students would experience success. As she commented in a journal entry early in the school year:

> I can't buy the idea that kids don't feel bad starting off with what they perceive to be failure. When they have homework they can't do or don't have the confidence to do then I have to intervene... I will help kids do more verbalization in class, get to the kids who didn't volunteer and guarantee them success by asking them to do things they couldn't fail to do right. I can't ignore that success breeds success. Too many are starting out with what I'm sure they perceive to be failure. [September 16, 1990]

In the classroom, this "structuring for success" often led the teacher to reduce a complex task to a set of subtasks which presented little challenge to students and provide limited insight into student thinking. The lessons became directive, guided by low-level questions that were, in Ellen's words, "not designed for deep thinking, just success." This pattern was evident in the videotape of the first day of the three-day observation. When given the pattern train shown in Figure 1, rather than providing students with time to investigate the pattern—building the next train in the sequence, predicting some future train, and making general observations

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regarding the pattern — the teacher began by asking questions regarding each figure in the train, guiding students in their observations and eliminating any potential for struggle or discomfort. The teacher-student exchange regarding the first train is shown in Figure 1.

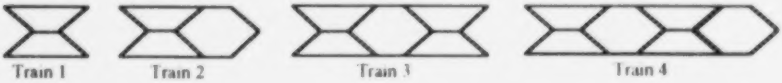

Figure 1.
Four Pattern Block Trains

T: Figure 1. Tim tell me one thing you and your partner noticed.

S: That there's 2 trapezoids and they're back to back and the small sides are back to back.

T: OK, would that be a pretty good description? He says "2 trapezoids back to back and the small sides are doing it." Anybody describe it differently, Mike?

S: A squished pop can.

T: And it looks like a squished pop can, all right. Keshia?

S: It has 4 equal sides.

T: The figure ends up having 4 equal sides there [pointing to the non-parallel sides of the two trapezoids]. And does it have another pair of equal sides?

S: Yes. And the outsides.

T: And the outsides.

S: Parallel sides.

T: The parallel sides are also equal. Right. Figure 2?

At a staff development session a few weeks later, teachers were given the opportunity to share a 10-minute segment of a videotaped lesson with their colleagues. Ellen volunteered to show a segment featuring the pattern block train shown in Figure 1. She indicated that she had discovered that her students weren't very good at observations, but that they had been verbalizing more since she broke it down for them, focusing their attention on each part of the sequence rather than on the entire series. One of the resource partners asked Ellen if the students had progressed to the point where they could make observations without "breaking it down." She asked Ellen how long you needed to break things down for students and questioned whether or not some of the observations would come out naturally if the students were given the opportunity to do so. Ellen went on to say that she hoped she did not always have to structure things, but that students were still at the comfort stage, and they needed this support. She believed that once students experienced success they would try harder and that then it would no longer be necessary to provide this support.

In her journal later that day, Ellen reflected on the comments that had been made at the staff development session: "I need to make sure I'm not structuring too much. It is easy to be too leading and feel OK about it because the kids seem happy. After all, many kids are happy with shut up and add." Over time, she continued to question this approach, wondering whether structuring the learning opportunities so that students were always successful would help students to become competent and confident problem solvers. As she noted in her journal:

> I have decided that when an activity is easy, maybe just reinforcing or practicing, the room is quieter. It is when they are being challenged that they get scared. I'm going to watch and see if this is so. I'm so used to the idea that that kind of confusion means I haven't introduced the lesson properly or have given the kids something too hard. Sometimes that is true, but sometimes it is necessary to go through panic before we find solutions. [November 13, 1990]

Later that month, after watching the videotape of the pattern train lesson in it's entirety, and responding to a set of reflection questions provided by the resource partners, Ellen commented:

> Students had ample opportunity to successfully predict visual pattern block trains in this lesson, but it was set up too much for success and not enough for the frustration that goes with problem solving...I now think I need to let them go through the frustration that goes with problem solving. The lesson probably wouldn't have looked as smooth, but I think it would have stretched the kids more. I am at a different point in my thinking than I was at the time of the lessons. [November 26, 1990]

This view of the fall teaching episode stands in sharp contrast to her perspective on the day of the lesson: "The lesson was all I could have asked of the kids...it is very exciting."

The teacher struggled to develop a practice that would honor both her concern that students not feel the frustration of not having immediate success and her interest in having her students learn to engage and solve challenging problems. Through reflection on classroom practice (i.e., watching videotapes and journal writing) supported and encouraged by the resource partners, and interactions with others providing support (i.e., resource partners), Ellen began to change her perspective regarding the dilemma and to consider ways of supporting student learning without reducing the complexity of the tasks.

Discussion

The teacher in this study was confronted with issues that are at the heart of instructional reform — how to challenge students in ways that will empower them as learners of mathematics and provide sufficient support to meet the challenges without reducing the demands of the task. It has been well documented that complex mathematical tasks are often implemented in ways that reduce the cognitive demands of the task (Doyle, 1988; Stein, Grover, & Henningsen, in press). This 'reduction of cognitive demands' often is the direct result of pressure exerted by students resulting from their frustration with the task (e.g., Romagnano, 1994). Goldsmith and Schifter (1993) suggest that an important question for teachers is whether or not "they are able to find ways to encourage and support students as they struggle with the limitations of their current ways of knowing" (p.11). An important question for teacher educators is how to create experiences that will help teachers build this capacity.

This study suggests that encouraging teacher reflection on practice, providing a non-threatening forum for discussing reflections, and providing on-going support to teachers during implementation may be important factors in building a teacher's capacity to cope with instructional reform. While the need for reflection and support are promulgated by many reform-oriented teacher education projects, the current study provides insight regarding the links between support, reflection, and change. Longer-term studies are needed that look at teacher change over time, that provide additional insight into how dilemmas "play out" over an extended period, and that begin to specify how teacher learning occurs through the management and resolution of dilemmas.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, 93(4), 373-397.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, *23* (2), 167-180.

Goldsmith, L. T., & Schifter, D. (1993). *Characteristics of a model for development of mathematics teaching*. Newton, MA: Center for the Development of Teaching, Educational Development Center.

Lampert, M. (1985). How do teachers manage to Teach? Perspectives on problems in practice. *Harvard Educational Review*, 55(2), 178-194.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Romagnano, L. (1994). *Wrestling with change: The dilemmas of teaching real mathematics*. Portsmouth, NH: Heinemann.

Stein, M.K., Grover, B., & Henningsen M. (In press). Enhanced instruction as a means of building student capacity for mathematical thinking and reasoning. *American Educational Research Journal*.

Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal*, *28*(3), 587-616.

RAISING VOICES: USING TEACHING CASES TO STIMULATE TEACHERS' THINKING AND REFLECTION IN MATHEMATICS EDUCATION

Linda Ruiz Davenport, Education Development Center
Annette Sassi, Education Development Center

This paper examines the reactions of 10 elementary teachers to a teaching case used as part of a professional development project's curriculum. Their written reactions to both reading the case and participating a two-hour discussion of it suggest that cases may stimulate teachers to think about their own understanding of mathematics, the mathematical thinking of children, and their roles as teachers. In addition, patterns in the data suggest that teachers' reactions to cases are strongly colored by their prior experience with case materials, their abilities to articulate the subtleties of reformed mathematics teaching practice, and where they are in their thinking about mathematics education reform.

The current mathematics education reform recommendations call for a practice that is different in kind from what we see in most classrooms today (Cohen et al., 1992; NCTM, 1989, 1991; NRC, 1989; Nelson & Hammerman, in press). Adopting a practice consistent with these recommendations requires developing a deeper understanding of mathematics, a new sense of what it means to learn mathematics, an appreciation of the mathematical thinking of which students are capable, and a sense of the mathematical tasks and investigations that support the development of powerful mathematical ideas.

Many teachers lack rich images of what this new mathematics teaching practice might be. They themselves did not learn mathematics this way, and many of them were not prepared to teach this way in their teacher education programs (Ball, 1988). Furthermore, the relative isolation of many teachers has resulted in few opportunities to visit innovative classrooms, even when these innovative classrooms exist (Ball, 1994; Barnett & Tyson, 1993).

Increasingly, teaching cases are proving to be a powerful vehicle for communicating possible visions of mathematics education reform. Cases provide situated images of this new pedagogy, allowing teachers to analyze its subtleties and complexities in classroom contexts (Barnett, 1991; Carter, 1993; Shulman, 1986; Shulman, 1992; Sykes & Bird, 1992; Witherall & Noddings, 1991). Cases help teachers learn to articulate the dilemmas of their own practice, thus helping them learn to voice their own perspectives, issues, and concerns (Schifter & Fosnot, 1993; Schifter, 1994). Furthermore, teaching cases help teachers learn to establish collaborative norms for thinking about, and talking about, their mathematics teaching practice (Featherstone et al., 1993).

Unfortunately we know little about the images that individual teachers create from these case materials. Nor do we know very much about differences between the understandings they construct from reading case materials, and what they further gain from participating in a structured case discussion. This paper explores what individual K-8 teachers in a professional development project seem to gain from reading a teaching case and then participating in a discussion about it with colleagues and a facilitator.

The Teachers

The teachers in this study were participants in the Teachers' Resources Network (TRN), a project designed to help K-8 teachers transform their mathematics teaching practice through the exploration of the resource materials currently available to the mathematics education community. These teachers met biweekly after school with a project facilitator and were encouraged to explore and discuss resource materials of particular interest to them.

Several teachers were very new to the teaching profession, and others had been teaching for over 20 years. They came to the project with a range of prior professional development experiences and were concerned about a wide range of teaching and pedagogical issues.

The Case and the Case Discussions

While most TRN meetings focused on resources that teachers selected, on two occasions the facilitator structured a whole group discussion around a teaching case that she brought to the group. This paper focuses on the first of these teaching cases, which was explored over a period of a month[1] last winter. The case, entitled *Inside Student Thoughts: Take One-Third*, is from a recently-published casebook in mathematics education (Barnett et al., 1994). It begins with a teacher posing the following mathematics problem to a group of 7th- and 8th-grade students: *On your own, draw a picture where you take 1/3 of 1 1/3. Hint: Start with a picture of 1 1/3.* The case then explores how students thought about this problem, how the teacher reacted to their thinking, and what the teacher thought about her lesson.

Prior to receiving a copy of the case, teachers were given as homework the fraction problem posed in it, worded exactly as it was in the case. They came to the next meeting prepared to discuss the problem, and were eager to share their thinking. Many said they also had posed the problem to friends, family, and even their own students. The discussion took most of the three-hour meeting. At the end of the meeting, the teachers were given a copy of the case and asked to read it and write their reactions for the next meeting.

The next three-hour meeting was dedicated to discussing the case itself. The conversation revolved around several main issues[2]: (1) what was difficult conceptually about the problem; (2) what role did language play in confusing or enlightening students about how they might solve the problem (e.g. how were students to know that "of" meant multiplication); and (3) to what extent was the teacher in the case intending to allow her students to "explore" and to what extent was she actually trying to "teach" them. These last two issues had been raised explicitly in the case by the teacher herself. During the last 20 minutes teachers were asked to write additional reactions or insights that came as a result of the discussion.

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[1] The discussion of the case actually continued for more than a month. Teachers continued to think about it, referred to it in many subsequent meetings, and one teacher even brought in student responses to the problem posed as late as the May meeting. However, only two meetings were specifically set aside to discuss the case with the group.

[2] The discussion was facilitated by the TRN project director, who used teachers reactions to the case as a starting place for the discussion. The description of the issues is couched in their terms.

We then analyzed the teachers' written reactions, looking for patterns across a set of 10 reactions written after reading the case but before the case discussion, and a set of 9 reactions after the case discussion.[1] These patterns are described below, along with excerpts of the actual writing of these teachers, used with their permission and identified with their initials.

Teachers' Reactions After Reading the Case

Reading the case did evoke for teachers images about the complexity of teaching. Their initial reactions to the case explicitly referred to several challenges often faced as teachers reconstruct their mathematics teaching: (1) confronting the limits of their own mathematical knowledge; (2) confronting children's mathematical thinking and reasoning; and (3) questioning their roles as a teacher.

Some teachers wrote about recognizing the limits of their own mathematical knowledge. Some expressed reassurance that they were not alone in their struggle to deeply understand the mathematics content. For instance, MMC wrote: "Maybe she [teacher in the case] has a hard time conceptualizing as I do." Others couched their awareness in more anxious terms. FCA wrote: "But how am I going to make sense of that to my students? How am I going to make sense of that to *me*?" YRC confided that she made the same mistake as one of the students in the case and then questioned "Why *is* it OK to divide in order to multiply?"

Some teachers thought about children's' mathematical thinking, often in connection with the confusing nature of the content. MMC conjectured, "I felt that by allowing the kids to 'explore' she [the teacher in the case] allowed them to raise issues and think 'incorrectly.' The more issues they raised the more confused [the teacher] became." One teacher, PDB, admitted her own difficulty in following the thinking of the students in the case: "There is no rhyme or reason given why Bob would represent one whole as nine circles." In contrast, another teacher, JPA, commented that the students' reasoning seemed "so logical." Several teachers felt that students were being confused by the language of the problem: "I think that this is one example which further reinforces the importance of the language of mathematics and using it with students often, carefully, and consistently." (MMB) Finally, one teacher, FCA, explicitly expressed her valuing of student thinking: "I think it's great that the teacher listened so well to what her students thought. I am trying to do that now."

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[1] Not all teachers were present for both meetings; there was a complete data set of "before" and "after" for only 8 of the teachers.

Questioning their role as a teacher is evidenced in the way teachers were able to place themselves in the case itself and think about what they would have done. MMB, for example, shared the tensions she felt in her own practice: "There is always the tension between what we need to provide as teachers and the need for allowing students to discover their own solutions and to hopefully internalize the concept through their own discoveries/constructions." Others were comfortable stating what they would have done in her stead: "I think the time spent should have been more on the language than on the fractions since that was where the difficulty lies" (MMB) and "Begin simple and gradually work up to the more difficult." (BCA)

Teachers' Reactions After Participating in the Case Discussion

The case discussion helped teachers see that they all faced similar struggles. Some, like MMC, identified pedagogical struggles: "After listening to the group I feel that many of us are locked into teaching the way we were taught . . . The bottom line is that we still resort to algorithms." Others, like FCA, highlighted mathematical understanding: "First of all I'm thrilled that other people spoke up to say that they also had difficulty conceptualizing the problem 1/3 of 1 1/3. So it was comforting to know that I am not out there alone."

One teacher (BAB) began to think differently about mathematics teaching and learning after the discussion: "Today's discussion helped me frame a new question. To what extent can you design/create a lesson or series of lessons that are open-ended (inquiry) based and also directed towards the discovery of a concept/skill? I realize even as I write this that this question is fraught with an error in that open-ended discovery by its nature is not directed toward any specific, predictable gain in skill....Maybe, as I look at my own teaching, there is a place for both."

Several teachers confessed that the discussion did not change their reaction to the case: "I still feel that understanding the concept of a whole and fractions and being able to work comfortably with different parts of a whole would have helped the students or anyone solving this problem–without using multiplication or division" (MMB); "I'm not sure that I feel any differently since I read over [my writing from the previous meeting]—I'm still feeling like I don't know everything I would like to know and probably never will...This lengthy discussion seems only to make my head spin but not clarify things for me." (JFA); "I guess I'm still my usual confused self." (MSA); and "I found this (discussion) to be just as confusing as reading the case." (YRC)

What We Are Learning About What Teachers Are Learning

Teaching cases are thought to be powerful because they provide situated images of teaching, help teachers to voice the issues and dilemmas of their own practice, and shape norms for collaboratively inquiring into teachers' practice. The TRN teachers identified with the images of classroom practice captured in the case, using them to think about mathematics content, student thinking, and the roles of the classroom teacher. For many, these images were an invitation to share aspects of their thinking about their own practice — be it their own mathematical confusions, their appreciation of students' thinking, or the struggles they face over when to explore and when to explain. By reading the case, they could recognize that at least one teacher faced struggles similar to theirs; by participating in a case discussion, they learned that they indeed all shared in these dilemmas.

This was the first teaching case explored by this TRN group, and for some it was a new and confusing experience. Some were unsure of the purpose of the reading or the discussion. MSA wrote that "Since I'm at the 4th grade level we never seem to get into fraction problems of this kind, and it really wasn't something that held my interest." Others were not accustomed to thinking critically about a classroom episode: "I think my problem with this case is that we are second-guessing someone who isn't here to expand of the article. We are projecting our own ideas and needs and assumptions on someone else, finding fault and criticizing without enough knowledge as to the teacher's expectation, class, or next steps." (YRC) Learning to use the cases as an opportunity to analyze a piece of teaching—indeed, even to think about teaching as something to be analyzed—is an orientation that seems to take some time to build.

We recognize that it is important to be cautious in making strong inferences about teachers' thinking and reflection based on their journal writings, as much depends on their abilities to express themselves in writing, as well as their willingness to take the time to do so. For example, journal entries ranged from as long as 20 lines to as little as two: "I found the article intriguing. I have an excellent activity with fraction circles that I did today." (BAB) and "Discussion was great! Ideas were good! Exchange was good!" (LJA) In each case, one of these brief entries was paired with a much longer, and quite thoughtful first or second entry.

On the other hand, teachers' writing can provide important assessment information about where teachers are in their thinking about mathematics education reform. For example, teachers in this study wrote openly of their own confusions and the confusions of their students. What is particularly informative is that almost all of them felt that these confusions were something to be remedied immediately. In marked contrast, only one teacher (KOB) wrote appreciatively about the role of confusion. In her response to reading the case, she wrote: "In regard to [the teacher's] query about letting students explore before a formal lesson, I believe this is extremely important. The level of thought, frustration, and cognitive dissonance she achieved with this class could not have been attained if she had first taught them formally about multiplying two fractions when you see the word "of." In her response to the case discussion, she commented about her colleagues: "I found there was a lot of focus on how we could help the kids 'get it' rather than on how to understand their thinking, and helping them go further with it."

Many interesting questions remain about the extent to which the images conveyed in a case help teachers reconsider fundamental aspects of their practice. There are also important questions about how teachers learn to participate in the dialogue about mathematics education reform, and share these aspects of their practice with colleagues. Both of these sets of questions are important facets of helping teachers reinvent their mathematics teaching practice.

References

Ball, D. L. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics, 8*(1), 40-48.

Ball, Deborah Loewenberg (1994). Developing mathematics reform: what don't we know about teacher learning — but would make good working hypotheses? Paper prepared for a conference on Teacher Enhancement in Mathematics K-6, Arlington, VA, Nov 18-20.

Barnett, C. & Tyson, P. (April, 1993). Mathematics teaching cases as a catalyst for informed strategic inquiry. Presented at the Annual Conference of AERA, Atlanta.

Barnett, C. (1991). "Building a Case-Based Curriculum to Enhance the Pedagogical Content Knowledge of Mathematics Teachers." *Journal of Teacher Education, 42*(4), 263 - 272.

Barnett, C., Goldenstein, D., Jackson, B. (1994). *Mathematics teaching cases: Fractions, decimals, ratios, & percents.* Portsmouth, NH: Heinemann.

Cohen, D. K., McLaughlin, M., & Talbert, J. (Eds.). (1992). *Teaching for understanding: Challenges for practice, research, and policy.* San Francisco: Jossey Bass.

Featherstone, H., Smith, S.P., Beasley, K., Corbin, D., & Shank, C. (1993). Expanding the equation: Learning mathematics through teaching in new ways. Presented to the American Educational Research Association, Atlanta, GA.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education.* Washington, D.C.: National Academy Press.

Nelson, B. & Hammerman, J. (in press). Reconceptualizing teaching: the teaching and research program of the Center for the Development of Teaching, EDC (working title). To appear in M. W. McLaughlin (ed), *Professional Development in an Era of Reform.*

Shulman, J. (Ed.) (1992). *Case methods in teacher education.* New York: Teachers College Press.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher,* 15(2), 4-14.

Sykes, G. & Bird, T. (1992). Teacher education and the case idea. In G. Grant (Ed.) *Review of research in education, Vol. 18* (pp. 457-521). Washington, DC: American Educational Research Association.

Witherell, C. & Noddings, N. (1991). *The stories lives tell: Narrative and dialogue in education.* New York: Teachers College Press.

Schifter, D. (1994). Voicing the new pedagogy: Teachers write about learning and teaching mathematics. *Center for the Development of Teaching Paper Series.* Newton, MA: Education Development Center.

Schifter, D. & Fosnot, C.T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform.* New York: Teachers College Press.

TENSIONS AND STRUGGLES: PROSPECTIVE SECONDARY MATHEMATICS TEACHERS CONFRONTING THE UNFAMILIAR

Bridget Arvold, The University of Georgia
Maureen Albright, The University of Georgia

Being open to new ideas is instrumental for the professional development of teachers. Using numerous case studies of prospective teachers' beliefs as a backdrop, we investigated the tensions and struggles four prospective teachers encountered as they confronted the unfamiliar. We interpreted both the formation of their belief systems and the interactions of their beliefs with the unfamiliar as our participants engaged in a discussion, reflection, and activity-rich pro-active teacher education environment. Through the tensions and struggles associated with multiple interpretations of mathematics and issues related to multiculturalism, we gained insights into the complexities of becoming a teacher during this time of mathematics education reform.

In light of recent reform efforts in mathematics education, teachers and teacher educators are striving to extend their own reflective, analytic, and adaptive mind sets and to encourage others to do so as well. Such mind sets are necessary to help teachers create a rich, comfortable, and empowering environment for their students (NCTM, 1991). Discussions, reflective journal writing, and experiencing learning and teaching through new methods provide opportunities for opening minds to new ideas (NCTM, 1989).

The manner in which people address new ideas is complex. One contributing factor in this process is the individual's tendency toward open-mindedness, a putting aside of predispositions and forestalling premature judgment in an attempt to come to understand new ideas and new perspectives of familiar ideas. Although existing beliefs and knowledge influence an individual's ability to attend to new ideas and understand them, existing beliefs can also inhibit one's intellectual growth. Since many teachers and prospective teachers can profit from gaining a broader range of perspectives, being open to new ideas is instrumental in their professional growth.

As part of the RADIATE[1] research team, we investigated the tensions and struggles of prospective secondary mathematics teachers participating in a proactive teacher education program. The program shared the underlying philosophies expressed in the NCTM *Standards* (1989, 1991). Paramount was the philosophy that educating is not disseminating information but facilitating each individual's accommodation of existing knowledge to new ideas within the context of a social setting. Reflection upon multiple perspectives of mathematics, teaching, and learning permeated the environment. Although the program's curricular design was responsive to student needs, the sharing of multiple perspectives was fundamental to all course activities.

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[1] RADIATE (Research and Development Initiatives Applied to Teacher Education) is directed by Thomas J. Cooney and Patricia S. Wilson and funded by the National Science Foundation (DUE9254475) and the Georgia Research Alliance. Opinions and conclusions expressed herein are not necessarily those of the funding agencies.

Framework

Multiple lenses provided insights into the beliefs and tensions of the prospective teachers. The dynamics of the belief systems in terms of strength of psychological commitment and clustering of beliefs (Green, 1971) aided us in identifying and interpreting the nature of the prospective teachers' beliefs. Within these systems, the received and connected procedural ways of knowing (Belenky, Clinchy, Goldberger, & Tarule, 1986) and the progression from dualism to relativism (Perry, 1970) provided insights into how beliefs and knowledge were held and into the sense of ownership related to specific beliefs. As tensions arose, the idea of a continuum from close-mindedness to open-mindedness (Rokeach, 1960) provided insights into how our prospective teachers handled these tensions, and the degree to which they chose to ignore the unfamiliar or to keep their minds open as they struggled with new ideas.

Methodology

The present study, as part of the RADIATE project, involved 15 prospective teachers with whom we worked from April 1994 through June 1995. Data were collected during the prospective teachers' participation in specially designed classes integrating pedagogy and content, during their student teaching experiences and during the culminating education seminar. Participant data were collected through initial surveys, nine guided interviews, handwritten and electronic journals, classroom and field observations and appropriate artifacts such as tests and reports. The instructor's January statement of goals and concerns supplemented the data from the participants. All members of the seven-person researcher team shared their respective constant comparative analyses (Glaser-Straus, 1967) during weekly discussions. In our analysis of tensions and struggles arising from encounters with the unfamiliar, we first categorized data contributed by the entire team. In this report we used a theoretical sampling (Glaser-Straus, 1967) to focus on the unfamiliar Standards-driven encounters most often noted by the prospective teachers. We explicitly discuss the tensions and struggles precipitated by the multiplistic-oriented functions unit and a multi-cultural lesson. Since the studies of our own particular participants provided somewhat representative data, we focused our attention on the tensions and struggles of Harriet, Carl, Alice, and Shannon.

Tensions and Struggles

The prospective teachers' cookbook expectations of the program contrasted greatly with the design of the curriculum and the instructor's stated goals "to shake people up and challenge them to think beyond the surface. I want them to become reflective about their assumptions, their reasoning, and their behaviors ...While I want students to be comfortable and to learn in a friendly, open environment, rattling ideas takes priority over comfort." Our participants entered this unfamiliar classroom environment with varied backgrounds. After losing interest in "boring" accounting, Harriet entered the teaching program to become a certified secondary mathematics teacher. She believed she had already learned all she needed to know from her mother, an experienced middle school mathematics teacher. It was her mother's voice she shared with us. Carl shared a different voice of experience. It was after fourteen years in a managerial workforce that he sought an undergraduate degree and teacher certification. His volunteer mathematics tutoring of confused student employees and his caring nature piqued his interest in improving the teaching profession by joining its ranks. Carl believed field experiences would be his teacher. Unlike Harriet and Carl, Alice had always wanted to teach. Her early practicum experience gave rise to both her dissatisfaction with traditional teaching techniques and a multitude of questions about the process of teaching. She hoped for an inspiring learning experience. Lacking a vision of her future, Shannon seriously considered the recommendations of her high school teachers, and entered a teacher education program. Like Alice, once Shannon realized that there were alternatives to lecturing, she poised herself to "listen and digest" as she encountered "the whole Standards outlook on things."

Material on functions served as an introduction to the program. Besides providing an opportunity to integrate the learning of content and pedagogy, the material fit the instructor's stated goal of helping "students realize that there is a great deal of high school mathematics that they do not know." The functions material built upon multiple strategies for problem solving by scaffolding activities that included investigations of multiple representations and categorizations of functions, analysis of case studies and reflections on pedagogical issues. Most often the activities included the modeling of real life situations and follow-up analyses enriched by the use of graphing calculators, and algebraic, geometric, and spreadsheet computer software.

Perceiving the function material as irrelevant, Harriet's tensions were limited to her coping with the curriculum. "It was like what we saw and what we are going to see is going to be much higher than what high school students are going to see which means it was like it was there for our enjoyment and completely almost forgot about our students." Harriet placed herself in the teacher role as she spoke of "our" students but her product-oriented mathematics blinded her of the opportunity to gain a deeper understanding of mathematics and teaching. Her more dualistic orientation (Perry, 1970) inhibited her seeing the value of multiple perspectives. Carl viewed the activities involving multiple strategies and multiple perspectives as no more than a catalogue of activities for use in the classroom. Carl's orientation toward partitioning his beliefs and knowledge (Green, 1971) instead of making connections was evident in almost every aspect of our study. For example, Carl's failure to understand the derivative nature of multiplication and addition during a lesson on completing the square precipitated tension during his student teaching. When he finally realized that his student's "2 times 13/2" meant the same as his "13/2 plus 13/2," he did not contemplate his negative response to his student, but rather was irritated at not having been told him about this fact in his earlier education. He claimed no ownership of his mathematics. After a brief troublesome period contemplating how he might fill the gaps in his mathematical knowledge, his pride in a high score on the state teachers' exam eased his tension. This instance reflected how his strong psychological beliefs (Green, 1971) held from a dualistic perspective (Perry, 1970) supported his abandonment of tension and struggle. The dualistic tendencies of both Harriet and Carl forestalled most tensions and obstructed not only the relevance of pedagogical issues but an analysis of the multiplistic and connected nature of the rich mathematics.

Both Alice and Shannon entered the program confident in the richness of their mathematics only to become perplexed by the function-related activities. Alice expressed dissatisfaction with the recurrent discussions of the maximum volume box problem while Shannon struggled to become proficient with the unfamiliar technology, trusting that soon the purpose for all this mathematics in a "methods class" would be explained. Although the experiences with the functions material did not conflict directly with their knowledge of mathematics, it underscored an unrecognized perspective of mathematics. Although neither Alice nor Shannon came to understand mathematics as one's own construction, upon reflection each of them came to recognize the lack of depth in their own received (Belenky et al., 1986) mathematical knowledge. Even though Shannon expressed her newfound freedom in mathematics, "I like stumbling around with math ... experimenting and playing around," she and Alice struggled with these new disconnected pieces of mathematics. Later Alice and Shannon credited the subsequent study of the Standards and their research involvement for helping them gain a perspective on what they had been experiencing. Their understanding of the philosophy of the Standards added coherence to the multiple ideas precipitated by the functions material by uniting many of the belief clusters (Green, 1971) that they had been forming.

Both the functions material and aspects of the multi-cultural lesson permeated the entire program. The multi-cultural lessons were driven by readings and discussions using the prospective teachers' own interpretations of culture. Harriet welcomed the study of culture even though she, as an African-American, had "not learned a whole lot" in class. She was already "aware and experienced" in cultural issues. She was comfortable with most cultural issues because they substantiated the knowledge imparted upon her by her select group of authority figures, her received knowledge (Belenky, 1986). There was one particular reading and classroom discussion that incited her to question and ultimately modify her belief about color-blindness. "I have come to realize that color-blindness is not a very powerful or profitable approach to relating to other people [students]." Her reflective re-evaluation of an existing belief illustrated an infrequently observed move toward open-mindedness (Rokeach, 1960) in an area of deep psychological commitment (Green, 1971).

Harriet did not adapt her existing perception of culture as race to the context of the mathematics classroom, and Carl's perception of culture as the origins of mathematics remained void of the new context as well. Harriet and Carl each protected their existing beliefs from contamination, a more dualistic orientation (Perry, 1970). Carl hinted at why he dismissed the relevance of culture. "I'd have a difficult time right now focusing on multi-cultural because there's so many other things ... Yes, it's something I'd like to master but certainly would not go into my initial teaching methods. ... Multi-cultural was just thrown in because an instructor wanted it, [but] I did use an aspect of it when it just happened to come up." Culture became just another disconnected belief cluster (Green, 1971), another item on his list of things to "master."

Alice's initial tension also reflected her limited view of culture in mathematics. "Math is math. Culture doesn't change the way you do math." Shortly thereafter she widened her perspective of mathematics as process and reflected upon the influence of culture in the creation of mathematics. Widening again, she reflected upon the cultures of her students. Alice and Shannon each struggled with similar issues as they came to the realization that their inability to observe culture during field observations was due in part to their negative view of "labeling people." In their struggles to integrate conflicting ideas, they reflected a more relativistic orientation (Perry, 1970). Alice chose to unveil her struggles in addressing the needs of her non-English speaking student through her senior research project, and evidenced an area of initial commitment (Perry, 1970) in her growing concern that many teachers are unaware and insensitive to the cultures of their students. Shannon focused on the importance of treating "any student with respect, no matter if their views are different," and "the teacher's responsibility to make sure students do not put down other students for their differences." She valued students gaining a better understanding of mathematics through its historical context and contemplated culturally rich research projects. Alice and Shannon both attempted to incorporate the voice of the student into their more connected procedural knowing (Belenky et al., 1986).

As we focused our analysis on tensions arising from encounters with new ideas, we tended to neglect analysis of how these new ideas actually relieved existing tensions. As existing tensions diminished others took their place. Alice's tensions relating to the inadequacies of traditional teaching lessened as she struggled to create a new vision of teaching. Shannon's tensions eased as she discovered that learning mathematics need not be constrained to the formalist rigor she had come to previously accept. Carl focused his tensions associated with his desire to improve the teaching profession as he concentrated on cooperative learning as his solution to poor teaching. Harriet entered with few tensions and left the program still questioning the appropriateness of the program itself.

Summary

This study suggests that our prospective teachers' orientations toward tensions were directly related to their inferred belief systems. More dualistic orientations tended to ward off tension whereas more relativistic orientations allowed the consideration of multiple perspectives. A more relativistic orientation also nurtured struggles involving analysis and the integration of novel ideas. In belief systems with an plethora of belief clusters, new ideas remained segregated from existing beliefs or entered a single cluster instead of becoming connectors between clusters. The abundance of multiplicity in the program may have created too much interference for viable attending and connecting.

Discussion

We as researchers, must open our minds to new ideas and new perspectives in order to better understand our prospective teachers and help them learn. Our own limited domains trammeled our analysis of the complexity of the dynamic environment. Although we gained a better understanding of how existing belief systems, tensions and struggles influence the professional growth of teachers, care must be taken to account for the value of theoretical perspectives in order not to fall into the trap of what Bauersfeld (1988) refers to as "theoretical autism."

References

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In National Council of Teachers of Mathematics (Eds.), *Perspectives on research on effective mathematics teachng.* Reston VA: Lawrence Erlbaum.

Belenky, M. F., Clinchy, B. M., Goldberger, N. R. & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice, and mind.* New York: Basic Books.

Glaser B. G., & Straus A. L. (1984). *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine De Gruyter.

Green, T. (1971). *The activities of teaching.* New York: McGraw Hill.

National Council of Teachers of Mathematics. (1991). *Professional standards for the teaching of school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Perry, W. G. (1970). *Forms of intellectual and ethical development in the college years.* New York: Holt, Rinehart, & Winston.

Rokeach, M. (1960). *The open and closed mind.* New York: Basic Books.

MATHEMATICS REFORM IN APPALACHIAN SCHOOLS

Genevieve A. Davis, Kent State University
Trish Koontz, Kent State University

Since 1990 Kent State University professors have collaborated with educators in an Appalachian County in a research project to improve mathematics teaching and learning inside the classroom as well as to help change belief systems of teachers and parents about how children best learn mathematics. This research project was designed to examine this change process in rural educational settings.

This research project provided opportunities for teachers, administrators and parents to effectively communicate with each other. This innovative reform movement has developed strong collaborative alliances within buildings and across districts that have continued to sustain themselves beyond the initial three years of NSF funding. This project has proven to be successful because its vision and design are grounded in sound theory based on constructivist views of teaching and learning.

As teachers' beliefs began to embrace constructivist views, the following practices were documented by collaborating university faculty:

- Teachers encouraged children to explore mathematical concepts.
- Teachers and children dialogued about their mathematical explorations.
- Teachers were more accepting of children invented procedures.
- Teachers provided time for children to reflect and communicate about their mathematical experiences through dialogue and journal writing.
- Teachers recognized the need for and practiced authentic assessment.
- Teachers' beliefs about the nature of mathematics changed significantly.
- Teachers developed the confidence to defend their beliefs about constructivist practices to colleagues and parents.

This research has confirmed that in order to affect change in teachers' beliefs about the nature of mathematics teaching and learning, inservice efforts need to include ample knowledge about constructivist learning environments, mathematical content knowledge, *and* pedagogy.

PORTFOLIOS AS A TOOL FOR REFLECTION IN TEACHER PREPARATION PROGRAMS

Elizabeth H. Jakubowski, Florida State University
Grayson H. Wheatley, Florida State University
Evangelina Diaz Obando, Florida State University

In order to assist teachers to have alternatives to teaching the way they were taught, teacher preparation programs have moved to including experiences that perturb prospective teachers into reflecting on past experiences and developing a vision of what mathematics classrooms, teaching and learning for the twenty-first century might be. It is asserted in this paper that the notion of reflectivity is a viable element for the improvement of the process and profession of teaching mathematics.

The concept of reflectivity is a reasoned, principled response through either pre-planned or spontaneous but conscious action in which awareness of past experiences and understandings are linked with present experience to lead to new understandings and appreciations. Reflective activity can emphasize a professional's basic freedom of choice, which is implicit in the concept of professionalism. A key feature of reflection is the need for prospective teachers to learn to exercise that freedom rather than merely conform to the influence of the professor.

Through reflection, the learning experiences of prospective teachers can be extended into what Dewey (1933) calls a learning loop. There is a continual reestablishment of relationships between experience and understanding thereby encouraging limitless opportunities for explorations into issues associated with teaching and learning mathematics. Reflection becomes a process for learning how to learn rather than performing a prescribed set of actions. When reflection is used in prospective teacher education programs, the prospective teachers are being subtly encouraged and predisposed to incorporating inquiry and evaluation as an habitual practice in all teaching experiences.

In developing a middle grades mathematics teacher preparation program we wanted to include elements which would cause the students to probe for deeper meanings in their reading, their study, and their actions. Two years ago when program development began, portfolios were to be included as a way of assisting prospective teachers to develop images of alternative assessment. However, as we have moved through one cycle of the program we have found that using portfolios has allowed us to encourage the students to incorporate inquiry into their teaching experiences. By examining three participants in the first cycle of the program, this paper will provide a description of how reflectivity, especially through portfolio activities, has enabled these prospective middle grades mathematics teachers to develop into inquiring professionals who are demonstrating during their internship programs the ability to be responsible and reflective professionals.

Reference

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process.* Boston: D.C. Heath.

A MODEL FOR EXAMINING THE CHANGING BELIEFS OF A HIGH SCHOOL PHYSICS TEACHER INTEGRATING MATHEMATICS THROUGH TECHNOLOGY

Marsha Paulus Nicol, The Ohio State University

Mike Smith, a high school physics teacher, decided to investigate the use of graphing calculators in the classroom. As a participant observer, I used a researcher-developed model for understanding teacher change to focus on Mike's changing beliefs about mathematics, education, and graphing-calculator usage. The theoretical framework for the study incorporated Kuhn's (1970) theory of paradigm shifts in the midst of revolutions, Vygotsky's (1978) and Piaget's (Eggen & Kauchak, 1992) developmental learning theories, and Anchored Instruction (The Cognition and Technology Group at Vanderbilt, 1990).

Mike's dramatic and revolutionary change came as a result of several factors. As colleagues and students convinced him that he needed to become knowledgeable about graphing calculators, his belief system was shaken and disequilibration occurred. As a result, he and I participated in a week-long professional development institute that emphasized the use of the Texas Instruments' (TI) Calculator-Based Laboratory (CBL) and the TI-82 graphing calculator in an integrated mathematics and science classroom environment. Immersed for a week in mathematics, science, and technology in a familiar setting, Mike's learning became anchored; and as his understanding of concepts began to deepen, his mathematics and education paradigms shifted.

Mike and I worked together preparing for CBL presentations and for his physics classes. As we ran experiments and discussed analysis and interpretation of our resulting data, he admitted that he learned much from me. Vygotsky's theories of learning from a capable peer were evidenced.

Mike first took ownership of his change. He then continued to experience much disequilibration, but he continually took steps to restore his equilibrium. Mike is a very reflective teacher, and reflection was the driving force behind his change. As a result, his beliefs changed, which forced change in his classroom practice.

References

Cognition and Technology Group at Vanderbilt. (1990). Anchored instruction and its relationship to situated cognition. *Educational Researcher, 19*(6), 2-10.

Eggen, P., & Kauchak, D. (1987). *Educational psychology: Classroom connections.* New York: Macmillan.

Kuhn, T. S. (1970). *The structure of scientific revolutions.* Chicago: University of Chicago Press.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

A MODEL TO ASSIST IN IMPLEMENTING CHANGE IN TEACHER PRACTICE

David R. Erickson, The University of Montana

The reform documents of the professional mathematics organizations, the National Council of Teachers of Mathematics, NCTM, and the Mathematical Association of America, MAA, project a vision for a new, changed school mathematics environment. Although NCTM and MAA make no prescriptions for obtaining this vision, a constructivist epistemology provides a theoretical framework for both investigating and implementing necessary changes.

During a two-year case study of a sixth-grade teacher of mathematics, a model for teacher change emerged with eight intertwined factors: goal, conflict, vision, commitment, support, action, reflection, and executive control. The first six factors are represented by nodes suspended within a metacognitive sphere. Bi-directional arrows that represent reflective activity link each node. Vignettes from the naturalistic study document how these eight factors are important to this teacher's attempts to change her teaching practice.

The focus of this poster is on the most important of these factors, executive control. This represents the steering mechanism for the change process and is a metacognitive activity. The strength of this activity, together with all the factors identified, determines the level of success and the length of time required for change to occur. Evidence of this executive control surfaced more readily in situations for which curriculum, pedagogy, and assessment were all addressed simultaneously. When one of these areas was unresolved, not yet aligned with the teacher's vision, progress was slowed. Having an overseer, a person who provides a direction to take at each of the factors, is the role assumed by this metacognitive activity.

Further research is necessary to test the hypothesis that identification of the manifestation of these eight factors by individual teachers will lead to an increase in successful implementation of desired change in teaching practice. Does a direct, explicit awareness of the executive control factor contribute to a decrease in the time required for making changes?

REFLECTIVE APPROACH AND CONTINUOUS TRAINING OF ELEMENTARY SCHOOL TEACHERS OF MATHEMATICS: AN ANALYSIS OF THE CLASSROOM MICROCULTURE EVOLUTION

Louise Poirier, Université de Montréal
Nadine Bednarz, Université du Québec à Montréal

A number of analyses of classroom interaction patterns have highlighted the implicit elements behind the social regularities (occurrences) emerging from some school cultures (Bauersfeld, 1980; Voigt, 1985). These analyses tie in well with the work on the didactic contract (Brousseau, 1986; Schubauer Leoni, 1986) which explains the system of reciprocal expectations observed in specific teaching situations; whereby the teacher's actions with respect to a given problem influence how the students will grasp a specific knowledge. The students decipher the situation given to them and execute the problem following their own interpretation of the rules in place as well as abiding by the implicit expectations they feel in this particular instance.

Given the fact that students simply react to what they think is the teacher's implicit expectation, how can one establish a didactic contract wherein students truly participate in the construction of their own knowledge? A number of recent studies have investigated the conditions of evolution of this classroom microculture in mathematics (Cobb et al. 1994). To change this classroom culture and to influence its evolution requires a thorough understanding of what really happens, as well as allowing teachers to reflect on the actions and interactions in the classroom. Since this study uses a socio-constructivist approach (Bednarz et al. 1993), the practices implemented in the classrooms are aimed at altering the traditional didactic contract defining the respective roles of the teacher and the students.

Objective. As part of a collaborative research project in an elementary school, teachers (1st, 2nd and 3rd grade) are invited to reflect on the interventions in their mathematics classrooms. What is being proposed here is regular, planned alternation of in-class experimentation of the approach with group (teachers and researchers) reflection on this process. In this presentation, the evolution of the teachers thinking will be emphasized by showing the change in the classroom culture.

Method. The reflections of teachers, as expressed during exchanges between researchers and teachers have been recorded on a regular basis over a two year period. In addition, their classroom teaching practices have been videotaped also on a regular basis during the same period. These observations provide an analytical basis for retracing the different phases which teachers go through as well as illustrating the doubts and changes in their ways of thinking and teaching, as they attempt to create a new didactic contract within the classroom. This analysis highlights also the evolution in the classroom culture fostered by the changes in the didactic contract made by the teachers.

References

Bauersfeld, H., (1980) Hidden dimensions in the so-called reality of a mathematics classroom. *Educational studies in mathematics*, 11, 23-41.

Bednarz, N., Janvier, B., Poirier, L., Bacon, L., (1993) Socioconstructivist viewpoint on the use of symbolism in mathematics education. *The Alberta journal of educational research*, vol. XXXIX, march, 41-58.

Brousseau, G., (1986) *Théorisation des phénomènes d'enseignement des mathématiques*. hèse de doctorat d'état, Université de Bordeaux 1, Bordeaux, France.

Cobb, P., Perlwitz, M., Underwood, D., (1994), *Construction individuelle, acculturation mathématique et communauté scolaire*, Revue des sciences de l'éducation, Vol. XX, no 1, 41-61.

Schubauer-Leoni, ML., (1986) *Maître-Élève-Savoir : analyse psychosociale du jeu et des enjeux de la relation didactique*. Thèse de doctorat, Université de Genève, Suisse.

Voigt, J., (1985), Patterns and routines in classroom interaction. *Recherches en didactique des mathématiques*, 6(1), 69-118.

THE SETTING FOR PROBLEM SOLVING: A TEACHER'S BELIEFS

E. S. Senger, Louisiana State University

While reform in mathematics teaching involves both internal beliefs and external practices, the distinction is important for studying the process of change and teachers' personal avenues toward professional development. Thompson's (1992) study and review of the literature suggest that teachers' conceptions and beliefs regarding mathematics highly influence their classroom practice. Fennema & Franke (1992) found that some forms of teacher knowledge and belief, develop through teaching practice.

Elementary teachers are faced with issues of change in the teaching of mathematics. Whether such changes are deep and lasting, or superficial depends on the process of reflection and reframing that the teacher has the opportunity and willingness to pursue (Richardson, 1990, Russell & Munby, 1991).

This study involved participant observation of one fourth-grade teacher, Pamela, and her process of change, both internal and external over the course of one school year. Interviews, videotaped observations, and discussions unpacked Pamela's beliefs and practices regarding teaching and learning mathematics while she struggled with issues of reform. Qualitative analysis revealed her focus at different times during the study.

The content and pedagogical concerns revolve around Pamela's beliefs and classroom tasks in problem solving, and the roles of teacher and student in learning to process and solve problems. The journey with Pamela through her dilemmas and attempts at change adds insight into her progression from teacher-as-answer-giver to teacher-as-witness of children's diverse ways of investigating problem situations.

References

Fennema, E., & Franke, M. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.147- 164). NY: Macmillan.

Richardson, V. (1990). Significant and worthwhile change. *Educational Researcher*, 19(7), 10-18.

Russell, T., & Munby, H. (1991). Reframing: The role of reframing in developing teachers' professional knowledge. In D. Schon (Ed.), *The reflective turn* (pp.164-188). NY: Teachers College Press.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D.Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.127-146). NY: Macmillan.

TEACHER CHANGE IN A STAFF DEVELOPMENT SETTING: CASE STUDIES OF MIDDLE SCHOOL MATHEMATICS TEACHERS

Terese A. Herrera, The Ohio State University

The purpose of this study was to document the influence of a mathematics inservice on practicing middle school teachers. My specific focus was the unique, individual character of the change process. Rather than rely on aggregated group effects, I employed qualitative methods suitable for registering the response of the individual participant and for documenting in detail the *process* of change as it occurred.

Three seventh grade teachers, participants in an intensive, long-term mathematics inservice, contributed their perspectives of change through interviews, lesson plans, assessment instruments, portfolios, and classroom observations. In my role as participant observer, I accompanied these teachers for over a year, gathering data on their conceptions of teaching and learning of mathematics prior to and during the summer inservice, and then throughout the following academic year.

The six-week inservice was sponsored by Ohio's State Systemic Initiative, Project Discovery, which offers science and mathematics institutes for teachers throughout the state. What is modeled is a "discovery" or inquiry method. Instead of lectures on mathematical content, the instructors immerse the participants in problematic situations. They directly encounter and explore the mathematics embedded in these situations. How or whether a teacher decides to implement the inquiry method is left to the professional discretion of the teacher. Seminars during the following year bring teachers together to discuss their experiences with inquiry teaching.

Data analysis produced case studies describing personal teacher change as experienced by the individual teacher, a perspective usually absent from investigations of staff development. Social context played a large role in the six-week experience, as did student reaction in the following year.

In a cross-case analysis, patterns emerged that allowed me to tentatively describe teacher change in terms of interaction between the inservice experience and the individual's personal construct of "math teacher."

On a more pragmatic level, knowledge of mathematics proved to be a determining factor in the teachers' interpretation and implementation of the inquiry method modeled in the inservice.

Teacher Conceptions of Mathematics

COMING TO TERMS WITH CONCEPTUAL KNOWLEDGE: ONE TEACHER'S JOURNAL

Christine L. Ebert, University of Delaware

This study will describe the emerging mathematical understanding of one student enrolled in an elementary mathematics content course. Through the medium of her journal entries, a portrait of Elena's struggle to overcome her difficulty with mathematics and construct conceptual knowledge has emerged. Though these struggles are not unique, Elena's story is at once both poignant and extraordinarily compelling. She gives voice to the child within struggling to make sense of this "mysterious information presented by adults which emphasized procedural skills over all," the adult attempting to negotiate this "return to meaningful learning," and the metacognitive monitor that reflects lucidly on the "mental paralysis" that she experiences even when "one has worked hard and made great effort to control one's reasons and thoughts." By examining Elena's emerging view of conceptual knowledge and her reflections on this process, we may also access information about viable means of facilitating this process.

Theoretical Framework

The current reform movement in mathematics education suggests that teacher subject-matter knowledge is an important component of the new view of mathematical competence. The task of investigating the construction of conceptual knowledge and the subsequent transformation of that subject-matter knowledge into pedagogical content knowledge is extremely complex (Shulman, 1986). As researchers, we struggle to construct tasks that will provide useful information about both of these processes. Within the content and methods classes we seek to design experiences that will facilitate the construction of conceptual knowledge. If we are to provide these opportunities for prospective teachers to construct the conceptual knowledge suggested by the reform documents, then we must examine cases of that construction process through in-depth portraits of individual teachers (Ball, 1988). Elena's journal entries provide important information about her struggles with the mathematics, her reflections on that struggle, and reflections on her mathematical journey.

Methodology

The data source for this study consists of one prospective elementary teacher enrolled in the first of two courses in mathematical content at a major university in the mid-Atlantic states. The course, which was designed to focus on problem solving and conceptual understanding, consisted of two weekly lectures and two weekly problem solving sessions. This class of approximately 180 students met with the instructor for lectures in a large auditorium and with one of the two teaching assistants for problem-solving sessions in a class of approximately 25 students. The course focused on the "construction of conceptual understanding of the elementary mathematics curriculum" (M251 Course Syllabus, 1993).

The emphasis on problem-solving was reflected through the allocation of two classes each week to solving problems with a partner. On the first day of problem-solving, all students took a skills test which consisted primarily of 36 arithmetic and pre-algebra problems. Based on the scores from this test, students were assigned a partner for the semester. Pairings were made such that the ability differential for all of the pairs was constant. During the problem-solving sessions students were either assigned problems from the textbook or given a problem-set to work on for that particular day. Problems were never assigned in advance. The role of the instructor during these two weekly sessions was to circulate around the room discussing the problems and problem-solving strategies and offering suggestions whenever appropriate. The "homework" in the course consisted of doing as many problems as necessary to gain understanding of the concepts.

Results

Elena is a returning adult student in her mid-forties who has worked as an artist and medical illustrator prior to seeking certification to teach art in the elementary school. Elena's journal was chosen for analysis because she consistently recorded her mathematical growth and struggles. She also included a wide variety of other sources from Polya's "little book", *How to Solve It*, to a *New York Times* article about Bob Moses' work on the Algebra Project. All of her quizzes and tests contained additional notes concerning the problems with which she had difficulty and the progress she was able to achieve. The richness of her journal mathematically and the compelling way in which she expresses her reflections throughout this journey provide important and intriguing information about the construction of conceptual knowledge.

Initiating the Conversation

Disposition/ Beliefs about Mathematics. Elena initiated the conversation through the medium of her journal with a quote from Polya's *How to Solve It* that she uses to illustrate her openness to this experience and express the hope that it's "never too late to experience the grain of discovery in the solution of any problem' whenever that susceptible age should occur" (Elena 2/10/93). She also provides her own mathematics history in the initial journal entry.

> It seems amazing that meaningful learning is considered to be "new"? Perhaps it is a new emphasis for mathematics? I certainly was not taught in a way that made mathematics meaningful for me. As a young child I believe that I experienced much of what might be considered the very worst. Math made no sense. The adults presenting this mysterious information emphasized procedural skills over all. "Do it my way, do it correctly and you will obtain the right answer." I am still struggling with the mental/emotional baggage from my earliest introduction to math in elementary school. These words come to

> mind - fear, confusion, discouragement, embarrassment, boredom. Needless to say I do not want to convey any negative attitudes or "hang-ups" to youngsters. I've worked hard as a parent to avoid this and I am striving now to overcome the attitudes that thwart my own progress as a student. (Elena 2/10/93)

Fears About Doing Mathematics. While this particular mathematics history is not especially unique, Elena does seem to be able to convey an important distinction between meaningful learning and her own experiences - that the emphasis focused on procedures and that her particular experience generated significant negative feelings. At this point in her reflections it is clear that she is "striving" to overcome the attitudes that thwart her progress. In the following quotation, Elena reflects on the "skills test" and gives voice to the child to describe her initial "return to meaningful learning."

> I took the skills test today. Some things worked, some things didn't. How long has it been since I thought about what an equation *is*, let alone construct one!...it continues to frustrate me that in math my mind is not at its best - even when I know what to do. It is embarrassing. I often feel that in this realm, I am a nervous 8-yr. old. (Elena, 2/11/93)!

She also writes that she "does not know how to go about describing sequences." A few days later she writes about her attempt to solve the initial problem in the text.

> DESCRIPTION = FORMULA! FORMULA = ALGEBRAIC DESCRIPTION (EXPRESSION) OF SEQUENCE INFORMATION
>
> I find moving methodically, slowly, through these sequences, watching how the numbers "make sense" to be extremely satisfying. I can't remember ever being taught this way. It makes my brain feel good. (Elena 2/15/93)
>
> $n(n+1) = 98{,}282$
> $n^2 + 1 = 98{,}282 - 1$
> $n^2 = 98{,}281$
> $n = 313$ He is on page 313.

The Metacognitive Monitor Speaks. An important contribution of Elena's journal is the multiplicity of perspectives that she conveys as she reflects on these experiences and her attempts to gain conceptual understanding of the mathematics curriculum. Her initial responses are those of the adult negotiating this return to learning and gathering as many and as varied a collection of resources as she is able to make sense of this experience. However, given that her childhood experiences with learning mathematics were so negative, the voice of the frightened, frustrated, and discouraged child is also quite prominent. In addition, when she can stand back and reflect on the mathematics she is doing, quite frequently the voice is one of metacognitive monitor - dispassionate and reasoned. She writes about examining her thought processes and her response to the first quiz.

> Quiz returned - I did better than expected - what a relief! Making tables is now seared in my brain (as is the formula for the area of a circle - square that radius!). *Determined* to be calmer the next time. (Elena 2/24/93)

On this first "written assessment" Elena did quite well (19/25 or 76.5%). Although the work and the progress that she makes does not reveal constant growth but the typical fragility inherent in the learning process, she expresses her arrival at this small plateau in an extremely poignant and revealing way.

> Taking in small amounts of information, experiencing small amounts of understanding at an ordered gentle pace is what I need at this time. I almost feel as if something once broken or walled-off is healing or opening. I am amazed by my need for plenty of time and how luxurious it is to be able to return again and again to some small idea and really hold it in my mind...there it is...and it makes sense to me now. (Elena 2/25/93)

This statement is particularly revealing because it stands in such sharp contrast to her previous statements. Although she may have understood intellectually that mathematical ideas could be held in one's mind, considered, and understood, she had not personally experienced this sense of conceptual understanding. Following the return of the quiz, mathematics was still rules and formulas that must be "seared in my brain...square that radius!" rather than concepts and ideas that could be considered and understood. Similarly when she reflects on the value of communicating about mathematics for children, her statement suggests her own wistfulness for this experience. Only in this most recent statement does she indicate a personal and individual understanding that mathematics can indeed make sense to her and the wonderment and sense of peace that she now feels as she experiences learning in this way.

Observations And Conclusions

It is clear throughout the journal that Elena believes that mathematics should make sense and that doing mathematics should be largely a sense-making proposition. She also indicates a clear understanding of the importance of sufficient time and social interaction in the construction of mathematical knowledge. While this process of struggling with conceptual understanding of the mathematics evokes childhood memories of those same struggles, her ability to reflect on the mathematics from the child's point of view is a real strength. The very nature of these struggles has provided her with the opportunity (she has the natural disposition) to consider central issues in learning and teaching mathematics - the intrinsic value and beauty of mathematics - the importance of constructing knowledge for oneself

- the value of social interaction in the learning process - the importance of children working out their own representations. Elena's journal is also clearly an example of an individual who was pre-disposed to significant self-reflection. It is also unique and idiosyncratic in that she was an "older" student, married, and a mother of two children. She herself makes some of the distinctions between herself and the "twenty-year-olds."

The task of investigating the construction of conceptual knowledge and the subsequent transformation of that subject-matter knowledge into pedagogical content knowledge is extremely complex. As researchers, we struggle to construct tasks that will provide useful information about both of these processes. Elena's journal provides a rich source of just this type of information. Elena's story suggests that serious reflection about learning is very hard and sometimes very painful. However, the value of giving voice to the child within and the metacognitive monitor provides an invaluable lens through which teachers may focus on learning.

References

Ball, D.L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring with them to teacher education.* Unpublished doctoral dissertation. Michigan State University, East Lansing, MI.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

HER MATH, THEIR MATH: AN IN-SERVICE TEACHER'S GROWING UNDERSTANDING OF MATHEMATICS AND TECHNOLOGY AND HER SECONDARY STUDENTS' ALGEBRA EXPERIENCE

Rose Mary Zbiek, University of Iowa

This case study investigates an experienced secondary school mathematics teacher's understanding of mathematics ("her" math) and decisions she makes about her students' classroom experiences ("their" math). This report focuses on the competing roles of her growing understanding of novel technology-rich mathematics and her decisions about activities and expectations in this algebra course in light of her beliefs about learning and teaching. Data document developments in her mathematical understanding and classroom practice during her first 13 months of teaching Computer-Intensive Algebra[1] as a participant in the Empowering Secondary Mathematics Teachers in Computer-Intensive Environments project (CIME).[2]

Framework

A practicing teacher's understanding of school mathematics includes a blend of her knowledge and beliefs about formal mathematics, about pedagogy, and about how people learn mathematics. Numerous prior studies (cf. Ball, 1991; Carpenter, Fennema, Peterson, & Carey, 1988) document the impact of practicing teachers' understandings and beliefs on their classroom decision-making. Other works investigate the relationship between teachers' understanding of mathematics and their students' achievement. This literature suggests that teachers' understandings of mathematics affect their classrooms and their students' learning environment in complex ways.

As Fennema and Franke (1992) note, when positing a framework for research in teacher knowledge, there is a need for research that explores the relative roles of knowledge of mathematics, pedagogy, and learning with respect to beliefs and current context of the teacher. If technology is an integrated part of school mathematics and curricula change to reflect its presence, teachers are teaching mathematics that is new to them with a focus on process rather than product in a technology-intensive mathematics classroom filled with open-ended activities and mathematical explorations (NCTM, 1989). Technology then suggests a need to study a teacher's understanding and doing of mathematics with technology and the classroom learning environment she fosters.

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[1] A revised version of Computer-Intensive Algebra is now distributed by Janson Publications as Concepts in Algebra: A Technological Approach.

[2] CIME is funded by the National Science Foundation under award number TPE-9155313, M. K. Heid and G. Blume, Principal Investigators. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

Methodology

Context. The current study follows one teacher through her entire experience as one of 60 participants in Empowering Secondary Mathematics Teachers in Computer-Intensive Environments (CIME). CIME is a teacher enhancement project that implements and tests a model for the continuing education of secondary school mathematics teachers. The CIME experience begins with a four-week summer institute that concentrates on developing teachers' mathematical understandings and technology knowledge and on engaging them in using alternative forms of assessment to better understand students' conceptions of mathematics. The first summer workshop also introduces them to *Computer-Intensive Algebra* (CIA) as one example of an innovative, technology-intensive first-year algebra curriculum. This curriculum – built around the function concept, mathematical modelling, use of multiple representations, open-ended exploration, and constant access to computer algebra systems – differs greatly from teachers' prior experiences with first-year algebra. CIME teachers return to their schools to implement CIA and then attend a one-week institute during the next summer.

Instruments. Data reported here come from four sources as collected over 13 months. Ten interviews address understanding and doing of mathematics in the presence of computing tools, understanding and beliefs about how learning occurs and about what it means to understand mathematics, and documentation of classroom activities. Six series of observations (four in CIA classes and two in geometry classes) with pre- and post-observation interviews address the nature of mathematics in the classroom and the teacher's changing practice. The remaining data are a set of one journal entry per week written by the teacher and copies of course materials created and used by the teacher.

Analysis. Analysis of data began with the coding of all interview transcript passages that included any discussion of mathematics. Patterns arising from these yielded tentative hypotheses about the subject's understanding and beliefs about mathematics and her perceptions of students' understandings and learning of mathematics. The findings exemplified here are hypotheses that survived comparison with results of a similar analysis of the classroom observation and journal data.

Subject

"LeAnne" is certified to teach secondary mathematics and taught for 22 years before her CIME experience and this study began. She has undergraduate degrees in both mathematics and elementary education, mathematics certification at both the elementary and secondary levels, and a master's degree in secondary mathematics education. LeAnne teaches in a suburban/rural high school of approximately 850 students in grades 9 through 12. Teaching only at the high school level for the last decade, LeAnne had a fairly stable teaching assignment consisting almost exclusively of geometry courses. She never taught an algebra course using CIA materials prior to the CIME experience but verbally espoused CIA goals of technology use and exploration.

Findings and Discussion

In the paragraphs that follow, a summary of LeAnne's expressed views about learning, doing, and teaching mathematics followed by a description of her classroom environment lead to a seeming contradiction between what she espouses and what actually occurs in her classroom. Subsequent consideration of the LeAnne's views and actions however address the extent to which she seems to alleviate the contradiction as her understanding and beliefs grow and change throughout the academic year.

Expressed views. Throughout the year, LeAnne expressed consistent views about mathematics learning, teaching and curriculum. She frequently spoke of learning as discovery and teaching as facilitating, as exemplified in her assessment of a sample teaching scenario during one interview:

> [The teacher in this scenario is] questioning them…she's not saying…what the answers are or what they need to do to find them…she's having them, ah compare their answer with what information they have…to maybe help them think through what they should get. And this is what I do a lot;…very few times do I actually give the kids answers…But I ask them questions for them to think through what the answer should be.

The importance of reasoning through mathematical problems as opposed to simply knowing outcomes also came through clearly in her stated goals for the geometry course. These expressed values appear consistent with the goals of CIA and CIME and set expectations about how LeAnne herself would approach computer-intensive mathematics and orchestrate her classroom. What then seems to be the nature of LeAnne's mathematics and what characterizes her students' classroom experience?

"Her" mathematics. LeAnne herself never used a computer algebra system (CAS) to solve real-world problems, complete modelling tasks, or investigate function families prior to CIME and the current study. However, she spent many hours preparing for the class by using Calculus T/L II[3] (the CAS available in her classroom) and working through the computer labs in the CIA curriculum materials. She became adept at using the technology and used it during interviews in ways and for mathematics tasks that transcended as well as matched the CIA curriculum. At the end of the year, LeAnne could quickly use Calculus T/L II to produce and use symbolic rules, tables and graphs of functions, to edit these things and to re-organize these images on the screen. She also developed a deeper sense of exploring both situations modeled by a function and families of functions presented in the abstract. Evidence of the level of expertise she achieved is in an interview task at the end of the year: Describe the effects of changing the values of

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[3] Calculus T/L II is distributed by Brooks/Cole Publishing Company, Pacific Grove, CA.

a, b, and c on the graphs of functions of the form $f(x) = a^x + f(b,x) + c$. She begins with $a=b=c=1$ and enters $f(x)=1$^$x+1/x+1$; the display appears as $f(x) = 2 + f(1,x)$. She noted:

> I to any power will give you 1, so it's adding the 1 plus 1… *[She creates s table as in Figure 1.]* Okay, at 0 I get undefined, and at half, 4. Then it keeps on going down. *[She produces graph in Figure 2.]* So if I graph it; it's a hyperbola (sic). Okay. Ah, what I'm going to do is change a to maybe 2 and see what happens. *[Teacher entered $f(x) = 2$^$x+1/x+1$ and produced a table and then the graph in Figure 3, while losing the definition of the first function, $f(x) = 2 + f(1,x)$.]*…I should have called [this new function] a different function name because I lost the original.

LeAnne sketched and labeled each of these two graphs. She then tested $a = -2$ by producing the table and graph in Figure 4. She noted that in this table "there are more 'undefineds' here." After she created the graph, she discussed the conflict between the graph and the table, sketching what she claimed is a better graph (Figure 5). She then zoomed in on the portion of the graph for $0.5 \le x \le 2$ but got an error message and noted:

> But yet on my table it's saying that ah, let's see. At 1 y shouid be a 0; so there should be a point at 1. Hmm. Okay, let's. Change this [value of a to] -1; see what happens.

LeAnne explicitly compared the graph and table, noting a discrepancy and predicting the pattern for negative values. She then tested one more value of a ($a = -1$) and concluded:

> When it's positive you get ah, two hyperbolas (sic), and it keeps increasing which means making a smaller one [on the right side]. And then when it's negative it keeps going off into infinity every other number. I mean ah, whole number it gives you a point and then the next x which is, I've got it set at 2.5 at half then it goes undefined.

LeAnne seemed to have a feasible attack to the problem. She used the tool fluently to achieve her goals, occasionally editing her previous work or ideas. When the tool produced unexpected outcomes, she stopped and interpreted them or pursued them further. She explained in detail what she did with the CAS and why her actions and results made sense. Her agile use of the CAS was apparent. However, she consistently stated conclusions based on two examples (e.g., try $a = 1$ and $a = 2$ and conclude about all $a \ge 0$). If asked to further justify conclusions, LeAnne relied on one additional example (e.g., $a = -1$) and only occasionally reasoned abstractly. This was "her math."

"Their" mathematics. In September, students met in the lab and worked through explorations. LeAnne described the lab as a noisy place where students

x	y
0	undefined
0.5	4
1	3
1.5	2.66667
2	2.5
2.5	2.4
3	2.33333
3.5	2.28571
4	2.25
4.5	2.22222
5	2.2

Figure 1.

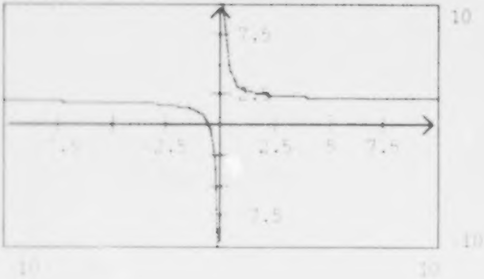

Figure 2.

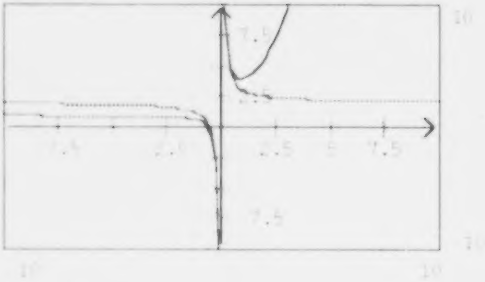

Figure 3.

x	y
0	undefined
0.5	undefined
1	0
1.5	undefined
2	5.5
2.5	undefined
3	-6.66667
3.5	undefined
4	17.25
4.5	undefined
5	-30.8

Figure 4a.

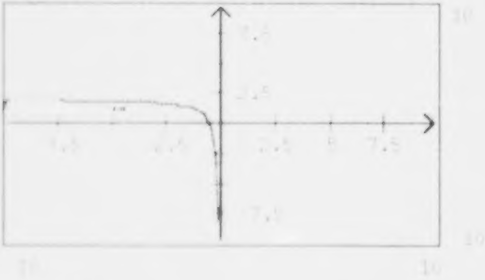

Figure 4b.

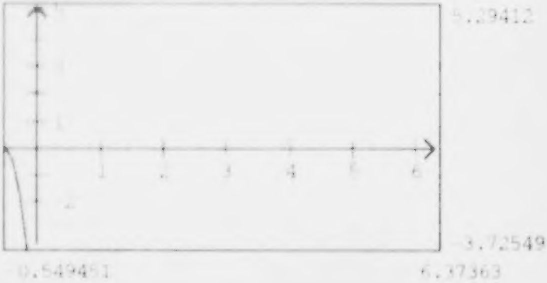

Figure 4c.

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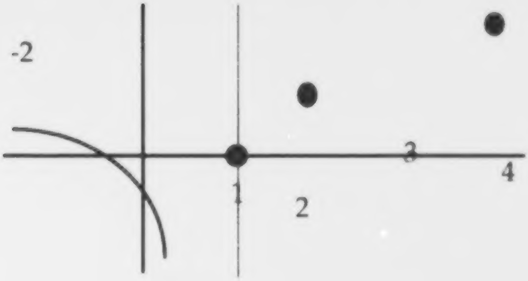

Figure 5.

asked many questions, puzzled over mathematical tasks and occasionally struggled with tool syntax – challenges LeAnne experienced while working through activities to prepare for class.

By the middle of the school year, LeAnne made some changes. Class began in a non-lab classroom where, with occasional computer demonstrations, LeAnne presented carefully prepared notes about how to use the CAS in the lesson. The notes prescribed keystrokes and commands that students would need to answer almost every problem they would encounter. Students then went to the lab and smoothly executed the lesson. For example, one CIA lab explores temperature in degrees Celsius as a function of temperature in degree Fahrenheit, with F(C) = f(9,5)C + 32. Students began this "exploration" by assembling in the non-lab classroom. LeAnne gave notes about using direct solve commands to determine the value of C given the value of F(C) and using computation commands to compute F(C) given the value of C. LeAnne and her students referred to these as "Finding Celsius" and "Finding Fahrenheit," respectively. Exchanges between teacher and student lab pairs during the lab experience then fell mainly into one of two predictable teacher-led patterns: determining whether a problem required "Finding Celsius" or "Finding Fahrenheit," and dwelling on the keystrokes needed. One example is LeAnne's exchange with two students as they sought F(56):

L: *They're looking for Fahrenheit. Don't you have Fahrenheit? [She scrolls up the screen to F(C)=f(9,5)C + 32.] Right. So let's reuse this, Fahrenheit. [She clicks on it.]*

S1: *I think so.*

L: *Then go into EXPRESSTON. Okay?*

S2: *Okay.*

L: *EXPRESSION. [Student clicks on EXPRESSION option.]*

S1: *Okay, for this we put, ah, 56 and then this. [LeAnne points to syntax notes for computing F(C). Student enters F(56) and the CAS responds with f(664,5).] 664 over 5.*

In addition to the notes, LeAnne created supplemental worksheets to provide practice with the tool. One worksheet showed printouts of LeAnne's CAS work to answer CIA questions. The students' task was to replicate her work, checking that they got the same results. The classroom mathematics experience moved from experimentation and diversity of approaches to precision and rapidity of task completion.

Initial comparison. Although LeAnne talked about valuing exploration, conceptual development and reasoning why, her students spent class time taking notes on LeAnne's explorations and then following these notes algorithmically. She knew the open-ended, exploratory mathematics environment reflecting her CIA goals, but she needed a less chaotic classroom. Her need for orderliness influenced on-going changes in her classroom. The result was a blend of *her* exploration and *their* organized activity.

Conclusion

In LeAnne's classroom, "their" math was "her" math at the beginning of the course. By the end of the year, their math became a well-orchestrated march through her tool-based tasks to lead to her mathematical conclusions. A myopic view of this observation however neglects to recognize the growth over one year in LeAnne's understanding of mathematics, agility with technology, and awareness of key aspects of innovative and radically different school mathematics curricula.

References

Ball, D. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. E. Brophy (Ed.), *Advances in research on teaching: Teachers' subject matter knowledge and classroom instruction* (Vol. 2, pp. 1-48). Greenwich, CT: JAI Press.

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in early arithmetic. *Journal for Research in Mathematics Education, 19*(5), 385-401.

Fennema, E., & Franke, M. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan Publishing Company.

THE INTERPLAY OF MATHEMATICAL UNDERSTANDINGS, FACILITY WITH A COMPUTER ALGEBRA PROGRAM, AND THE LEARNING OF MATHEMATICS IN A TECHNOLOGICALLY RICH MATHEMATICS CLASSROOM

M. Kathleen Heid,The Pennsylvania State University

As teachers begin to implement mathematics curricula that capitalize fully on computing technology and that are focused on concepts and applications instead of on execution of by-hand symbolic manipulation routines, their well-established routines of thinking about mathematics and its teaching no longer apply in seamless fashion. This case study, a part of which is reported here, examines the ways that an experienced teacher who participated in CIME, a four-week program on the teaching and learning of mathematics in technology-intensive environments, confronted some of the mathematical issues inherent in technology-intensive mathematics. This report gives some insight into one teacher's understanding of functions, independent variables, and parameters, and the ways that this understanding interacts with her use of the new computing tools.

Researchers (Fennema and Franke, 1992) have suggested important components of teachers' knowledge that impact on their teaching and their students' learning: knowledge of mathematics (Ball, 1988; Lampert, 1989) and mathematical representations (Hiebert and Wearne, 1986), pedagogical knowledge (Clark and Peterson, 1986; Shulman, 1986), and knowledge of how students come to understand mathematics (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In computer-intensive environments, additional components of teachers' knowledge that impact on their teaching and their students' learning may include knowledge of the use of technology for the exploration of mathematics and knowledge of the technology itself.

The Empowering Mathematics Teachers in Computer-Intensive Environments project (National Science Foundation award number TPE 9155313) is a multiple-year teacher enhancement/research project which focused on developing secondary mathematics teachers' abilities to implement computer-intensive mathematics curricula. Teachers involved in the project (Computer-Intensive Mathematics Education or CIME) completed several courses connected with their teaching of *Computer-Intensive Algebra*[1] (CIA) (Fey, Heid, et al., 1991), a radically reformulated beginning algebra curriculum that is built around the concept of function, employs calculators and computers as tools for student exploration, and develops fundamental concepts of algebra (e.g., variable, function, equivalence, system) through mathematical models of realistic situations. The CIME course experiences (one four-week course the summer prior to their teaching CIA and one one-week course the following summer) had three integrated components: mathematics; assessing students' understandings in technologically rich mathematics classrooms; and issues of teaching and learning in computer-intensive environments.

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[1] The *Computer-Intensive Algebra* Project was funded by grants from the National Science Foundation under award numbers DPE 84-71173 and MDR 87-51500 to The University of Maryland (Principal Investigator: James T. Fey) and award number MDR 87-51499 to The Pennsylvania State University (Principal Investigator: M. Kathleen Heid). The CIME project was funded by the National Science Foundation to The Pennsylvania State University through award number 91-55313 (Principal Investigators: M. Kathleen Heid and Glendon W. Blume; Faculty research associate: Rose Mary Zbiek). Any opinions, findings, conclusions, or recommendations expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation. First distributed through the Office of Technology Liaison at The University of Maryland, *Computer-Intensive Algebra* is currently being distributed as Fey, J. T. & Heid, M. K. with Good, R., Sheets, C., Blume, G., Zbiek, R. M. (1995).*Concepts in Algebra: A Technological Approach*. Dedham, MA: Janson Publications.

As teachers begin to implement mathematics curricula that capitalize fully on computing technology and that are focused on concepts and applications instead of on execution of by-hand symbolic manipulation routines, they find that their well-established routines of thinking about mathematics and its teaching no longer apply in the same seamless fashion. The case study reported here examines the ways that an experienced teacher who participated in the CIME program thinks about the new mathematics, the ways she interacts with computing tools, the ways she attempts to understand what her students are understanding, and the ways she converts her new experiences into a teaching/learning situation for her students.

Subject and data

The focus of the case study was Sara, a teacher who had taught mathematics, almost always first-year algebra, for over 20 years in the same rural high school. The primary data used as a basis for this case study consists of verbatim transcripts from a variety of sources over a thirteen-month period: task-based, scenario, and documentation interviews with Sara, eight observation cycles focused on the CIA class Sara taught, small group sharing sessions in which Sara participated during the summer courses, and sessions during both summers during which Sara helped plan and execute task-based interviews with a ninth grade student who had completed a *CIA* course.

We conducted three types of interviews[2] with Sara during the summers preceding and following her first year of teaching CIA. Task-based interviews (TBI at the beginning of Summer 1, TBII at the end of Summer 1, and TBIII during Summer 2) were designed to get a picture of Sara's understanding of mathematical concepts underlying CIA and her use of technological tools to explore those concepts. Scenario interviews (SCI at the beginning of Summer 1, SCII at the end of summer 1, and SCIII during summer 2) were designed to tap Sara's abilities to understand students' mathematical understanding as seen through interview transcripts provided for her. A documentation interview (DOC) during the second summer provided data on Sara's perception of teaching CIA.

We conducted four rounds of observations of the CIA class that Sara taught. Each round consisted of several days of observations. Pre-observation conversation and post-observation conferences along with the observations, were focused on Sara's instructional decision-making. Finally, we analyzed portions of what happened during the summer courses. We analyzed what Sara said about teaching CIA during small group sessions, and we studied the ways in which she attempted to assess student understanding through task-based interviews she and several others conducted both summers.

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[2] Interviews and observations were designed and conducted by M. Kathleen Heid, Glen Blume, and Rose Mary Zbiek. Analysis was aided by Mathematics Education doctoral students Barbara Edwards, Wilhelmina Mazza, and Barbara Edwards.

Results

Analysis of the data is currently ongoing, but preliminary results suggest possible tensions related to teaching mathematics in technologically rich environments. Several results address the effects of a teacher's developing understanding of mathematical concepts, of the use of computing tools, and of new ways to think about teaching and learning. An example of these effects is discussed below.

As Sara thought about, talked about, and taught a functions-oriented algebra course, her personal understanding of function came to the fore. Sara saw little use for function notation, often using explicit function rules rather than more generic function notation. Early in December, for example, Sara was beginning a total class discussion of a CIA problem which involved attendance at a talent show as a function of the price of a ticket. The function rule with which the class was working was $a(t) = 1.05(800 - 50t)$ and the class was finding the ticket price that yielded various attendance values. The following interchange ensued:

Sara: What was the input variable in this situation?

S1: Ah,... the input variable was the price of a ticket.

Sara: Okay. Okay, S2, what was the output variable?

S2: Attendance.

Sara: Okay, the output variable was attendance. And S3, do you remember what another form for the rule was that we were looking for yesterday?

S3: **a** equals one point oh five times the quantity eight hundred minus fifty **t**...

Sara: Okay, **a** equals ... so we know that instead of writing **a** of **t**, we can also just write that as **a** equals when we're wanting to find the attendance. When would you write it as simply **a** equals? Which command would you be using when you would do that?

S: The solve command.

Sara is suggesting to her class that they should find the attendance for a given price (say $8) by "solving" the equation $a = 1.05\ (800 - 50*8)$. Even though the program with which Sara's class was working would have allowed the user to ask the program to "evaluate" a(8), Sara prefers not to use function notation and finds a way to get the numerical answer without such notation. Interestingly, Calc T/L II[3] is a program especially designed to force the user's attention on the objects with which they are working. Before asking for some particular symbolic manipulation, the user must choose the object with which he or she is working. To evaluate the function a(t) for $t = 8$, the user would (1) select "function," (2) define the function, a, from the function window, then (3) select "expression," (4) write a(8) from the expression window, and (5) evaluate it. Sara was proposing what was, for her purposes, a shorter method: (1) select "function," (2) define the function, a, from the function window, (3) redefine the function a as $a = 1.05 (800 - 50*8)$, and then (4) write resulting redefined function, which would be displayed in evaluated form. The fact that the Computer Algebra System her CIA class used was predicated on a function as object approach was no help to Sara since she was reticent to explore the computer program and used it to get answers even if the methods producing those answers made little conceptual sense. Her use of the function concept suggested a "process" rather than an "object " concept. This tendency to view function as process along with her aversion to function notation played itself out as Sara encountered families of functions.

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[3] Calc T/L II by J. Douglas Child is distributed by Brooks Cole; Pacific Grove, CA.

Prior to teaching CIA, during the first CIME summer Sara was just beginning to deal with families of functions, at first allowing only families with familiar names (e.g., linear, quadratic). She took a "function as process" approach to exploring families of functions with which she had no previous familiarity. For example, in investigating the effects of a on $f(x) = f(a,x) + bx + c$, Sara started by assuming a b-value of 5 and a c-value of -5. She continued, saying "Well, let's just let x be 2, okay?" She then calculated the value of the resulting expression, $f(a,2) + 5(2) - 5$, for $a = -2$ and $a = -4$, and concluded that the function decreases as a decreases since $f(-4,2) + 5(2) - 5 < f(-2,2) + 5(2) - 5$. The fact that she took a numerical instead of a graphical approach to her exploration may have been thought to be related to her relative inexperience at that time with graphics programs and their use in teaching algebra. The following summer, however, after having taught CIA to a low-ability group of ninth graders for a year, her approach to exploring functions was not very different from the first summer's approach. In exploring the function $f(x) = a^x + f(b,x) + c$, she decided to let b be equal to 1, let c be equal to 0, and let x be equal to 2. She then calculated and examined values of $a^2 + f(1,2)$ as the value of a increased in a manner similar to her exploration the previous summer. She continued the exploration, this time seeming to reverse the role of the parameter and the independent variable completely, graphing $f(x) = x^2 + f(1,2)$, and treating the original a as if it were the independent variable instead of the parameter. Because Sara has fixed the value of the original independent variable at $x = 2$, she is examining a different function than was originally intended and concludes that changing b and c have the same effect on the function. She noted that, in so doing, "whether I change the b or the c, it has the same effect." Because the function notation itself has little meaning for Sara, the fact that she is examining $f(x) = a^x + f(b,x) + c$ is no different from her examining $f(a) = a^x + f(b,x) + c$.

For Sara, teaching mathematics in a technology-intensive environment meant encountering new mathematics or encountering old mathematics that takes on new importance. In many traditional mathematics textbooks, there was no confusion about the meaning of the function notation. In $f(x) = 4^x + f(5,x) + 6$, it was clear

that the independent variable was x. In those environments, Sara and her students would have had to confront the meaning of the function notation. In the technology-intensive environment surrounding the teaching of CIA, both Sara and her students were confronted with situations in which clearer understandings of functions and function notation were needed. Perhaps because Sara was not one to explore the tool on her own and because she could find ways, however conceptually inappropriate they might have been, to generate numerical answers without using appropriate notation, her understanding and use of function notation seemed not to improve substantially over the year.

Sara's emerging understanding of functions and avoidance of function notation, her reluctance to explore the capacity of the computer, and her lack of experience with families of functions combined to produce a confusing perception of one of the central CIA mathematical concepts. Other data suggests that this set of circumstances had significant effects on her students' mathematical understandings.

References

Ball, D. (1988). Unlearning to teach mathematics. East Lansing: Michigan State University; National Center for Research on Teacher Education.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P. & Loef, M. (1989). Using knowledge of children's mathematical thinking in classroom teaching: An experimental study. *American Educational Research Journal*, 26(4), 499-532.

Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd edition, pp. 255-296). New York: Macmillan.

Fennema, E. & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Fey, J. T., Heid, M. K., et al. (1991). *Computer-Intensive Algebra*. College Park, MD: University of Maryland and The Pennsylvania State University. [Now distributed as *Concepts in Algebra: A Technological Approach*. Dedham, MA: Janson Publications.]

Hiebert, J. & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert, (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 199-223). Hillsdale, NJ: Lawrence Erlbaum.

Lampert, M. (1989). Choosing and using mathematical tools in classroom discourse. In J. E. Brophy (Ed.), *Advances in research on teaching* (Vol. I). Greenwich, CT: JAI Press.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF MULTIPLICATION INVOLVING FRACTIONS

Diane S. Azim, Washington State University

This research focuses on preservice elementary teachers' understanding and reconstruction of understanding about multiplication in the fraction domain. At the start of the study, 44% of the 50 preservice teachers studied reported that they had a method for reasoning about multiplication with fractions; 28% were able to describe a situation modeled by multiplication with a fraction operator. Although reasoning individually, the preservice teachers revealed common dimensions of understanding — about taking fractional parts of non-unit wholes, and about numerical effects, referents for results, and the invariance of multiplication — as they reconceptualized multiplication with fractions. "Sense" of multiplication and "sense" of fraction relationships were forms of reasoning that supported the reconceptualization process.

With a renewed interest in conceptual understanding of mathematics as a fundamental purpose for classroom instruction, the conceptual understandings of teachers and prospective teachers emerge as an extremely important issue. Recent studies of teachers' knowledge of rational numbers concepts and procedures have revealed such findings as the fact that, "Ten to 25 percent of the [218] teachers missed items which we feel were at the most rudimentary level. In some cases, almost half the teachers missed very fundamental items" (Post, Harel, Behr, & Lesh, 1991, p. 186). Research into preservice teachers' understandings of multiplication with decimal numbers has resulted in similar findings (e.g., Graeber & Tirosh, 1988; Harel, Behr, Post, & Lesh, 1994).

The current research focuses on preservice teachers' understanding of — and re–construction of understanding about — multiplication with fractions. Multiplication with fractions is included in United States textbooks in the middle school grades, and teachers are expected to teach this topic with a conceptual orientation, having knowledge of its potential for modeling real world situations. This study researches prospective teachers as they enter an elementary mathematics methods course (following the conclusion of their mathematics content coursework), and addresses the following research questions: (1) How do the preservice teachers reason about multiplication with fractions as they enter the methods course? (2) What common dimensions of understanding about multiplication with fractions do the preservice teachers evidence as they construct understanding? (3) How do individual preservice teachers construct understanding? (4) What forms of reasoning appear to influence or support their re–construction of understanding about multiplication in the domain of fractions?

These questions focus on preservice teachers' understandings about multiplication with fractions from four perspectives: their entry level understandings or forms of reasoning, the content–related structures of understanding that they collectively reveal as they construct new understanding, perspectives on individual preservice teachers' methods of constructing and re–constructing understanding, and theoretical speculation about reasoning that supports the re–construction of understanding about multiplication in the fraction domain. The theory, or theoretical framework, within which this research is conducted is the research–based theory that whole number multiplication must be re–conceptualized in the rational number domain (Greer, 1994; Hiebert & Behr, 1988):

> It is likely that there are not smooth continuous paths from early addition and subtraction to multiplication and division, nor from whole numbers to rational numbers. Multiplication is not simply repeated addition, and rational numbers are not simply ordered pairs of whole numbers. The new concepts are not the sums of previous ones. Competence with middle school number concepts requires a break with simpler concepts of the past, and a reconceptualization of numbers itself. (Hiebert et al., p. 8)

The research questions are framed within this theory. It was the purpose of this research to infer prospective elementary teachers' understandings about multiplication with fractions, their methods of constructing and re–constructing understanding,[1] common structural dimensions in their re–construction of understanding, and particular forms of reasoning supporting their re–conceptualization of multiplication in the fraction domain.

Methodology

Fifty preservice elementary teachers enrolled in two sections of a required state university mathematics methods course taught by the researcher contributed research data in a three–phase qualitative research design:

Phase I (weeks 1–4): Entry–level assessment of understandings of preservice teachers through one–hour individual audiotaped interviews focused on their work on a written inventory that requested them to create and solve a word problem modeled by each of four given fraction and whole number multiplication expressions.

Phase II (weeks 5–10): Instruction about multiplication with whole numbers and fractions—conceptual models for multiplication, numerical patterns in multiplication with whole numbers and fractions, and situations modeled by multiplication—in order to support the preservice teachers in construction and re–construction of understanding (rather than to provide and measure the effects of a treatment); collection of coursework; and keeping of field notes of the class sessions about multiplication.

Phase III (weeks 11–16): Inferring of understandings and construction processes in the preservice teachers' development of understanding, through one–hour audiotaped individual interviews during which they were asked to describe their understandings about multiplication with fractions and to conceptually interpret a sample of fraction multiplication expressions, through continuing coursework, and through class field notes.

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[1] Students both constructed understanding about whole number multiplication and re–constructed understanding about whole number multiplication in the fraction domain.

During and following Phases I–III, the data were interpreted and analyzed to address each research question. Processes of analysis, including interpreting, inferring, and categorizing forms of reasoning and structures of understanding, were utilized. Review and evaluation of the interpretations and findings by a cohort of mathematics education researcher experts, and triangulation of the data through the complementary data sources for each preservice teacher, were utilized.

Findings

As they entered the methods course, 8 (16%) of the 50 preservice teachers were able to create word problems modeled by all four of the whole number and fraction multiplication expressions on the written Inventory [24 x 37, 7 x $\frac{1}{4}$, $1\frac{1}{2}$ x $\frac{1}{3}$ and $\frac{2}{3}$ x $\frac{3}{4}$]. Eighteen students[2] (36%) were not able to create a word problem for any of the three fraction multiplication expressions on the Inventory, and an additional 18 students (36%) were able to create a word problem for only one of the three fraction multiplication expressions, the expression with one whole number factor [7 x $\frac{1}{4}$]. In other words, 36 students, 72% of the students in the study, entered the methods course unable to describe a situation that would be modeled by multiplication with a fraction operator. In addition 4 students entered the methods course unable to construct a word problem appropriately modeled by the whole number multiplication expression 24 x 37.

Twenty-two students (44%) entered the methods course reporting that they had been taught or had discovered a form of reasoning about multiplication with fractions less than 1 [the forms being that multiplying by a fraction less than 1 reduces other numbers, divides other numbers, or takes a fraction of other numbers]. Each of the 14 students who succeeded in creating a word problem for one or both of the Inventory multiplication expressions with a fraction operator was among this number. The other 28 students in the study (56%) reported that they had no method for reasoning about multiplication with fractions as they entered the methods course. Their knowledge was strictly procedural.

Structural Dimensions of Understanding

Structural dimensions or benchmarks of learning common to the preservice teachers as they constructed and re-constructed understanding about multiplication during the study are described as follows:

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[2] The preservice teachers are referred to alternatively as "preservice teachers" or "students." They were students in the methods course in which this study was conducted.

Understanding the Numerical Effects of Fraction Multiplication

Students provided evidence of constructing knowledge about the numerical results of multiplying with fractions: that multiplying by a number less than and greater than 1, respectively, reduces or enlarges other numbers. During their second interviews, 48 of the 50 preservice teachers discussed the numerical results of fraction multiplication in relation to the multiplicative identify, 1 (The remaining two students were confused in their understanding.). However, not all students could interpret fraction multiplication expressions using the commutative property, or in terms of the influence of each factor in the expression on the other factor; fewer than 50% of the preservice teachers departed the study demonstrating the flexibility to interpret expressions in two directions.

Conceptualizing a Fractional Part of a Non–Unit Whole

For one–half of the students, conceptualizing a fractional part of a quantity other than one discrete unit (other than 2/3 of 1 cup or 2/3 of 1 hour, for example), involved new learning. Even though 14 of the students in the study had learned or discovered that they could reason about multiplying with fractions using the word of (e.g., 1/3 of 6 for 1/3 x 6), 10 of these students, in addition to other students in the study, experienced difficulty physically representing and making sense of expressions such as "*2/3 of* 3/4" (2/3 x 3/4). Learning to operate on a non–unit quantity when the operator is a fraction less than 1 (such as learning to take 2/3 of 3/4 rather than 2/3 of 1 unit) emerged as an important benchmark in the preservice teachers' continuing development of understanding of multiplication with fractions. As students constructed understanding, some students experienced difficulty, similarly, conceptualizing a fraction greater than 1 operating on a non–unit whole, such as $1\frac{1}{3} \times \frac{3}{4}$ or $2\frac{1}{2} \times 4$. All students revealed during their second interviews that they had constructed understanding, in some form, of the taking of a fractional part of a non–unit whole. Methods students used to conceptualize this process with operators less than and greater than 1 differed.

Interpreting Referents of Results

Approximately one–third of the students experienced difficulty identifying the referents, or units of measure, for fraction multiplication results. They experienced difficulty understanding the whole (or referent unit of measure) to which the result of the multiplication referred. Most commonly, students who experienced difficulty interpreted the referent for the result of multiplication using the original quantity being reduced or enlarged: 3/4 x 1/2 = 3/8, for example, was interpreted as 3/8 of "1/2 unit" rather than "3/8 unit." Some difficulties in interpreting referents seemed related to the fact that students attempted to make sense of fraction multiplication through representing the numerical answers to, rather than the conceptual processes involved in, fraction multiplication expressions — such as in attempting to represent the answer of 3/8 (and interpreting this as 3/8 within the 1/

2 unit), rather than representing the conceptual process of taking 3/4 of 1/2 and then interpreting the referent for the 3/8. Reasoning about and representing the conceptual processes involved in fraction multiplication expressions supported students in interpreting referents for multiplication results.

Understanding of Multiplication as an Invariant Process

One-third of the students described multiplication, during their second interviews, evidencing a conceptualization of multiplication as an invariant process—modeling the same situations or illustrating the same conceptual models whether with whole numbers or fractions. Other students, who could interpret multiplication expressions with fraction and whole number operators, interpreted as distinctly different the concepts or models involving whole number operators and those involving fraction operators—frequently a repeated addition or equal groups model for whole numbers and a "breaking down" concept (in the words of several students) for fraction operators. Conceptualizing multiplication as an operation modeling the same concepts or situations with whole number and fraction operators was difficult. Although students could discuss the numerical effects of multiplying by numbers greater than and less than 1, interpreting multiplication with operators both greater than and less than 1 using the same conceptual model (e.g., equal groups, multiplicative compare, and area) was difficult.

Individual Forms of Reasoning and Constructing Understanding

The construction processes revealed by preservice teachers as they developed understanding of the four structural dimensions described above were distinctly unique (see Azim, 1995). During their second interviews, students' reasoning and understandings could be categorized in three categories: students constructing concepts for the first time during the interview—19 students (38%); students reconstructing concepts—17 students (34%); students building more complex constructions on their own—14 students (28%).

Forms of Reasoning Supporting Subjects' Construction of Understanding

Two particular forms of reasoning seemed to support students' construction of understanding of multiplication with fractions throughout the study: (1) their multiplication "sense," or forms of reasoning about multiplication (particularly with whole numbers), and (2) their fraction "sense," or sense of size relationships between fractions or fractional quantities. Students who had a more clearly developed sense of multiplication with whole numbers (having one or two even implicitly constructed conceptual models to draw on) and who could discuss fractional quantities in relation to each other (such as the relationship that 1/2 is one-third of, or one of 3 equal parts in, $1\frac{1}{2}$), drew on these senses to construct meaning for fraction multiplication. Students who demonstrated a limited understanding of multiplication with whole numbers (having a very weak or no concept of this operation), and a limited sense of fraction relationships, experienced greater difficulty interpreting fraction multiplication. The most powerful finding of this study was the repeated observation of the influence of the interaction of the levels of development of these two forms of reasoning on the preservice teacher's re–construction of understanding about multiplication. This finding is parallel to a summary by Sowder (1992) regarding good *estimators*: "They demonstrate a deep understanding of numbers and operations, and they continually draw upon that understanding" (p. 375). Deep understanding of multiplication and of fractions supported preservice teachers in their construction of understanding about fraction multiplication.

Other Theoretical Connections

The data in this study support Greer's (1994, p. 77) observation that, "The invariance of multiplication... over the numbers is a powerful idea that potentially can be harnessed to overcome the limitations of intuition." Students constructed understanding (and started to construct understanding) about multiplication as an invariant operation through different reasoning processes; knowledge that multiplication is invariant supported them in their attempts to re–interpret multiplication with fractions. Students also revealed evidence of using both quantitative reasoning (reasoning about quantities without numerical reference) and numerical reasoning (reasoning about numbers evaluating quantities)—two forms of reasoning theorized by Thompson (1994)—in their re–construction of understanding. Students who constructed each of the four dimensions (described above) to greater degrees evidenced both numerical and quantitative reasoning in the construction process.

References

Azim, Diane S. (1995). *Preservice elementary teachers' understanding of multiplication with fractions.* (Doctoral dissertation, Washington State University, Pullman, 1995).

Graeber, A. O., & Tirosh, D. (1988). Multiplication and division involving decimals: Preservice elementary teachers' performance and beliefs. *Journal of Mathematical Behavior, 7*, 263-280.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp.61-85). Albany, NY: SUNY Press.

Harel, G., Behr, M., Post, T., & Lesh, R. (1994). The impact of the number type on the solution of multiplication and division problems: Further investigations. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 363-384). Albany, NY: SUNY Press.

Hiebert, J., & Behr, M. (1988). Introduction: Capturing the major themes. In J. Hiebert & M. Behr (Eds.), *Research agenda for mathematics education: Vol. 2. Number concepts and operations in the middle grades* (pp. 1-18). Reston, VA: National Council of Teachers of Mathematics.

Post, T. R., Harel, H., Behr, M. J., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177-198). Albany, NY: State University of New York Press.

Sowder, J. (1992). Estimation and number sense. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 371-389). New York: MacMillan.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: SUNY Press.

THE ROLE OF ONE TEACHER'S MATHEMATICAL CONCEPTIONS IN HIS IMPLEMENTATION OF A REFORM-ORIENTED FUNCTIONS UNIT

Gwen Lloyd, University of Michigan
Melvin R. (Skip) Wilson, University of Michigan

This paper links the conceptions of an experienced high school mathematics teacher to aspects of his first implementation of a reform-oriented curriculum during a six-week functions unit. The teacher exhibited comprehensive understandings of the function concept, dominated by graphical representations and a covariation description of function. These features contributed to classroom emphases on the use of multiple representations to understand dependence patterns in data and characteristics of different types of relationships.

The current reform movement in mathematics education places new demands on teachers to offer students varied classroom opportunities to develop deep understandings of the function concept. The envisioned treatment of functions in grades 9-12 includes modeling real-world situations using functions; representations and interpretations of relationships using tables, graphs, equations, and verbal descriptions; translations between multiple representations of functions; and recognition of the variety of problem situations that can be modeled by the same type of function (National Council of Teachers of Mathematics [NCTM], 1989). There is little empirical information about how secondary mathematics teachers cope with the complexity of change as they attempt to incorporate such recommendations into their teaching practice. This paper documents the influence of one veteran high school teacher's mathematical conceptions on his instruction during a six-week functions unit using the reform-oriented curricular materials of the Core-Plus Mathematics Project.

Previous empirical and theoretical work about teachers' and students' understandings of the function topic contributed to the conceptualization of this study (e.g. Even, 1990; Leinhardt, Zaslavsky, & Stein, 1990; Norman, 1992; Vinner & Dreyfus, 1989). Our framework also builds on the growing body of research and theory related to teachers' knowledge and beliefs about mathematics and teaching that has begun to support the notion that teachers' conceptions contribute significantly to their instructional practice (Fennema & Franke, 1992). In particular, recent investigations have reported that experienced teachers' knowledge acts as a critical filter in the interpretation of reform-oriented mathematics curricula (Gamoran, 1994; Wilson, 1990). Taken as a whole, this literature emphasizes the need for further consideration of the complex relationship between teachers' conceptions and their instruction in the mathematics classroom.

Research Design

This study followed an interpretive case study design to investigate the conceptions of Mr. Allen, a 14-year veteran high school mathematics teacher, as part of a larger, ongoing project examining the experiences of three teachers implementing the Core-Plus materials for the first time. During the 1994-95 school year, Mr. Allen voluntarily used the Core-Plus Course 1 materials in a single class of ninth grade students. The public school district where he teaches is located in a small urban community in the Northeast United States. Data were collected between September 1994 and January 1995 using interviews, observations, and classroom artifacts. All interview data were transcribed.

At the beginning of the school year, Mr. Allen participated in two video-taped baseline interviews. In the first interview, referred to as the function sort, he was given the task of interpreting and organizing a stack of 32 cards depicting mathematical relationships that varied along these dimensions: family of function, representation, and particular characteristics such as functionality or continuity. These differences supplied a challenging set of situations for Mr. Allen to analyze and multiple criteria on which to base an arrangement of the cards. The second baseline interview was used to further investigate his informal and formal descriptions of the function concept and his orientations toward teaching about functions.

Mr. Allen's classroom was observed on a daily basis for 26 consecutive lessons (11/11/94 - 12/22/94) while he implemented the Core-Plus *Patterns of Change* unit that focuses on varied representations and explorations of real-world functional relationships. Detailed fieldnotes of his teaching were taken, and worksheets, quizzes, and tests were collected. A videocamera and remote microphone followed Mr. Allen as he moved around the classroom, capturing both whole-class and small group instruction. Observations were supplemented by four interviews during which Mr. Allen was asked to comment on certain instructional decisions. At the conclusion of his instruction with *Patterns of Change*, he watched selected videotaped segments from the function sort interview and reflected on his experience with the unit as it related to the particular segments.

Mr. Allen's Conceptions Prior to Teaching the Unit

Given a choice of formal textbook definitions of function including both correspondence and covariation descriptions, Mr. Allen favored definitions involving univalent correspondences between sets. However, his informal characterization of function involved "the relationship between two things and ... how a change in one affects the other," a covariation relationship that he claimed is most clearly viewed in a graphical display. Despite his contrasting concept image and definition, Mr. Allen demonstrated flexibility in his thinking as he appreciated the different utilities of the two notions of function: his covariation image describes "how the two variables are working together," but the more restrictive univalent correspondence definition is indispensable because "how [the variables] work together determines whether it's a function or not." Although Mr. Allen made occasional use of his formal concept definition, for example to determine the functionality of unfamiliar relationships, his covariation image dominated his verbalizations and actions in the function sort interview, during which he repeatedly asked himself the question, "As one changes, what happens to the other one?" and relied heavily on graphical representations to interpret the function sort situations. His analysis of each relationship consisted of first developing a general sense of how a change in one variable affects the other, and then attaching a label that indicates the appropriate family of function. For instance, to understand a verbal description of the area and diameter of a pizza, he first expressed that "as the diameter would increase, the area would increase by that factor to the second power," and then concluded that the relationship was "quadratic."

Mr. Allen's efforts to develop appropriate labels also illustrate his graphical proficiency. When he examined cards showing graphs, he created family labels (e.g. linear, quadratic, etc.) on sight by immediate recognition. In contrast, he found the information in verbal and tabular representations to be less accessible and thus made frequent translations to graphs before labeling them. For example, although examination of the table shown in Figure 1 led Mr. Allen to observe that there are "some square numbers on the bottom," he did not achieve a conclusive family label until he created a graph to help him "see" the relationship.

x	-2	-1	0	1	2
y	1	0	1	4	9

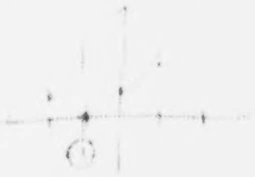

Figure 1. A table of "perfect squares" and Mr. Allen's corresponding graph.

Looking at his graph, he announced that "it's parabolic centered around negative 1," and concluded that the table represents a quadratic relationship. Mr. Allen's visual strengths also facilitated his construction of a card ordering, based on traditional teaching sequence, that began in the cards showing graphs and resulted in similar core orderings in each of the four representations of the sort as follows: linear, polynomial, exponential and logarithmic, and trigonometric. The consequent organization allowed Mr. Allen to point out connections between different representations of the same family of function.

To summarize briefly, covariation and graphical representations dominate Mr. Allen's thinking about functional situations. Covariation notions guide him toward the development of a complete description of a dependence pattern, including association with a major family of function. Graphs offer Mr. Allen the best display of covariation, serve as the primary format to which other representations are translated for analysis, and thus act as the source for connections between different representations of the same family of function.

Links Between Mr. Allen's Conceptions and Instruction

Mr. Allen's strong covariation image played a crucial supporting role in his adaptation to the Core-Plus approach to functions that is summarized in this excerpt from the *Patterns of Change* Student Text: "In many cases, we can describe the relation between two variables by saying that one variable is a function of another, particularly if the value of one variable depends on the other" (p. 4). Although his formal correspondence definition and typical classroom introduction of function appeared to be closely tied to traditional materials and activities, Mr. Allen's flexible understandings enabled him to comfortably enact the less formal covariation approach laid out in *Patterns of Change* and place emphasis on dependence relationships throughout his instruction. He repeatedly engaged students in discussions framed by the very same questions that he had applied to his own thinking during the function sort: "*Is* there a relationship?" and "*How* are the variables related?" The use of these questions put his own images into action in the creation of opportunities to encourage students to interpret features of a variety of dependence relationships.

Consistent with the dominance of visual representations in his own thinking about functions, Mr. Allen gave precedence to graphs in his implementation of the unit, portraying them as crucial tools that offer the optimal display of patterns of dependence. As he pointed out to a student, "The table gives you times and heights, but the graph gives you *the relationship* between time and height." The privileged position of graphs among the representations he used in the classroom was further evidenced in his numerous additions of "investigative graphing" tasks to assignments, and his urging of students to make effective use of the graphics calculators to quickly create visual representations.

Although Mr. Allen's conceptions and instruction were dominated by graphical displays of relationships, his classroom actions also demonstrated a high regard for explorations of multiple representations of problem situations. Mr. Allen's overriding interest in the determination of covariation patterns contributed to his tolerance of representations other than graphs. He appreciated the centrality of the variety of tables, graphs, equations, and verbal descriptions in the *Patterns of Change* activities because of the different information that each representation provides. As he explained, although graphs are the most helpful to him personally, "some people might be able to see the relationship with an equation, and ... making a table can maybe help them." In accord with his belief that further representations offer increased opportunities for students to understand a relationship, Mr. Allen frequently supplemented the *Patterns of Change* materials with extra representational tasks, including development of both recursive and explicit rules, construction of tables and graphs, and writing verbal descriptions of covariation patterns.

In addition to his classroom focus on the variety of perspectives provided by multiple representations, Mr. Allen emphasized the links between different displays of the same relationship. Reflecting the strength of the connections that he exhibited during the baseline interviews, Mr. Allen repeatedly communicated to students that the "rule, table, and graph all show the same thing." He capitalized on graphs in his instruction as a starting point from which to stress connections between multiple representations of the same situation, and to accentuate the features that distinguish different families of functions.

The main themes just discussed may be best illustrated by an example of Mr. Allen's classroom interactions that occurred during a *Patterns of Change* lesson involving the development of tables, graphs, and the explicit equations $I=2.50T$ and $P=2.50T-450$ relating a theater's daily income I and profit P to the number of tickets sold T. After students had investigated this situation in their groups, Mr. Allen used the two linear graphs for I and P versus T (as shown in Figure 2) as a focal point for a whole-class summary discussion.

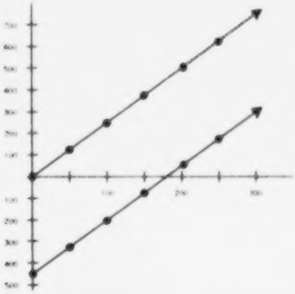

Figure 2. Reproduction of Mr. Allen's display of two graphs on the chalkboard.

Mr. Allen first drew attention to the constant rate of change in the covariation relationship between I and T by noting the increasing graph and asking, "*How* is it going up? If you sell 1 ticket, how much income do you get?" This ongoing form of questioning allowed him to highlight that "it is going up by the same amount each time," and to subsequently connect the constantly increasing data values to the linearity of the graph: "That's why you see the dots lining up." Having illustrated the constant rate of change in I, Mr. Allen attempted to demonstrate the same feature in P by asking "Does anyone notice anything about these two lines?" Mr. Allen related the students' visual observations that the lines are straight, parallel, and increasing back to the table data and equations: "For every ticket you sell, both are going up by 2.50 for each ticket sold. Notice in the equations, both are multiplied by 2.50." In a similar manner, he pointed out the different "starting points" for I and P through comparison of the graphs and equations. Thus with his demonstrations rooted in graphical displays, Mr. Allen identified distinguishing features of linearity and established links between their emergence in the varied representations of the theater situation.

The above illustration exemplifies the tight connections between Mr. Allen's conceptions and his instructional emphases that were evidenced throughout his implementation of the unit. His graphical proficiency and personal focus on patterns of covariation empowered him to utilize the Core-Plus materials to facilitate classroom interactions that constructed explicit ties between representations and types of functional relationships.

Discussion

Our findings corroborate those of other studies suggesting that teachers' comprehensive, well-organized conceptions contribute to instruction characterized by an emphasis on conceptual connections, powerful representations, and meaningful discussions. In stark contrast to the teacher in the study of Stein, Baxter, and Leinhardt (1990) whose limited knowledge of functions led to narrow instruction marked by *missed opportunities* to highlight connections between concepts and representations, Mr. Allen applied his flexible understandings to his implementation of the Core-Plus unit in ways that *created* such opportunities. Moreover, this study illustrates the notion that teachers who can make connections between different approaches to content can adjust their teaching to accommodate ideas that are not traditionally emphasized in the school curriculum. Because Mr. Allen was able to reconcile the Core-Plus approach to functions with the prominent features of his own conceptions of functions, the *Patterns of Change* materials furnished a way for him to translate his understandings into new but comfortable classroom strategies.

This paper focuses on the impact of Mr. Allen's conceptions of function on his instruction with the Core-Plus materials, but there were certainly other important influences on his teaching. For instance, throughout the year, he faced a tension between teacher direction and student independence. As Mr. Allen himself suggested, finding a suitable balance will involve developing greater familiarity with the Core-Plus materials and classroom behavior in a more student-centered classroom. The resolution of this struggle is particularly important in light of Mr. Allen's deep conceptions: the more comfortable he can become with his newly-defined role as teacher in the reform classroom, the more freedom and energy he will be able to devote to the application of his comprehensive understandings to even more meaningful opportunities for student learning.

References

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521-544.

Fennema, E. & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Gamoran, M. (1994, April). *Content knowledge and teaching innovative curricula.* Paper presented at the Annual Meeting of the American Educational Research Association. New Orleans, LA.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

NCTM. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Norman, A. (1992). Teachers' mathematical knowledge of the concept of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA Notes, Vol. 25, pp. 215-232). Mathematical Association of America.

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal, 27*(4), 639-663.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*, 356-366.

Wilson, S. M. (1990). A conflict of interests: The case of Mark Black. *Educational Evaluation and Policy Analysis, 12*, 309-326.

EXPERIENCED TEACHERS DO NOT GIVE UP EASILY: A TEACHER'S STRUGGLES WHEN TEACHING CONDITIONAL PROBABILITY FOR UNDERSTANDING

José (Alberto) Contreras, The Ohio State University

Objectives

In this presentation I describe and discuss some teaching episodes in which an eighth grade algebra teacher; Mr. Kantor, provided meaningful explanations of the concept of conditional probability (CP). I attempt to understand Mr. Kantor's explanations and what they reveal about the complex cognitive skill of teaching. To this end, I focus on (a) Mr. Kantor's content knowledge (CK) and pedagogical content knowledge (PCK) related to CP, and (b) how this topic is structured in the textbook.

Participant and Setting

Mr. Kantor holds secondary certification and has been teaching mathematics for about five years at a middle school in a suburban school district known for high student achievement.

Theoretical and Empirical Background

Shulman (1987) theorizes that teachers transform their personal understanding of subject-matter knowledge to make it understandable to students. That is, he contends that teaching is a pedagogical process. Doyle (1992), on the other hand, argues that this process is both a curricular and a pedagogical process. This process draws theoretically upon three main types of knowledge: CK, PCK, and curricular content knowledge. One means through which teachers transform a curricular topic is the use of explanations. Teachers' explanations are receiving increased attention in research on teaching (e.g., Borko et al., 1992).

Data Collection

I relied mainly on videotaped lessons to gather information about Mr. Kantor's explanations. These data were supplemented with questionnaires, open and semistructured interviews, stimulated recall interviews, and written documents.

Data Analysis and Results

Mr. Kantor posed several problems involving conditional probability in which he provided some meaningful explanations of why $P(A \cap B) = P(A) \cdot P(B \text{ given } A)$. The analysis also shows that his CK is rich and connected, that his PCK is growing, and that his CK, PCK, and the textbook have an impact on his teaching.

Conclusions

These findings suggest that knowledgeable teachers can use their CK and PCK to provide conceptual explanations, and also provide information about how teachers' knowledge affects their instruction.

References

Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194-222.

Doyle, W. (1992). Curriculum and pedagogy. In P. W. Jackson (Ed.), *Handbook of research on curriculum* (pp. 486-516). New York: Macmillan.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

TEACHERS' PERCEPTIONS OF THE ROLE OF LIMITS IN THE TEACHING OF CALCULUS

Linda Simonsen, Montana State University

The main goal of this research was to investigate high school advanced placement calculus teachers' subject matter and pedagogical perceptions by examining the following questions: What are the teachers' perceptions of the concept of limit, the role of limits, and the teaching of limits in calculus? Are these teachers' perceptions associated with their participation in a technology enhanced calculus reform project focused on staff development?

A multi-case study approach involving detailed examination of six teachers was used. The sample consisted of three "project" teachers who had participated in a particular calculus reform project and three "independent" teachers who had not participated in any calculus reform project. The data collected and analyzed included questionnaires, interviews, observational field notes, videotapes of classroom instruction, journals, and written instructional documents. Upon completion of the data collection and analysis, detailed teacher profiles were created with respect to the questions above. The results of this study were then developed by searching for similarities and differences across the entire sample as well as comparing and contrasting the group of project teachers and the group of independent teachers.

Results indicated that the teachers perceived calculus as a linearly ordered set of topics in which the concept of limit formed the backbone for appreciating and understanding all other calculus topics. The teachers felt the intuitive understanding of limits was essential to the further understanding of calculus. However, little class time was devoted to developing this intuitive understanding and little emphasis was given to drawing connections between limits and subsequent calculus topics. The independent teachers devoted much time to discussing formal epsilon-delta definition and arguments. The complex relationship between teachers' perceptions and classroom practice appeared to be affected by the significant influence of the teachers' goals of preparing students for the AP exam and college calculus and the authority given to the calculus textbook.

Differences between the group of independent teachers and the group of project teachers were found related to the following factors: (a) devotion to the curricula, (b) planning, (c) use of multiple representations, (d) attitude towards graphing technology, (e) classroom atmosphere, (f) examinations, (g) appropriate level of mathematical rigor needed for teaching calculus, and (h) the stability of perceptions. These factors, however, were not fully attributed to participation in the given calculus reform project.

CARTOGRAPHIES OF COGNITION: REPRESENTING TEACHERS' CONCEPTUAL RELATIONSHIPS

Avril M. von Minden, West Virginia University

In the current investigation, knowledge structure representations of university mathematics professors, mathematics educators, and public school teachers were compared. Their perceptions of the semantic relationships in a set of mathematical concepts were elicited by means of a concept mapping task and a similarity judgments task. Their responses were submitted to Pathfinder network analysis. The teachers answered structured interview questions about their ideas on what constituted important mathematical content and pedagogical concepts, and they responded to a series of tasks with varied task constraints.

The prediction was that because of their training in both mathematics and pedagogy, the Mathematics Educator group would respond to the requirements of both the constrained and the unconstrained tasks in ways that would demonstrate their expertise in both domains. Their performance across all tasks would infer the possession of a superior structural knowledge of mathematics content and pedagogy to that possessed by the other groups. The findings confirmed this prediction.

Analysis of the quantitative and qualitative data indicated superior performance across tasks for Mathematics Educators. In contrast, Mathematicians organized mathematics content concepts similarly to Mathematics Educators. Their pedagogical decisions, however, were most similar to Middle School teachers. High School teachers both proposed and organized content concepts similarly to Mathematics Educators and made instructional decisions that were most similar to those of Mathematics Educators and Elementary School teachers. These three groups appeared to differ in their conceptualizations of teaching and learning. Mathematics Educators, High School, and Elementary School groups' similarity appeared to originate in their conceptualizations of teaching as facilitation of conceptual change and learning as interactive. The Mathematicians and Middle School groups appeared to conceive of teaching as transmission of knowledge and learning as accumulation of knowledge in an effort to satisfy external demands.

Two-thirds (67%) of the participants organized concepts in the concept mapping task differently than they did in the similarity judgments task. Pathfinder network representations depicted a greater number of links for concepts organized under the constraint of the mapping task. The inference is that the concept mapping task may be capturing aspects of structural knowledge that are missed by using a similarity judgments task.

PRESERVICE TEACHERS' CONCEPTIONS OF PROBLEM SOLVING LEAD TO LEARNING EXPERIENCE MODEL

Peter Appelbaum, The William Paterson College of New Jersey
Rochelle G. Kaplan, The William Paterson College of New Jersey

This poster session reports on an investigation of the mathematical thinking of one cohort of 96 elementary preservice teachers conducted during the semester preceding their enrollment in an intensive practicum field experience and mathematics methods course. Each subject was assessed on one of four strands of mathematics curriculum (numeration, patterns and functions, geometry, and statistics/probability) utilizing problems adapted from the mathematics section of an academic competency exam given to all eighth grade public school students in the state of New Jersey. Both the format and the content of the assessment was consistent with the *Standards* recommended by the National Council of Teachers of Mathematics and involved the use of higher order thinking skills applied to problem solving contexts in mathematics. Demographic data and information about previous college mathematics courses taken by subjects was also obtained.

The assessment problems contained a mix of multiple-choice and open-ended questions. In addition to indicating a single correct answer, subjects were asked to show how they solved each problem using words, numbers (including algebra), and/or pictures (diagrams). They were also asked to provide alternative approaches for solving each problem. Randomly selected subjects were subsequently clinically interviewed about their test protocols.

Solution strategies for all responses were coded and videotapes of the clinical interviews were analyzed for strengths of informal conceptions underlying procedures, misconceptions about mathematical relationships, and the nature of mathematics. The results of this study indicated that many preservice elementary teachers possess a working knowledge of arithmetic procedures, but do not necessarily apply these procedures in a reasonable or reflective manner. The character of their conceptions will be presented in the "poster" using: a) graphs of patterns of responses based on our statistical analysis of the data; b) samples of the subjects' written responses and an accompanying visual display of our interpretations of these responses; and c) a diagrammatic model for a recommended learning experience in mathematics education for preservice teachers based on the findings of this study.

TEACHERS' KNOWLEDGE OF REPRESENTATIONS IN THE DOMAIN OF ALGEBRAIC MULTIPLICATION: THE CASE OF MR. KANTOR

José (Alberto) Contreras, The Ohio State University

One of the underrepresented areas of research on teachers' knowledge is teachers' knowledge of representations. For example, Fennema and Loef (1992) ask: "Do teachers know the representations of the content they ordinarily teach?" (p. 154). Because most mathematical ideas taught in school can be represented using a variety of situations, a related question is: What kind of representations would teachers provide if we asked them? And, because students, and teachers alike, do not ordinarily receive formal instruction using examples of representations, another question worthy of investigation is: How difficult is it for teachers to provide representations of mathematical ideas? The results of an in-depth case study to be displayed and discussed in the poster session shed some light on the answers to those questions.

The research reported here was part of a larger project whose purpose was to investigate how two middle school mathematics teachers use their pedagogical content knowledge when teaching multiplication and division in algebra. Mr. Kantor, the participant whose knowledge of representations will be the focus of this presentation, holds secondary certification. He has been teaching mathematics for about five years at a middle school located in a suburban school district known for high student achievement.

I relied mainly on questionnaires and videotaped teaching episodes to gather information on Mr. Kantor's knowledge of representations in the domain of algebraic multiplication. These data were supplemented with interviews that were audiotaped and transcribed. Through a content analysis of the lessons of the textbook related to algebraic multiplication about 41 main mathematical ideas were found. For each of those ideas, Mr. Kantor was asked to provide, when appropriate, a mathematical definition, a pictorial representation, a story problem representation, and a mathematical proof.

The data analysis indicates that Mr. Kantor knows most of the representations related to algebraic multiplication. For example, when he was asked to illustrate with a story problem that a negative number times a positive number is a negative number he said: "How much more or less did you weigh five days ago if you have been gaining two ounces per day? You weighed ten ounces less." However, he didn't provide a mathematical proof of some theorems. For example, he failed to prove that $a(-1)=-a$.

A matrix showing each of the four types of representations provided by Mr. Kantor for each mathematical idea asked and the degree of difficulty he had in providing them is being designed. The matrix will be displayed in the poster session presentation.

Reference

Fennema, E., & Loef, M. (1992). Teacher's knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Teacher Education

BEYOND RECIPES AND BEHIND THE MAGIC: MATHEMATICS TEACHING AS IMPROVISATION

Annette M. Sassi, Education Development Center, Inc.
Lynn T. Goldsmith, Education Development Center, Inc.

This paper proposes that post-reform mathematics teaching may be characterized as "improvisational." It uses observations of an extended mathematical investigation from a summer institute for elementary teachers to examine four aspects of improvisational practice: 1) the structuring of the activity; 2) planning and preparing that is both reflective and anticipatory; 3) an attentiveness and responsiveness in the moment; and 4) an improvisational understanding of the content itself. The paper concludes that this conceptual framework may help both teacher educators and researchers understand better how to help teachers learn this way of teaching because it preserves rather than simplifies its complexity.

In response to the mathematics education reform movement, many teachers are grappling with how to reconstruct their mathematics teaching. The reforms being advocated take away much of the certainty of teaching mathematics that teachers have known in the past and replace it with an indefiniteness that often leaves teachers feeling that they must invent their math classes on the fly. Yet, while these reforms challenge traditionally structured classrooms, they are not meant to suggest that teachers abandon all sense of organization and order.

The challenge for teacher educators is to communicate and model how post-reform teaching might look when one of its major characteristics is its lack of prescription. While there are no recipes for creating these new forms of teaching, neither is it a matter of teaching solely by intuition or "feel." There are pedagogical and epistemological issues to which teachers must learn to attend closely: for instance, how to recognize an opportunity for a rich discussion that wasn't planned; how to determine if a child's mathematical argument is rich enough to explore more deeply; how to anticipate the kinds of questions that will get students engaged in a substantive mathematical inquiry. It is crucial to help teachers develop a deep sense of what this teaching is about so that they do not feel as if they've abandoned certainty in favor of a free fall into a pedagogical abyss.

To succeed at this task we need conceptual frameworks that preserve rather than collapse the complexity of attending to the particularities of individual classrooms—one of the hallmarks of "constructivist" teaching. While the new pedagogy encourages teachers to confront this complexity in their classrooms, it offers few theoretical constructs of what might be entailed in responding to it. In this light, work in philosophy on practical reason and judgment is especially relevant. Nussbaum (1990) in particular, provides a rich analysis of the situated complexity of deliberating and choosing well. She argues for the priority of the particular and

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This paper is based upon work supported by The National Science Foundation under Grant No. ESI–9254479. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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holds that good deliberation must take into account the contextual features of the situation. She argues further that the person engaged in practical deliberation must inevitably improvise, balancing her own general knowledge with the particularities of the given situation. Several writers have offered practical insights into the workings of improvisational activity in different domains. In extending this notion to teaching, we have drawn from analyses of musical improvisation (Sawyer, 1992; Sudnow, 1978) and discussions of improvisational qualities of other largely interactive endeavors such as qualitative research (Oldfather & West, 1994) and play (Sawyer, in press).

This paper uses observations of an extended mathematical investigation from a *Mathematics for Tomorrow* (MFT) summer institute for elementary teachers to explore the claim that teaching is improvisational. Our analysis focuses on teachers of teachers, rather than on the practice of schoolteachers. While the elementary teachers participating in the MFT project are just beginning to consider new ways to teach, the practice of the MFT staff exemplifies many of the key tenets of this new pedagogy. Since our interest lies in understanding some important dimensions motivating and organizing constructivist teaching we felt we needed to turn to mature (though still developing) examples of such teaching to explore its character.

Method

The mathematical investigation serving as the focus of analysis is called *Starfish.* This is an adaptation of *Xmania*, an exploration of number systems used at SummerMath for Teachers (Schifter & Fosnot, 1993). The exercise was part of the July 1995 Summer Institute for MFT teachers, and was the first extended mathematical investigation that teachers undertook as participants in this two year, professional development program.

Data for the analysis presented here comes from the materials used during the Starfish activity, field notes of observations of teachers' work during the Starfish exploration and of teacher educators' interactions with them, and notes of a post-institute debriefing meeting with the instructors regarding their teaching during the Starfish exploration.

The investigation itself begins with the instructors posing a problem within the context of a fantasy society. Teachers are told that a famous mathematician of Starfish society unexpectedly died just as she was to make public her newly invented number system. The existing system represented the quantities between 0 and 26 with the symbols 0, A, B, C, D, E, F,...Z, and then referred to any quantity larger than Z as "lots." While the details of the new system died with the professor, she left some sketchy information about it: the system can represent any quantity exactly, it can be used to perform any arithmetic operation, and it uses only the symbols A, B, C, D, and 0. Teachers are then asked to develop a system consistent with these claims and with the aid of some artifacts which the Professor left behind. There is no explicit discussion of how teachers might use these "artifacts" (which are base 5 blocks), although they are instructed to use these materials as an integral part of constructing the system. With this introduction, teachers begin their explorations in groups of three to five.

The goals of this activity are several. It aims for teachers to investigate fundamental issues about number systems and place value and to experience some of the issues children encounter as they learn our base 10, place value system. These goals form a constant mathematical structure around which the teacher educators design specific exercises. The activity also aims to give teachers experience in collaborative mathematical inquiry, and the chance to see the range of ideas that develop as different people work on the same problem.

Analysis

Our reading of the literature on improvisation and our prior observations of the MFT teacher educators lead us to propose four key factors characterizing improvisational practice: 1) the structuring of the activity; 2) planning and preparing that is both reflective and anticipatory; 3) attentiveness and responsiveness in the moment; and 4) understanding of the content itself. This paper considers these four factors in the context of the Starfish investigation.

Structuring for possibilities. Consider first what characteristics of this activity allow it to have "improvisational potential?" An insight came during a discussion when two teachers asked why the teacher educators structured it so tightly. Why weren't they simply given some beans and ask to construct number systems? A number of the teachers felt that such constraints bound, rather than facilitate, creative explorations. Yet, improvisational arts have very defined structures. In improvisational music, for instance, certain musical structures such as a fixed chord progression bound where the improvisations can—or can't—go at particular times in the piece.

Similarly, structuring the place value investigation by providing materials which suggest grouping by fives, and using alphabet letters in place of Hindu-Arabic numerals effectively pulls people away from the terrain that they know (organizing quantities into units of 10) and moves them into more unfamiliar territory (thinking about how to organize and represent quantity). As another teacher later pointed out, had they been given just beans they would have been tempted to construct a base 10 system because they already knew how to group by powers of 10. Had they been able to rely on this understanding, they would not have been challenged as deeply to build from scratch their understanding of units, groupings, and naming of units. Because teachers were charged with describing their systems in terms of both the base 5 blocks and the symbols, they were constrained from using the symbols in a base 10 form—for instance, using upper and lower case letters to get nine symbols. By structuring out an easy reliance on the familiar, the teacher educators opened up possibilities for the deep exploration of the unfamiliar.

The organization of an improvisational activity also has to allow for creativity in the moment. The structuring of Starfish has such characteristics. When the teacher educators planned the activity they did not know what, exactly, would

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emerge from teachers' work. The particular materials and their presentation purposely leave open the possibility of inventing many number systems. By providing four symbols (in addition to zero) and four block sizes, the teacher educators construct a mathematical ambiguity that can be resolved in a number of ways. Should the symbols represent numbers of blocks (A blocks, B blocks… A rods, B rods, etc.), or should they represent the various sized blocks themselves (A is a unit, B is a rod, C is a "flat")? One choice may lead to the construction of a base system, another to a Roman numeral-type system.

Planning and Preparing. While the teacher educators had a good sense of the kinds of discoveries that teachers were likely to make, they still needed to be ready for surprises. This readiness to respond to the ongoing work involves a kind of planning and preparing that incorporates considerations of possible scenarios and responses to them. By analogy, we can turn again to improvisational music. Preparing for an improvisational performance does not involve running through an exact arrangement. (In fact, this is impossible by definition, since there is no exact arrangement of the music to practice, only sketches.) Instead, the musician anticipates what might happen with other musicians, tries out possible families of responses to them, and investigates new musical spaces in anticipation of confronting them in the performance.

In many ways, preparing for a math activity like Starfish requires similar planning and preparation. The teacher educators used past experiences to anticipate different kinds of outcomes and directions, and conjectured about possible responses to these circumstances. For example, teachers study the different kinds of systems that different groups created and talk as a whole group about the different systems. The teaching staff want the teachers to consider some key issues about number systems, for example: what is gained or lost mathematically with different systems; how important is efficiency; what resemblances to each other do different systems bear? But the teaching staff are never sure what systems will be created. Consequently, in anticipating the discussion the teacher educators imagine a range of possible scenarios. In the event that only base five systems would have emerged, for instance, the staff imagined that they would stimulate consideration of the questions above by asking teachers to talk about their false starts—the systems that people started to create but abandoned because they felt that they weren't working.

Furthermore, planning for Starfish is ongoing. For example, while the teacher educators planned to have the teachers do worksheets involving arithmetic operations in the different invented systems, they did not finalize the particular problems for the worksheets until they could assess the range of group understanding about the different systems. This year the instructors designed the worksheet to incorporate some specific mathematical misconceptions about place value they observed in this particular group of teachers; the worksheets had never before included these kinds of problems, because they had not surfaced as requiring attention until now.

Mindfulness and Responsiveness. To work well, this kind of planning and preparing requires teachers to be especially attentive and responsive to their students in the moment. One cannot plan an activity with contingent elements and then fail to listen and watch for contingent outcomes. In the case of constructing the worksheets on operations. the teacher educators were able to incorporate the teachers' misconceptions precisely because they were alert and attentive to such things. In designing these exercises the teacher educators were especially responsive to the unusually wide range of mathematical abilities represented in this particular group of participants. While some teachers were struggling with counting in the different systems, others were trying to figure out how to develop a division algorithm for base 5. The teacher educators' response to this range was to develop different worksheets to challenge people at different levels of understanding: one dealing with issues about counting; one focusing on understanding addition and subtraction; and one challenging teachers to think about division, fractions, and "pentimals." While the latter was appropriate for some of the teachers in this group, it would not be appropriate for all groups of teachers who explore Starfish. In fact, this was the first time in dozens of experiences teaching Starfish that several participants made extended investigations of division and fractions.

Content Knowledge and Context. Teaching which predicates basing instructional experiences on the current state of students' knowledge cannot be done well unless the teacher, herself, understands the content in deep and flexible ways. To draw another analogy to music, the improvisational musician's knowledge of music is fluid and situational; she understands the structure and meaning of the music and knows how to mobilize this understanding to compose in the moment. In a similar way, these teacher educators know mathematics in a fluid, non-fixed way. They are able to see the various elements of the domain as interrelated elements of a continually constructed and re-constructed discipline rather than as discrete facts, operations, or procedures. (cf. Schifter 1994).

This kind of knowledge is necessary to be improvisationally responsive to what arises in the moment. The teacher educators need to know the intellectual terrain well enough to follow what others are doing and to grasp, in the moment, if students' thinking is consistent with the terrain or not. They also need to understand enough about how understanding develops so that they can image a student's likely path of understanding (Fennema, Carpenter, & Loef, 1993). Finally, they need to understand the kinds of mathematical and pedagogical interventions that will help students to develop these ideas (Shulman, 1987).

In the case of the Starfish activity, the teacher educators were able to help participants work through ideas by following their reasoning, asking questions and posing situations which were designed to push on their thinking, and redirecting investigations when they felt it necessary. This was possible to do because the teacher educators were prepared mathematically for the multiple directions the activity could go. Had they been less "fluent" in base 5, less familiar with the kinds of understandings that teachers typically construct when engaged in this work, or less able to make connections between teachers' thinking and the important mathematical ideas inherent in the exploration, they would not have been able to help keep participants' thinking as active and focused on moving toward greater understanding than they were.

Conclusion

Capturing what teaching within the new pedagogy entails is especially difficult because this kind of teaching has no easy recipes or prescriptions. Furthermore, to codify it would effectively undermine the principles upon which it is based. This paper has taken to task the challenge of articulating some of the fundamental characteristics of teaching from a "constructivist" perspective by characterizing it as "improvisational" and looking at four aspects of improvisational practice in the context of an extended mathematical investigation. If the characterization of post-reform teaching as improvisational proves through further investigations and refinements to be robust, then this framework may indeed contribute to a greater specificity of what this kind of teaching entails and what is required to learn it.

References

Fennema, E., Carpenter, T., & Loef, M. (1993). Learning to use Children's Mathematical Thinking: A Case Study. In C. Maher & R. Davis (Eds.) *Relating Schools to Reality*. Needham Heights, MA: Allyn & Bacon.

Nussbaum, M. C. (1990). The Discernment of Perception: An Aristotelian Conception of Private and Public Rationality. *Love's Knowledge: Essays on Philosophy & Literature*. Oxford: Oxford University Press.

Oldfather, P. & West, J. (1994). Qualitative Research as Jazz. *Educational Researcher*. 23(8): 22-26.

Sawyer, K. (in press). *The Performance of Pretend Play: Improvisational Creativity and Conversation in the Nursery School.*

Sawyer, K. (1992). Improvisational Creativity: An Analysis of Jazz Performance. *Creativity Research Journal*. 5(3): 253-263.

Schifter, D. (1994). Teachers' Changing Conceptions of the Nature of Mathematics: A Developmental Model. *Proceedings of the Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (296-302), Baton Rouge, LA: PME-NA.

Schifter, D. & Fosnot, C.T. (1993). Reconstructing Mathematics Education: Stories of Teachers Meeting the Challenge of Reform. New York: Teachers College Press.

Shulman, L. (1987). Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review*. 57(1): 1-22.

Sudnow, D. (1978). *Ways of the Hand: The Organization of Improvised Conduct*. New York: Harper & Row.

BUILDING BRIDGES TO MATHEMATICS FOR ALL: A SMALL SCALE EVALUATION STUDY

Joanne Rossi Becker, San José State University
Barbara J. Pence, San José State University
Dianne Pors, East Side Union High School District

This paper reports on a small scale evaluation study of a staff development project for high school mathematics teachers that focussed on new curriculum, updating pedagogy, and examining issues of equity. After two years of the project, interviews were conducted with five previous participants who have had varying amounts of staff development. Interview data were compared and supported by classroom coaching visits and teacher responses to an Instructional Practices Scale. Although all teachers could provide the rationale for "algebra for all," there were significant differences in the changes they had made in their classroom instruction that were related to the number of years of staff development in which they had been engaged. This study confirms that real change comes slowly with in-depth involvement in staff development over a long period of time.

For a number of years we have been engaged in research in inservice education for secondary teachers focussing on teachers' beliefs, their relationship to classroom practices, and how changes in one may foster concomitant changes in the other (Becker & Pence, 1990a; 1990b; Peluso, Becker & Pence, 1994). As Grouws (1988) pointed out, there is little information available about the overall design features of inservice education which maximize changes in teacher beliefs and classroom behavior. Grouws has called for studies which focus on the impact of various features of inservice education on classroom practice. This paper extends previous work to add to the body of knowledge on inservice education.

The Program

The staff development program is now in its third year, and includes several facets (see Peluso, et al., 1994 for more details about the program). In addition to a 13-day intensive summer institute, the project includes classroom coaching, five followup workshops during the academic year, and purchase of manipulative materials, software and graphing calculators for the schools. Administered in conjunction with the College Board's Equity 2000 project, "Building Bridges" was designed to provide all high school mathematics teachers in two districts with basic staff development to focus on: equity in mathematics; innovative curriculum materials; use of technology; and new modes of pedagogy. This program was created to help teachers meet the challenge of eliminating tracking and placing all ninth grade students in algebra 1/course 1 by Fall 1995 as mandated by the Equity 2000 project.

This project and the research reported in this paper were supported by grant # 758-7 of the Dwight D. Eisenhower Mathematics and Science Education State Grant Program administered by the California Postsecondary Education Commission. The opinions expressed are those of the authors alone and do not represent those of the Commission.

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Goals of this Study

In this study we were interested in ascertaining which aspects of the "Building Bridges" program had the most impact on teachers; whether the program was able to facilitate change in teachers' beliefs about the teaching and learning of mathematics; and why some teachers from this project chose to continue into a two- or three-year project.

Methodology

The main source of data collection was in-depth interviews with the subjects. The five subjects were selected according to several criteria. First, we wanted to have a diversity amongst the teachers that was representative of the teachers we had in the project over the first two years. Second, we wanted teachers who had only participated in the "Building Bridges" project as well as others who had chosen to apply for the National Science Foundation project (NSF) which provides a continuation of the "Building Bridges" project. Potential interviewees were identified according to these criteria, their responses on the Instructional Practices Scale (Becker & Pence, 1990a), and experiences of staff during classroom coaching. An attempt was made to select teachers who exhibited a range of change in their teaching based on these criteria.

The interview questions probed goals of the project as teachers perceived them; difficulties their schools were experiencing implementing Equity 2000 goals; measures their schools were taking to alleviate the difficulties; what a typical mathematics class of the subject was like; a description of a rewarding lesson; their perceptions of their role as mathematics teachers; their beliefs about teaching and learning mathematics and how that has changed; the role of technology in teaching mathematics; the most and least useful aspects of the program; and, for those teachers who joined the NSF project, why they did so and what they hoped to gain from it in the next year.

Subjects

Five teachers were interviewed for this study. This included four female and one male teacher; one African American, one Vietnamese American, and three European Americans; and one non-NSF teacher, two who joined NSF for three years, and two who joined NSF for two years. Two of the teachers, Aretha and Kim, have about five years teaching experience; two, Kathy and Belinda, have approximately 15 years experience; and the fifth teacher, Brandon, has 25 years teaching experience.

Analysis

The interviews were audiotaped and transcribed fully. Responses were analyzed for patterns in responses within and across questions. Results were also triangulated using qualitative information from classroom coaching visits and quantitative data on the Instructional Practices Scale. Due to space limitations results from the interviews which were confirmed by other data will be the focus of the discussion.

Results

Several components were highlighted as having an impact on participants. The paper will examine each of these components through the statements of the interviewees.

Equity 2000 Goals. All of the participants seemed to glean from the staff development the goals of the project and rationale for putting all ninth grade students in algebra 1/course 1. There also seemed to be agreement that these were worthy goals.

> Brandon: My understanding is, as presented in a workshop ...is that one, ... there had been a study... that students who at least go through geometry in high school had a better chance of getting into college and succeeding. And one of the primary goals was to make sure that students had exposure, that they would start off in algebra 1... but at least through geometry so they would have a fighting chance...So that was the primary goal of Equity 2000 to get students into algebra and make algebra accessible to students, particularly those who had been knocked out previously.

Typical Mathematics Class. There were some marked differences in the "typical" mathematics class of teachers who had been involved in staff development, through our "Building Bridges" and NSF programs, for one, two or three years. In this sample, Brandon had one year of staff development, Karen and Aretha had two years, and Belinda and Kim had three years.

Brandon described a fairly traditional class as his typical day.

> Brandon: In a typical day, say it's inequalities and equations, is to give them a wide range of examples.... I would do so many, because I engage them from the beginning, I would do so many and they would do so many primarily on the board ... and then have at least 15 minutes for them to work on homework in class.

Contrast that to what Aretha and Belinda say; both are making much more use of cooperative groups and Belinda is actually using two different innovative curriculum materials, one for algebra and another one for geometry.

> Aretha: To begin my class I have a warmup question, one or two.... take 10 minutes for the warmup.... That's just a problem from yesterday, you know typical problems, a little review before I go on. And how I organize my classroom we work in groups all the time, all the time, except for the days we have tests.

However, Aretha still does some "traditional" teaching as well as group work, spending about 15 minutes of each class providing explanations before the group work.

Belinda has moved much further away from a traditional style of teaching, using College Prep Math for her geometry course and Computer Intensive Algebra for algebra 1.

> Belinda: Ok. In geometry I'm teaching the CPM Davis geometry, year two, and it's all group work. Kids are in groups of four. I place them in the groups... In the algebra 1 I do some lecturing as well as going around helping the kids on the computers. In the geometry I circulate around the room and I listen and I make sure that everybody in the group has the same question before I'll answer the question.

Role of the Teacher. The question of whether the program was able to facilitate change in teachers' beliefs about the teaching and learning of mathematics may be the most interesting result. Changes in teachers' beliefs proceed slowly. This study involved individuals who had varying years of institute participation. The change process can be seen as evolving as we look across the one, two, and three year experiences. For the person who has been involved for only one year, there is an awareness of suggested changes but little acceptance. The responses include references to what "they" say versus what "I" think.

> Brandon: One thing I'm finding more of is that they want me, when I say they I guess the powers to be have decided how it ought to be done, more of a facilitator than someone just delivering information. I don't have a problem with that as long as we understand that I think some days need to be different.

Aretha and Karen both spoke about struggling. Although they were both relatively new teachers and said that they had not changed because they had encountered similar ideas in their teacher education program, they were experimenting. This means that they were still maintaining a traditional classroom structure but they were trying various modifications in both curriculum and classroom organization. Even more important, they were reflecting on these experiments and their own transitions. At this point, they may have more questions than answers. The questions deal with their role as the classroom teacher, their curriculum, and the organization of their classroom.

> Karen: When I went to college the ideas as far as the changes in teacher's role were taught to me and I started teaching with that idea. But I think I got a better idea by going to the summer institutes . . . I still do lecture, but I do have portions of it where they're working and I'm trying to help them. It's really changed my view in so many things -- is there really a vital thing that I have to be teaching any more?

Kim and Belinda, however, have moved to more action-oriented thinking regarding their students and their role with other faculty.

> Belinda: It [staff development] has reoriented me to a new way of teaching...And it involves the kids more because they are more active in the learning...I just think the more involved they are the more

> they'll learn. Obviously it has affected me a lot because I changed from a lecture teacher to a facilitator.

Use of Technology. All of the teachers found that the staff development had changed their views of the role of technology in teaching mathematics. But of course the longer staff development had allowed Aretha, Karen, Kim and Belinda to spend more time thinking about and using that technology. As a result they seem more willing to use technology, especially calculators, in algebra I as well as upper level classes, and have begun to question what exactly we should be teaching in light of the ready availability of technology. The project's ability to purchase graphing calculators for all schools over the three years has greatly facilitated teachers' incorporation of them into their teaching.

> Brandon: But the graphing calculator, there's no two ways about it, it has opened up a whole new world. That has really changed.

> Kim: Yeh the technology for me is, made me stop and think about what was really important to teach.

Most Useful Aspects of Program. Teachers seem to agree that the networking aspect of the project was one of the most important features to them. They feel they have little time during the academic year to meet with other teachers and learn from them, and the summer institute particularly provided that opportunity.

> Kim: Well getting together with colleagues and talking was very important. Seeing their views and how they did things was really important because since I didn't have any previous experience with someone teaching me a certain way I had to see what other people who were taking risks were doing and seeing how they were doing it and trying it and going back. So that was real important.

Technology parts of the workshops were also very valuable although sometimes frustrating for teachers who had little or no access to computers.

> Karen: Oh the most useful I would have to say all of the lessons we have had on the graphing calculator I have used quite a lot. In fact I discovered I actually know quite a lot about graphing calculators. And that's been really helpful.

Why the NSF Program. Finally we discussed with Aretha, Karen, Kim and Belinda, who had chosen to join the NSF program after completing "Building Bridges," why they had done so and what they hoped to gain from the next year of the project. Their positive experiences in the initial project stimulated their continuance; but perhaps more important, they felt the need for further professional growth and support as they tried to make curricular and instructional changes in their schools.

> Karen: Well toward the end of the first summer I had really had a good time and I likened it to going to camp which I never got to do.... I just really enjoyed it. To be honest with you I, it was nice cause it was half the time that summer school was, I was going to be getting paid

> for my time, and I knew that I was going to be leaving with a lot of good teaching things, and it satisfied professional growth for me. So really it did a lot of things for me.

For Aretha there was the added attraction of working on her own leadership skills.

> Aretha: Ok. First time I see it, learn to become a leader, I say ok I need this skill because I'm so shy I don't speak for myself much at all. And I'm very very shy, I'm quiet. So I thought this is for me, I'll learn to speak for myself and maybe some day I'll get up there and do something. So that's why I tried it.

Kim in particular noted the importance of a support system provided by the network of teachers formed in these institutes.

> Kim: I think that if I had not continued with NSF I probably would have just stopped and stayed where I was.... But with the continued programs and the support. And now what I'm seeing is this year at least we're all working on the same things, so we're starting to build the support in our schools.

Summary

This small-scale evaluation study substantiated what we and others (Clarke, 1994) have found previously: that short term staff development may have some impact on changing teachers' beliefs and practices, but that real change comes slowly with in-depth involvement over a longer period of time. Obviously the "Building Bridges" program had some effect on participants; even Brandon, who has perhaps not made a complete buy-in, has shown some change in use of technology and multiple representations in his teaching. He knows he should be working toward being more of a facilitator in class but the process of making that change seems difficult for him. For the other four teachers, there seemed to be a relationship between the length of inservice involvement and the extent of change in classroom instruction. All of these teachers seem to be on a journey toward implementing recommended reform in mathematics teaching; the more in-depth staff development we can provide, the more comprehensive that journey may be.

References

Becker, J.R. & Pence, B.J. (1990a). *Teachers' beliefs and classroom practices.* Paper presented at the Fourteenth International PME Conference, Mexico.

Becker, J.R. & Pence, B.J. (1990b). Linkages between teacher education and classroom practice. In Dossey, J.A., Dossey, A.E. & Parmantie, M. (Eds.) *Preservice teacher education: The papers of action group 6 from ICME 6, Budapest, Hungary.* Normal, Il.: Illinois State University, 201-207.

Clarke, D. (1994). Ten key principles from research for the professional development of mathematics teachers. In Aichele, D.B. & Coxford, A.F. (Eds.) *Professional development for teachers of mathematics.* Reston, VA: NCTM, 37-48.

Grouws, D. (May, 1988). *Overview of the role of research on mathematics teaching and learning in improving research in classroom instruction and teacher education.* Paper presented at the First Wisconsin Symposium on Teaching and Learning Mathematics, Madison, Wisconsin.

Peluso, T., Becker, J.R. & Pence, B.J. (1994). "Developing a community of risk takers." In da Ponta, J.P. & Matos, J.F. (Eds.). *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education. Vol. II.* Lisbon: University of Lisbon, 56-63.

CAN PROSPECTIVE TEACHERS BE AUTONOMOUS DECISION MAKERS?

Evangelina Díaz Obando, Florida State University
Elizabeth M. Jakubowski, Florida State University
Grayson H. Wheatley, Florida State University

The purpose of this study was to investigate the impact of a middle grades mathematics teacher preparation program on prospective teachers' practice during their internship. Using qualitative data collected from students enrolled in this course, it was found that prospective teachers during their internship displayed a great deal of autonomy in creating nontraditional learning environments and when negotiating with their supervising teachers.

Many calls (e.g., National Council of Teachers of Mathematics [NCTM], 1989; NCTM, 1991; National Research Council [NRC], 1989) to implement changes in content and pedagogy in elementary and secondary school mathematics suggest a shift from traditional practices to more student-centered activities, in which teachers become facilitators of student's learning rather than dispensers of knowledge.

Recent studies (Thompson, 1992; Brown, Cooney, and Jones, 1990) are suggesting that what teachers think about mathematics is largely influenced by their experiences long before they engage in a formal teacher preparation program. Contrastingly, Trowell and Wheatley (1994) found that college students with many years of traditional teacher-centered mathematics classroom experiences were able to negotiate rich learning environments in which problem solving, sense making, and collaboration were highly valued. It was our belief that if prospective teachers learned to become teachers in an atmosphere where they were allowed to "work at establishing a culture" in which they can negotiate meanings, explore different learning approaches, ask questions, listen to responses, and ultimately, make sense of mathematical situations in various forms then, their teaching practices can reflect the reform recommendations. Providing opportunities for prospective teachers to construct discipline specific pedagogical knowledge is one of the priorities of this project. Furthermore, we believed that participation in such a preparation program would facilitate the construction of adequate meanings for mathematical concepts as central to middle school mathematics (Jakubowski, Wheatley, Erhlich, 1993).

Previous research conducted by Shaw and Jakubowski (1991) described a model for teacher change that was used as a guide in this project not only with the prospective teachers, but also with the practicing teachers who may become role models for the prospective middle school teachers. The model includes four main constructs, "perturbation," "commitment," "vision," and "reflection" (Shaw & Jakubowski, 1991).

The work reported in this paper was supported by National Science Foundation Grant # DUE 9252705. All opinions, findings, conclusions, and recommendations expressed herein are those of the authors and do not necessarily reflect the views of the funder.

Throughout the teacher preparation program a variety of activities were developed to engage prospective teachers in order to facilitate their development of a vision of a teacher. These activities were informed by a constructivist perspective under which individuals actively construct their knowledge, rather than simply absorbing ideas spoken by another person, or somehow internalizing those ideas through practice and memorization.

Research Setting

This research is part of a larger research and development project. The aims of the project were:

1. To plan and implement courses in mathematics learning and teaching and mathematics for prospective middle school teachers based on problem-centered learning. The research group teams were created to revise or design each one of five mathematics and two mathematics education courses.
2. The development of a summer enhancement program for middle grades teachers who were also potential supervising teachers for school based experiences (e.g., Summer 1993, Summer 1994). These teachers were engaged in activities and tasks which were planned for the designed courses. The main goal of this was to engage these teachers in problem-centered learning and to provide the practicing teachers with opportunities to construct a vision of mathematics that was consistent with current calls for reform.
3. Students' participation in school-based experiences each semester prior to their student-teaching. During the last semester in the program, prospective middle school teachers are placed in middle schools for a period of fifteen weeks. An attempt is made to place them with practicing teachers who have participated in the summer program.

Methodology

Through the use of a qualitative design, several techniques were used to collect data that depicted various perspectives. Classroom observations, interviews with interns and supervising teachers, interns' reflections, audio and video recordings of interviews and class sessions, follow up interviews, meetings with interns participating in the study constituted the data.

Analysis and interpretation of data was done to identify relevant patterns to construct a framework for communicating what data collected revealed. Data collected were analyzed on a continuous basis throughout the study. Each interview and observation was recorded in memos, field notes, and transcripts. The data were categorized according to common themes that later on were grouped. To this end, explanations were constructed by the researchers, to elucidate the actions of the participants. Triangulation of data (Guba, & Lincoln, 1989) was accomplished by comparing data from teachers, interns, and a university supervisor to support assertions and to assure the viability of the interpretations.

Findings

The analysis provided rich descriptions of classroom events, relationships between courses the participants had taken during their teacher preparation program, and decisions the interns made as to what to do in the classroom during their student teaching.

Lani

Lani, is a white female in her early twenties, married with two children. Lani found herself dissatisfied in one of her early teaching sessions (at the very beginning of her internship, when she was mainly using a textbook), so she initiated negotiations with her supervising teacher about the possibility of implementing alternative teaching strategies.

She began to have students work on activities situated in a cooperative learning environment, where she organized the class into small groups and provided students with a task so that students could discuss it in their small groups and attempt to make sense of the task. An example of that occurred when she was teaching a unit on fractions. Planning to use tangram sets to help students make sense of fractions in a meaningful way (mathematically speaking) . She decided to have students construct their own tangram sets. Lani's main rationale was that by having students construct their own set, they would become more familiar with the different shapes in the set and with different relationships of the pieces. Therefore, when they later used their tangram sets to solve fraction problems, students' explanations would be more meaningful to them. It seems that Lani had the students' understanding of mathematics and student's enjoyment of mathematics as a primary goal of her teaching. The supervising teacher expressed repeatedly [e.g., bi-weekly evaluations, interviews], "Lani continues to establish a classroom environment that is conducive to learning. She engages students in a variety of meaningful learning activities...she has also incorporated other methods of teaching in her lessons...all activities thus far encourage students to construct their own knowledge...through discussion she constantly encourages students to reflect on their own knowledge...I am impressed with the meaningful activities which she has either made or found."

Uncharacteristic of interns, Lani organized a grade level activity she had developed and called a "A Pi Day." This required her to obtain the cooperation of other teachers on her team. In this activity students could submit 3-Dimensional shapes, a poster, a tessellation, and/or a mathematical puzzle. This activity was characterized by other school teachers, judges, and students as a huge success.

Andy

Andy, a white male in his early twenties, was a Mechanical Engineering major for two years prior to becoming a prospective middle school mathematics teacher.

He had done volunteer coaching of track and wrestling to middle school students. For his student teaching he was placed at a middle school with a teacher who had been involved in the project.

From observations of his classes it was evident that Andy cared a great deal about students' learning. Similarly to Lani, Andy had initiated negotiations with his supervising teacher in an attempt to implement alternative teaching strategies. His supervising teacher expressed the following, "I see great potential in his instructional methodologies. I like that he takes initiative, and my desire is that this internship will be a great learning experience for him." In another interview she said, "He is not relying on me, he has his things that he is trying to do (in his classroom), and I think that is coming from the middle school program [Andy's autonomy to initiate or to suggest ideas he wanted to implement in the classroom], what I do is to ask him to show me beforehand his ideas on what he plans to teach, for specific concepts...; He is very secure that I am going to support him. I may not agree with what he is doing, but I'm not going to tell him "no." I'll let him make his own decisions." The teacher's ways of describing Andy's style of planning and teaching was an indication of his confidence and initiative; because of his preparation, he was able to formulate effective learning environments for his students.

For example, on several occasions, during his teaching, Andy took students outdoors to conduct mathematics learning activities. He provided meaningful learning environments for students by using a variety of settings and materials. Also, Andy enriched his teaching by using segments of videotapes (e.g., "The Alhambra Past and Present: A Geometer's Odyssey," and "The Story of Pi"), manipulatives, and laser disks (from The Jasper Woodbury Series) as a way to facilitate students' understanding of the mathematical concepts being studied. When additional resources, not available at the school, were needed, he went to the Mathematics Education Department to get them.

Andy's style of teaching focused on developing connections between mathematical concepts and real-life situations. He said that students at middle school have so much potential that it would be a shame not to take advantage of it to bring mathematics into their real world. He also said, "If I don't enjoy teaching everyday in some manner I will convey this attitude to my students." This attitude typified most of Andy's actions in his classes. The creation of trust between teacher and students and connecting mathematics to real life situations were the foundation for Andy's teaching practices.

Kathy

Kathy is a white female in her early twenties that was about to get married. She believed that she could make special contributions to middle grade students because she is patient, caring and a good listener. It was evident during her interactions with students.

During their teacher preparation courses, Kathy, unlike the other two participants, was not as active in the classroom as Lani and Andy were. Nevertheless, Kathy was very involved in the program and her ideas were sharp and insightful. Kathy was not placed in a local middle school [for personal reasons] and therefore was with a supervising teacher who was not engaged in the middle school project and who Kathy did not already know.

While Kathy was quiet and more passive than Lani and Andy, she took initiatives in planning sessions with her supervising teacher. She prepared some activities before meeting with her supervising teacher, so that during planning time she could propose those activities as well as alternative instructional strategies such as small group problem solving. She also believed the main role of the teacher was to make sure that students understand the concepts being taught during class time. Hence, she used manipulatives to facilitate her students' learning.

She was very good at providing students with assistance, and during class discussions she usually allowed two or three students to share their approach to solving a problem. In fact, her area coordinator supervisor was very impressed with the type of questions that she asked, such as "how did you get it?," "could you explain how you constructed that shape?," or "what is another way to solve this problem?" These types of questions evidenced that Kathy was not only interested in an answer, but that she was very concerned about the process that would tell her how specific students were making sense of a situation. Accordingly, the learning atmospheres she created with her students were conducive to students' learning.

Kathy also initiated the use of personal mathematics folders. It included materials given and/or constructed by students that were relevant to the topic they were studying and students' definitions of the new terminology. She agreed with her supervising teacher that because of the additional work it represented, she would implement this folder only in one group. Her rationale for the inclusion of this activity was that students at middle school age needed to develop a sense of responsibility and to become more organized. This rationale was consistent with her philosophy of teaching in which she had emphasized the importance of providing students with opportunities to become decision makers. Kathy had been asked to keep such folders in her preservice courses.

Her supervising teacher was touched by Kathy's attitude toward teaching and highlighted Kathy's attributes as a beginning teacher by stating, "I have had many student teachers [from other universities] but Kathy is the best intern I have had so far. On a scale from one to five, I would give Kathy a five."

Conclusions

In this study we have learned that the experiences that Lani, Andy and Kathy have had during their teacher preparation program have been beneficial in helping them to develop images for the creation of learning environments where students could learn mathematics meaningfully. It was possible to trace many of their actions directly to their university experiences. Thus participation in this project did influence these prospective teachers since they all showed initiative in both their teaching practice and in negotiating with their supervising teacher during planning time.

From the participating prospective teachers' experiences we observed the potential for prospective teachers to become autonomous through participation in a preparation program similar to the one described herein. However, it remains to be seen whether this group of prospective teachers will be autonomous when they begin teaching professionally and no longer have the support of the middle school project or supervising teacher.

The summer program for middle school teachers helped them be more effective as supervising teachers by not only being receptive to innovative practices suggested by interns but by helping them develop successful lessons. Having teachers who share a vision of mathematics learning with the university mathematics education faculty is a crucial element in effective teacher preparation.

In summary, we found evidence that these interns were becoming "emancipated teachers" (as described by Grundy, 1987) who were eager to create alternative learning environments in their classrooms. When the conditions are provided and prospective teachers have had experiences consistent with those suggested by reform calls, we witnessed evidences that prospective teachers can be autonomous decision makers as suggested by curricular reform documents.

References

Brown, S., Cooney, T. & Jones, D. (1990). Mathematics teacher education. In W.R. Houston, M. Haberman, & J. Sikula (Eds.) *Handbook of research on teacher education* (639-656). New York: Macmillan.

Grundy, S. (1987). *Curriculum: Product or Praxis.* New York: The Falmer Press.

Jakubowski, E., Wheatley, G., & Erhlich, J. (1993). *Developing effective learning environments for prospective middle school mathematics teachers.* Paper presented at the 15th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Pacific Grove, CA, October 17-20.

Guba, E. G., & Lincoln, Y. S. (1989). *Fourth Generation Evaluation.* Newbury Park: SAGE Publications.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (1991). *Professional Standards for Teaching Mathematics.* Reston, VA: NCTM.

National Research Council (1989). *Everybody Counts.* Washington, DC: National Academy Press.

Shaw, K., & Jakubowski, E. (1991). Teachers changing for changing times. *Focus on the Learning of Mathematics, 3*(4), 13-20.

Thompson, A. (1992). Teachers' conceptions of mathematics: A synthesis of the research. In D. Grouws (Ed.) *Handbook of research on mathematics teaching and learning* (127-146). New York: Macmillan.

Trowell, S., & Wheatley, G. (1994). *The negotiation of social norms in a mathematics class.* Paper presented at the 16th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Baton Rouge, Lousiana, November 5-8, 1994.

CREATING A CULTURE OF INTELLECTUAL INQUIRY IN TEACHER INQUIRY GROUPS

James K. Hammerman, Education Development Center

Creating communities of discourse among teachers that tread the "delicate middle ground" of theory linked to actual classroom practice can serve the purpose of promoting teachers' ongoing professional development. Yet both teachers and teacher educators are navigating relatively uncharted territory in attempting to develop such communities. This paper examines some of the features of such a community of discourse, the dilemmas faced in creating and maintaining it, and poses some questions for further investigation.

For teachers to change their mathematics teaching practice in accord with the NCTM *Standards* (1989; 1991; 1995) they must make several interrelated shifts in their knowledge, beliefs, and teaching practice. They must develop new notions about the nature of mathematics and what it means to do mathematics; new views of how students learn, based on careful listening to students' mathematical thinking; and new beliefs about what classrooms ought to look, sound and feel like, and skills in creating and managing such classrooms. (Goldsmith & Schifter, 1993/1994)

Such changes take time. They require extended investigation, inquiry, and experimentation into issues of mathematics, learning, and teaching. This learning is ongoing—there are not answers to be acquired but rather, a complex terrain of practice to be negotiated and dilemmas to be dealt with (Ball, 1994; Nussbaum, 1990). This means that teachers must not only explore new ideas in professional development programs, they must also develop habits and inhabit structures which will enable them to continue their professional development over time. One method for doing so is through participation in a community of ongoing inquiry into practice. There, teachers develop ways of talking with one another that are both supportive and critical, basing discussions of issues in practice on analysis of descriptive data of various kinds (Carini, 1975; Cochran-Smith & Lytle, 1990, 1992; Heaton & Lampert, 1993; Lord, 1994; Watt & Watt, 1991).

This culture of careful description and deep inquiry into practice is novel for most teachers. It treads a delicate middle ground between practical "idea swap" sessions and abstract, theoretical conversations ungrounded in practice—two forms of sharing that often feel more familiar to teachers. Navigating this middle ground—creating real intellectual discourse and investigation tied to the particulars of teaching practice—requires new forms of discourse, new ways for teachers to interact, new assumptions about what's important to look at, and new skills. This is a lot to accomplish. It is made even more difficult because creating and sustaining a conversation in this middle ground is also novel for us as teacher-educators. It is

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This paper is based upon work supported by The National Science Foundation under Grant No. ESI-9254479. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author and do not necessarily reflect the views of the National Science Foundation.

fraught with dilemmas and requires many new experiments. Yet creating this conversation is important, not only for teachers to develop new forms of practice, but also to create new roles and communities for teachers, thereby increasing teachers' voices in political and research communities. (Cochran-Smith & Lytle, 1990)

In this paper, I will describe some of what occurs in such a group to create a conversation in this middle ground. In so doing, I will also describe some of the questions that I ponder as a teacher educator in understanding and facilitating the creation of this conversation, and will suggest further questions for investigation.

Project Context and Data

"Inquiry Groups" represent one of several components of the NSF-funded Mathematics for Tomorrow project. The project works for two years at a time with school-based teams of teachers from four Boston-area school districts, as well as with principals and district-level administrators from those districts. Teachers attend an intensive two-and-a-half week institute in each of two summers to explore mathematics themselves; to attend to student thinking about mathematics through analysis of clinical interviews, the results of whole class assessments, and of teacher-written "Episodes" (Schifter et al., In Preparation); and to explore ideas and techniques for developing classroom practice through inquiry. They then meet biweekly after-school, in schools, throughout the academic year in district-based Inquiry Groups—teams of 7 to 14 people who discuss a variety of issues concerning how institute ideas play out in actual classroom practice. A staff member also consults in each teacher's classroom four times each year, and there are four day-long workshops annually for all teachers.

The quotes in this paper come from an Inquiry Group discussion that took place in March, 1995, the second year of working with this group of teachers. The author was the teacher educator/ facilitator of this group throughout these two years. In addition to the author, there were seven teacher-participants attending this session and two researchers taking field notes. The conversation was audiotaped, and a transcription was made.

Findings and Questions

In analyzing this discussion, we notice several features that are interesting because of the questions they raise about the nature of this kind of community of discourse among teachers, and what it means to create such a community. 1) It is difficult to find a focus that helps the discussion enter and stay in the middle ground of theory embedded in particulars. Even when a question is initially posed well with solid data to explore, this is a new intellectual space to be in and the conversation often needs refocusing. 2) Teachers do not use solely the shared data brought to the group. Rather, they extend this data by describing experiences from their own classrooms that either support or tend to refute what the shared data shows. 3) Generation of alternative hypotheses about the data moves the conversation towards the "middle ground" by trying to make sense of the particulars of what occurred, though this is difficult and occurs infrequently. 4) Reflection on our own experiences in the Inquiry Group serves both as yet another data source, and as the potential basis for developing a self-reflective culture of ongoing inquiry among teachers. I will take up each of these issues in turn.

One important ongoing task for the group, in individual sessions and over the course of the year, is negotiating a focus for discussions. This task is important because it serves to frame the conversation within the balance of theory and practice that is necessary for critical colleagueship. Helping the group find this balance and develop criteria for deciding on what is more likely to produce this balance is a crucial task of my job as the teacher educator. This is not easy. In part, it represents a new intellectual space for teachers (and many teacher educators)—one which we can only really come to know by experiencing, but which we can only experience as we create it together in the group.

A lot of effort went into framing discussions well. The usual procedure was for a teacher (or pair of teachers) to take responsibility each time for bringing some data from her classroom and a question or set of questions to launch and focus the discussion. The teacher and the teacher educator would typically talk several times at length before the session about the data and the focus question. The intention here was to find data that was rich enough, and a question that was generative enough to stimulate a lengthy and interesting discussion. We were planning a day's curriculum for the group.

Ana, a first and second grade teacher who was the focus person here, had two clear questions: 1) the nature of student discourse and her role in facilitating it; and 2) how to increase participation especially among students who tended to be silent. Both of these questions were of interest to the group—they had been discussed several times in the past in different ways.

Ana also had more or less clear data, in this case three conversations which represented different experiments in facilitating discourse in order to include students' voices in the discussion. In the first conversation, Ana paraphrased almost every students' statement, but found this dissatisfying because, "It didn't give me a lot of understanding of what kids really understood, what their thinking was." In the second conversation, she had more focused questions and worked hard *not* to paraphrase. We had a quasi-transcript—detailed field notes, really—of this conversation which we distributed to the members of the group. Finally, Ana reported on a third conversation in which she asked students to reflect on their own about the ideas of the conversation before it started.

Yet even with interesting questions (albeit two, *separate*, interesting ones) and clear data, it was difficult to clearly frame and launch the discussion for the group. Several minutes were spent describing the different data, paraphrasing the focal questions, and essentially publicly negotiating the focus of the discussion. Even as the discussion progressed, different members of the group shifted and refocused the conversation.

As the teacher educator, I was left wondering how to decide which of these questions and tangents was more generative; which would lead us to link specific classroom actions with deeper understanding about implications for teaching and learning; and what kinds of moves I, and others, could make to refocus the discussion when needed. My sense was that by attending to the observable effects of specific actions, we could focus on what was going on in Ana's head at the time and the reasons for or implications of her decisions, rather than starting with hypothetical "What if you had tried this?" kinds of statements.

Yet an extended digression into the effects of students' family background on their participation—a teacher's move which at first seemed to be getting us off topic—also generated hypotheses, counter-examples, and some interesting analysis. What criteria can we use to predict whether a question or focus will move us to the middle ground? How important is the initial data? What are the features of the question itself? What moves can serve to shift the focus to this middle ground? It's interesting that the group decided to continue meeting monthly into the fall of 1995, and asked for help developing skills to focus conversations and keep them focused.

While negotiating a focus happens both before *and* during an inquiry group session, the other issues tend to arise primarily *in* the session itself. I will present a single, extended excerpt from one conversation to illustrate them. This excerpt occurred just a few minutes after the group had read Ana's quasi-transcript, roughly one-third of the way through the entire conversation.

Rhonda: And I too have the same problem you have which is paraphrasing for them. And I really make that effort not to do that.

Jim: So let's see. It would be interesting to look through this and see how often Ana said, "Why?" and what kinds of responses did she get from that. How often did she paraphrase for kids? How often did she ask kids to paraphrase?

Rhonda: I don't think you paraphrased at all.

Jim: So let's hear other people.

Kathy: Line 46. "So what Larry is saying...," and then you drop it. So that somebody...

Jim: Or I didn't write it down...

Kathy: Well, I like to think that you (laughter) started the question and then...

Ana: I think I did...Yeah. Like Amelia was saying last time that she caught herself starting and then catch herself. "OK, wait a minute, I shouldn't be doing this," and then let somebody else do it.

Kathy: I didn't even think that you caught yourself. I thought that you were just giving the kids a lead in. You know it's another way of saying...

Jim: Oh, that's interesting...

Ana: Either one could be. [Others: Right. Uh-huh.] Cause I do do that too, sometimes. I'm not sure what I did on this occasion—whether I caught myself or if it was a lead in...

Kathy: I don't think it's ever a bad [inaudible].

Jim: Well in some ways that's a way of asking for a paraphrase. You can do it in that way. "So what Larry is saying..." It could be a fill in the blank by paraphrasing.

Kathy: Also the, "Can you...?" Asking kids...I was talking to [inaudible] about it... I just have words... I can't ask a question like that in my room cause kids will say, "No I can't." So I say, "We need a volunteer to...." Or, "we need somebody to try to blah blah blah."

Jim: So you're looking at line 32 or 34?

In this excerpt, teachers ground their statements not only in the specific shared data in front of them—in this case the quasi-transcript (lines 3-5, 8, 29)—but also in their own classroom experiences and stories (lines 1 and 25–28). These stories may serve to "validate" the data in teachers' own experiences, and to provide supporting and sometimes disconfirming evidence for the hypotheses drawn from data (see Carter, 1993). As teachers develop this form of discourse, it will continue to be interesting to pursue how they view these different types of data. Is evidence from outside used to dismiss conclusions that might otherwise seem clear from the shared data? Does it provide the context needed to make the experiences seem more realistic and practical? Does it serve as the basis for generating alternative hypotheses? What other purposes does it serve?

Several alternative hypotheses are proposed to explain a piece of the data in this excerpt—in this case, the phrase "So what Larry is saying..." This was interpreted variously as someone interrupting to explain Larry's thinking (lines 8–9); incomplete data (line 10); Ana catching herself in the middle of a paraphrase (lines 12–14); "giving kids a lead in" (lines 15–16); and a subtle way of asking for a paraphrase (lines 22–24). Several of these were accepted as possible by Ana. Developing alternative hypotheses to explain teacher moves is a crucial piece of developing a critical frame of mind about teaching. It also expands a teacher's potential repertoire of actions. Yet this is often difficult to accomplish—mainstream teaching culture still reinforces teachers when they evaluate others' teaching, or when they give suggestions. (Lortie, 1975) What would encourage more of this kind of hypothesis generating behavior? What effects does it have on actual teaching practice?

People in the group often reflect on our own experiences together, and even on our own conversation, to provide examples for the discussion. In this excerpt, Ana refers back to a previous week's conversation (line 12) saying, "Like Amelia was saying last time..." Later in the session, Kathy points to something Jim had said earlier in that very conversation as an example of asking for a paraphrase in a more directive way.

Kathy: Well you do it. You did it just now. "Say more about that." Instead of, "Can you say more about that?"

This kind of reflecting back on our own conversation is interesting. How might it serve to help teachers find data relevant to teaching in a wider variety of situations? How might it be linked to the development of the self-reflectivity

needed to publicly develop criteria for behavior and a culture that will make the group self-sufficient in the long-run? What other functions does it serve?

Creating a community of discourse about teaching that links particulars of classroom practice to theory-building is a complex endeavor. Many questions remain about the nature of this discourse, the role of teacher educators in facilitating the creation of this discourse, and the skills and knowledge teachers need to sustain such discourse in the long-term.

References

Ball, D. L. (1994). *Developing Mathematics Reform: What Don't We Know About Teacher Learning—But Would Make Good Working Hypotheses?* Paper presented at the conference on Teacher Enhancement in Mathematics K–6, Arlington, VA.

Carini, P. F. (1975). *Observation and Description: An Alternative Methodology for the Investigation of Human Phenomena*. Grand Forks, ND: North Dakota Study Group on Evaluation, University of North Dakota.

Carter, K. (1993). The Place of Story in the Study of Teaching and Teacher Education. *Educational Researcher, 22*(1), 5-12,18.

Cochran-Smith, M., & Lytle, S. L. (1990). Research on Teaching and Teacher Research: The Issues That Divide. *Educational Researcher, 19*(2), 2-11.

Cochran-Smith, M., & Lytle, S. L. (1992). Communities for Teacher Research: Fringe or Forefront? *American Journal of Education, 100*, 298-324.

Goldsmith, L. T., & Schifter, D. (1993/1994). *Characteristics of a Model for the Development of Mathematics Teaching*. Reports and Papers in Progress, Newton, MA: Center for the Development of Teaching/ Center for Learning, Teaching & Technology, Education Development Center.

Heaton, R. M., & Lampert, M. (1993). Learning to Hear Voices: Inventing a New Pedagogy of Teacher Education. In D. K. Cohen, M. W. McLaughlin, & J. Talbert (Eds.), *Teaching for Understanding: Challenges for Policy and Practice*, (pp. 43-83).

Lord, B. (1994). Teachers' Professional Development: Critical Colleagueship and the Role of Professional Communities. In N. Cobb (Ed.), *The Future of Education: Perspectives on National Standards in America*, (pp. 175-204). New York: College Entrance Examination Board.

Lortie, D. C. (1975). *Schoolteacher: A sociological study*. Chicago, IL: University of Chicago Press.

NCTM. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

NCTM. (1991). *Professional Standards for Teaching Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

NCTM. (1995). *Assessment Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nussbaum, M. C. (1990). The Discernment of Perception: An Aristotelian Conception of Private and Public Rationality. *Love's Knowledge: Essays on Philosophy & Literature*, (pp. 54-105). Oxford: Oxford University Press.

Schifter, D., Bastable, V., Russell, S. J., Lester, J. B., Yaffee, L., Cohen, S., & The Participants in the Teaching to the Big Ideas Project. (In Preparation). Building a System of Tens. Newton, MA: Education Development Center, Inc.

Watt, M. L., & Watt, D. L. (1991). *Teacher Research, Action Research: The Logo Action Research Collaborative* Reports and Papers in Progress (No. 91-4). Newton, MA: Center for Learning Teaching & Technology, Education Development Center.

EISENHOWER NATIONAL CLEARINGHOUSE FOR MATHEMATICS AND SCIENCE EDUCATION

Marsha P. Nicol, The Ohio State University

The Eisenhower National Clearinghouse for Mathematics and Science Education (ENC) is funded through the U. S. Department of Education to provide K-12 teachers with a central source of information on mathematics and science curriculum materials. ENC was established in 1992 through a contract with The Ohio State University and is located in Columbus, Ohio.

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ENC will provide an alternative way to access the ENC catalog and its materials through two CD-ROMs to be distributed free of charge to every US school that requests a copy and that has or soon will have a CD-ROM player. CD-ROM products will include standards documents, teaching improvement information, and curriculum resources. ENC also produces print documents for educators and others interested in mathematics and science education. These materials include a minicatalog with curricular information about special topics, and a newsletter featuring information about the Clearinghouse and promising uses of technology in education. All print documents are available free on request as well as online.

To connect to ENC, use the telnet command to connect to enc.org or use Gopher to enc.org or World Wide Web client software to www.enc.org. If using a modem, dial (800)362-4448. Set your communication software to VT100 terminal emulation, no parity, 8 data bits, 1 stop bit, and full duplex. Once the connection is made press <return> or <enter> once to bring up the welcome screen. Then type *c* to connect and login as *guest* (lowercase, no caps). For more information, contact ENC at info@enc.org or phone (614)292-7784 or (800)621-5785.

"UNCONSCIOUSLY LEARNING SOMETHING:" A FOCUS ON TEACHER QUESTIONING

Emily Dann, Rutgers University
Ralph S. Pantozzi, Rutgers University
Elena Steencken, Rutgers University

The mathematical behaviors of a group of seventh grade students have been observed as part of a longitudinal study of how children build mathematical ideas. The children, having built representations of their solutions to a combinatorics task, are challenged by their teacher to explain and discuss their ideas, and to extend them to similar situations. This report focuses on how teacher questioning facilitates students as they 1) justify their ideas, 2) extend ideas to problems with similar structure, 3) make connections to previous tasks, and 4) generalize their conjectures in the context of isomorphic problems.

Several children have been observed over the course of a longitudinal study[1] of the development of mathematical ideas. During this time, they have been exposed to a constructivist classroom setting, where students are encouraged to build concrete representations and justify solutions to mathematical tasks. After two days of investigation into a combinatorics problem, students were asked how they felt about this activity. One student, Jeff, responded:

> I don't know, it feels, like, I know I'm, I'm, it's like I'm unconsciously learning something, like I know I'm doing something to figure something out, it's just that...yeah, like cause in math we'll go over a subject, and in science we'll say, well, we're learning about "Jane Adams" and we'll study her, but in this it's sort of like you just learn it over, sort of, while you're in the mid[dle]... [when you're] doing something.

It seems that Jeff has indicated an awareness that his learning was significantly different and unlike that which he has experienced in the past. Notice that he indicates that he has "unconsciously learned something." Although Jeff may not have been able to fully articulate his ideas, he did indicate that his learning occurred in the process of *doing* mathematics. The problem-solving activities in which he and his classmates were engaged had been designed to prompt students to search for meaning and build connections between previous, relevant experiences. It could be useful to analyze the details of the learning experience that prompted Jeff's response. This paper will detail a sequence of episodes in which teacher-student interaction and student-student conversations contributed to this process. We focus on the role of teacher interventions, through questioning and probing, while students are actively constructing ideas. We will present episodes that illustrate how teacher questioning plays a part in students' efforts to justify and extend those ideas.

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[1] This research is supported in part by a grant from the National Science Foundation #MDR-9053597 to Rutgers, the State University of New Jersey, Robert B Davis, and Carolyn A. Maher, directors. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Theory

Students, when investigating solutions to mathematical problems, often attempt to generalize their solutions based upon recognition of patterns. In addition, they gain confidence in their extensions as more evidence accumulates in support of their proposed pattern. This confidence may be confused with deeper understanding. For example, in a research study involving ten year old Stephanie's recognition of a doubling pattern after having built towers from plastic cubes selecting from two colors, a conjecture was made that 1024 unique towers could be built that are 10 cubes tall. (Maher & Martino, in press) One might have reasonably inferred from Stephanie's recognition of a doubling pattern that she connected the generation of towers from the pattern. Subsequent teacher probing, however, indicated that Stephanie could not match her generation of actual towers with her pattern recognition. In fact, in tracing the development of the idea, one finds that Stephanie took over one and one half years to build this understanding (Maher & Martino, in press). Vinner (1994) has reported similar findings; he cautions us that what may appear to be a meaningful generalization on the part of the learner may actually be "pseudo-conceptual" behavior. By this he means student-teacher and student-student discussion is based only on teacher cues and student guesses.

Research on teacher questioning (Martino and Maher, 1994) suggests that appropriate teacher intervention can facilitate students' building of justifications of problem solutions, particularly applied at points in time when students are cognitively ready to revisit their ideas. Questions which stimulate students to justify and generalize their ideas may give teachers a greater insight into children's thinking. Questioning used in this way may also provide an alternative model for students to consider as they engage in discussions about their own work. In fact, a long-term case study of one student, Jeff, who has been engaged in thoughtful mathematical problem solving since grade 1, has indicated qualitative differences in his ability to question other students and listen to their ideas (Maher, Martino, and Pantozzi, 1995).

The purpose of this research is to analyze the problem solving activity in which Jeff and his classmates were engaged that prompted him to claim that he had "unconsciously learned something." Specifically, we will focus on three episodes of student conversation that was triggered by teacher intervention in the classroom community that seeks to foster students' construction of mathematical ideas and creation of generalizations.

Background

The students in this study come from a working class school district in New Jersey which has been the site of an on-going longitudinal study[2] in classrooms centered around problem solving. Since grade 1, the students have participated in problem-solving sessions under the guidance of a teacher/researcher intermittently during the school year. The children were seventh graders at the time of this study.

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[2] The study is now in its seventh year.

Methods and Procedures

Thirteen children were seated around tables in two groups of four and one group of five. A camera at each table videotaped the activity. The classroom exploration took place over three days, consisting of two 80-minute sessions and a third session of 40 minutes. Videotapes were transcribed and analyzed by a research team, and the transcripts along with other records were used to produce a video portfolio[1] to trace the development, among individuals and groups of students, of ideas relating to the fairness of the games. The transcripts of the classroom sessions and follow-up student interviews, along with students' written work and assessments, researcher notes, and interpretations of students' work constitute the data for the study.

Design

As one strand of a longitudinal study, the children have worked on combinatorics activities. For this research, we report on the students' investigations of games of chance. The activity required that they determine the "fairness" of a game of chance involving rolling sets of dice and called for their determination of a suitable sample space. After speculating about the possible outcomes when rolling three 6-sided dice, the teacher/researcher introduced tetrahedral dice so that the students could more easily support the conjectures that they had made regarding the number of elements in the sample space. The episodes presented here refer to the students work with the tetrahedral dice.

Episodes

Three episodes from the videotape transcripts provide data for this study.

Episode 1: October 27, 1994. The students had shared their ideas about the number of possibilities when rolling three tetrahedral dice. They were then asked to consider two other cases, where three dice were rolled and when two dice were rolled. After some discussion, the students decided that there were four outcomes for one die and 16 outcomes for two dice.

1 Teacher: Now, what if I'm rolling three of them [the dice]?

2 Bobby: I've got it.

3 Magda: Sixty-four.

4 Michelle: No! Wait. It's more.

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[1] See Maher, C.A. & Martino, A.M. (under review) Conditions contributing to a young child's development of mathematical proof: A 4-year study. *Journal for Research in Mathematics Education.*

5 Bobby: Sixty-four.

6 Teacher: Why don't you write those out? I'd like you to show me a way of representing all those outcomes if you're rolling three.

Episode 2: October 27, 1994. Ankur suggested that rolling four dice would result in 256 possibilities. The students were again asked how it worked with fewer dice. Ankur responded to the case of rolling two dice by producing the first two columns of Figure 1. He made a tree diagram by connecting 1 to 1, 1 to 2, 1 to 3, 1 to 4, 2 to 1, etc., until there were 16 lines visible. When asked to interpret what he had done, he produced the 16 ordered pairs in Figure 2.

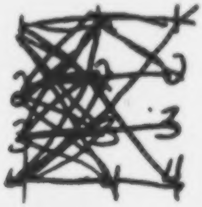
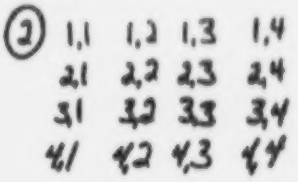


Figure 1



② 1,1 1,2 1,3 1,4
2,1 2,2 2,3 2,4
3,1 3,2 3,3 3,4
4,1 4,2 4,3 4,4

Figure 2

Ankur was asked to extend the idea represented by his diagram:

7 Teacher: Okay, so you've convinced me now that there are sixteen. Okay, now how would you do it if you were rolling three now?

8 Ankur: Sixty-four.

9 Teacher: I want to be able to see them in my head, so show me how you get these sixty-four. Show me, show me how it works.

10 Ankur: Add four more numbers. [Writing the third column of 1,2,3,4 in Figure 2.] This one [pointing to the third column in Figure 1] has four here...you count these too... [the 16 possibilities indicated in columns 1 and 2]

11 Teacher: Okay, can you work together, work out a way to show me how you generate [your method?] to get to two fifty-six? Continue what you're doing and also the way you could keep doing it. Show me the way you begin to think about it.

In both episodes, the students proposed generalizations based upon their previous findings. After the students discussed these generalizations, they created various types of tree diagrams, representing the number of possibilities that they proposed. Figures 3, 4, and 5 represent a portion of the students' written work.

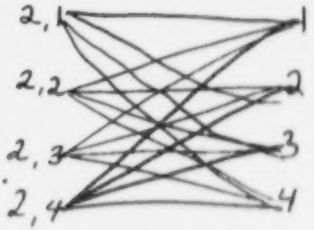


Figure 3

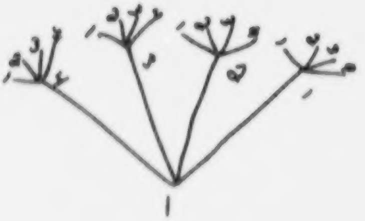


Figure 4

Episode 3: October 28, 1994. In this episode, the students discussed their written work in a small group interview. Michelle began by explaining her work in Figure 5.

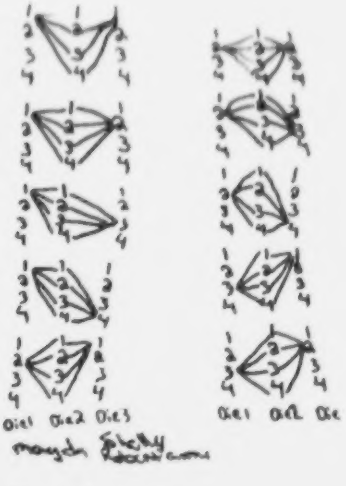


Figure 5

12 Michelle: This [Figure 5] is sort of a little like theirs [Figure 3] except theirs is separated.

13 Teacher: Can you explain it to me?

14 Michelle: If you roll the one on the first die... but then they all went to one [the third die shows 1] and then two and then three and then four [referring to the first four charts in Figure 5] and then if we did it with the twos and then threes and then fours [referring to the remaining charts - some not shown here]

15 Teacher: Oh, how neat

16 Teacher2: Can I ask something for clarification? These numbers [in the first column of each chart] go with the first die?

17 Michelle: Yeah.

18 Teacher2: And the middle ones are the second die? [the second column of each chart]Okay and how many would there be all together? How many outcomes? Possibly sixty-four? Help me to see that in this picture.

19 Michelle: Um, it just shows like..

20 Jeff: Sixteen for each [number of the first die rolled].

21 Ankur: Yeah.

22 Michelle: There are sixteen down here -yeah, it would be...

23 Ankur: Four here, four here, four here, four here [points to each chart in Figure 5] four times four.

24 Jeff: Four times four which got sixteen and then you multiplied.

25 Teacher: I don't see that Michelle multiplied though. Can you see it Michelle?

26 Ankur: These are all the combinations. [Referring to Figure 5 and additional charts, not shown.]

27 Michelle: This [the first four charts of Figure 5] is just for the one thing [die]. There's like sixteen for each like like, like if you rolled a one for the first die number on the chart.

28 Teacher: Huh, the question is, show us sixty-four. I guess I thought I saw it and now I'm not sure.

Later in the conversation, Michelle suggested how her diagram (Figure 5) might be extended if additional dice were rolled, while referring to her work in Figure 4.

29 Teacher: Now if you were to do it, for rolling it four times, what would your chart look like, how would you do it?

30 Ankur: Another four numbers on the side. [of the charts in Figure 5]

31 Teacher: Another four numbers on the side, what would that look like?

32 Michelle: I guess it would sort of be harder to...like...do it that way.

33 Teacher: Well could you do it though? Is it possible?

34 Michelle: Well it would be like this is, [Figure 4] ah...See here's our number one. I looked at their chart things. And like it's not exactly the same, but it sort of is and I remember we did um when we like did the towers we did like a tree thing. I don't know if any

of you people remember, but I remember when we did like a tree thing.

35 Teacher: What was the towers?

36 Michelle: When, I forget, I just knew when we were working on towers.

37 Jeff: Towers of three, and two different colors, how many can you make...

38 Michelle: And, yeah, then we did it in trees, so when we... I remembered that so we did it like that. And this is what you would roll on the first die [pointing to first tier of Figure 4] and this is like what you would roll on the second die [pointing to second tier]and this is what you could roll on the third die [points to third tier].

Near the end of the session, Ankur offered a general rule that he had developed.

39 Teacher: Do you want to tell us what that is? [Referring to a rule Ankur mentioned previously]

40 Ankur: Well it's the number of sides, that's a four, in this case, times, to the power of like the number of dice you have [Writes 4^4 on his paper.]

41 Teacher: Does that work, if you had a six-sided die and you were rolling it twice?

42 Ankur: That's six to the second power.

43 Teacher: Tell me, this [pointing to the base number]is the number of sides? And...

44 Ankur: This, the number here [pointing to the base number] is the number of sides. And this [pointing to the exponent] is the number of dice.

45 Michelle: But that's like four times four times four times four.

46 Ankur: Yeah.

47 Michelle: Oh, okay.

48 Teacher: Okay, so you had a general rule, didn't you? With x's and y's? You were showing me?

49 Ankur: Yeah.

50 Teacher: How does that work? Show me. Why don't you say...

51 Michelle: Is that [4^4]for the sixty-four?

52 Ankur: No, it's for two hundred fifty-six. Sixty-four is four to the third. Four times four is sixteen times four.

53 Michelle: Oh, OK.

54 Teacher: Ankur, suppose I had a twelve-sided die and I was rolling it three times, what would the rule be?

55 Ankur: Twelve times twelve times twelve, twelve to the third.

56 Teacher: [to other students] What do you think?

57 Jeff: I agree.

58 Teacher: So what's the general rule you're telling me?

59 Ankur: It's like if 4 is equal to x [writes x on his paper] and this is y [writes y as an exponent]

60 Teacher: So what does the x represent? Why don't you write it out?

Conclusions & Implications

These episodes indicate instances (lines 6, 9, and 11) where a teacher/researcher has posed questions that were designed to promote the interaction of students and prompt them into explanations and justifications of their ideas. We note that these questions were directed not only to elicit a response from one student, but to invoke the participation of others in the group. Instead of confirming the students' findings, the questions focused on further elaboration of their proposed generalizations (lines 31, 41). In response, Michelle made a connection to a previous task (line 34), and Ankur suggested a general rule (line 59).

At any point where students offer generalizations, teachers may make the decision to build connecting structures for students. It is in this "territory" where we believe that teacher decisions are critical. In the episodes presented, the students were given opportunities to revisit and share their ideas with each other. We suggest that Jeff's belief in having "learned unconsciously" might have arisen through his involvement in situations where the bounds of his inquiry were not externally framed. One implication is that appropriate teacher intervention, and ample time for students to build mathematical ideas, may be crucial tools in helping students build further connections between the deep ideas underlying their work.

The teacher questioning and student discussion presented here may serve to illuminate elements of "unconscious thinking." Such thinking may include the process of students' building powerful schemata through active reconsideration of ideas, prompted by teacher questioning.

References

Davis, R.B. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*. New Jersey: Ablex Publishing.

Maher, C.A., Davis, R.B. & Alston, A. (1991) Brian's representation and development of mathematical knowledge: A four year study. *The Journal of Mathematical Behavior, 10*, 163-210.

Maher, C.A. & Martino, A.M. (under review) Conditions contributing to a young child's development of mathematical proof: A 4-year study. *Journal for Research in Mathematics Education*.

Maher, C.A. & Martino, A. M. (in press). Young Children Invent Methods of Proof: The Gang of Four. In L. Steffe, P. Cobb, G. Goldin, B. Greer & P. Nesher (Eds.), *Theories of Learning*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Maher, C.A. & Martino, A. M. (in press). Conditions for Conceptual Change: From Pattern Recognition to Theory Posing. In H. Mansfield, (Ed.), *Young Children and Mathematics: Concepts and their Representations*. Curtin University of Technology, Australia: Australian Association of Mathematics Teachers.

Maher, C.A., Martino, A.M., and Pantozzi, R.S. (1995) Listening Better and Questioning Better: A Case Study. In L. Meira, D. Carraher, (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol. 3 pp. 82 - 89). Recife, Brazil: The Program Committee of the 19th PME Conference.

Vinner, Shlomo. (1994). Traditional Mathematics Classrooms - Some Seemingly Unavoidable Features. In J. Ponte and J. Matos, (Eds.) *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (pp. 353 - 360). Lisbon, Portugal: The Program Committee of the 18th PME Conference.

BECOMING A TEACHER OF MATHEMATICS: IMPACT OF THE PRE-SERVICE EXPERIENCE

Gavrielle Levine, C.W. Post Campus, Long Island University

The National Council of Teachers of Mathematics (1989) recommends revising the mathematics curriculum as well as the way in which mathematics is taught, starting in elementary school. One serious consequence of implementing these changes may be increased anxiety for teaching mathematics, resulting in interference with teacher preparation and diminished classroom effectiveness (Kelly & Tomhave, 1985). To address these concerns, the crucial relationships among teacher preparation, teaching effectiveness (i.e., teaching style), and anxiety for teaching mathematics have been examined.

A cohort of pre-service elementary school teachers enrolled in a mathematics methods course initially showed high levels of anxiety for teaching mathematics, which decreased at the conclusion of the mathematics methods course (Levine, 1993a) associated with shifts in anticipated teaching style (Levine, 1993b). Interviews with a subgroup of this cohort examined mathematics teaching style and anxiety for teaching mathematics during their experience student teaching and/or classroom teaching to assess the impact of the teacher training experience on classroom practice. Results of this investigation have implications for teacher preparation.

References

Kelly, W.P. & Tomhave, W.K. (1985). A study of math anxiety/math avoidance in preservice elementary teachers. *Arithmetic Teacher, 32*(5), 51-53.

Levine, G. (October, 1993a). *Assessing shifts in anxiety for teaching mathematics in prospective teachers.* Paper presented at the annual conference of the Northeastern Educational Research Association, Ellenville, New York.

Levine, G. (October, 1993b). *Relationships among anxiety for teaching mathematics, prior student history, and expectations for mathematics teaching style among prospective elementary school teachers.* Paper presented at the 15th annual meeting of the International Group for the Psychology of Mathematics Education, North American Chapter, Asilomar, California.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

PLANNING TO TEACH GEOMETRY WITH THE GEOMETER'S SKETCHPAD

Heréndira G. Galindo, Indiana University—Bloomington

The theoretical framework of this study draws from several correlational studies based on Myers and Briggs' theory (an extension of the theory of types of Carl G. Jung). These studies suggest that there are four types of teachers, namely, *nurturers*, *instructional managers*, *intellectual challengers*, and *facilitators*. Furthermore, there is evidence that teaching strategies, methods of teaching, and assessment methods of school teachers can be categorized according to these four types. The present study investigates whether this categorization could also be observed in computer environments.

Nineteen preservice high school mathematics teachers participated in the geometry workshop designed to serve as a context for the present study. The results presented here are based on four selected cases representing the four teacher types mentioned above. Participants were asked to design computer environments, that is, computer uses and classroom interactions. They had to give an example of how they would use the software to teach a geometrical relationship or concept and also a form to assess their activities. They were also interviewed and asked to describe the teaching strategies, methods of teaching, and assessment methods that they would put into practice in their computer environments. The data analysis suggests that all teacher types but *the instructional manager* adopt the role of facilitators when planning to teach geometry with the Geometer's Sketchpad. The *instructional manager* of this study plans to teach in the same way in both the classroom and the computer laboratory. He sees the curriculum as something fixed, unchangeable.

The most common type observed in school teacher populations is the *instructional manager*. The proportion of this teacher type is even greater in school mathematics teacher populations. Type theory suggests that this teacher type is not open to curricular changes and the present study supports this result; that is, it was observed that the preservice teacher with an *instructional manager* approach to teaching could not adopt new teaching strategies, methods of teaching, and assessment methods. But the point is not how to make these teachers change their approach to teaching but to help them to adapt to new social conditions and demands. Under the positivist paradigm the relevant question was "How does A get B to respond as A desires?" Under the new more humanistic paradigms the concern is "What can A do to help B achieve self–realization?" (Lynch, Norem–Hebeisen, & Gergen, 1981). The question that arises here is: How can this principle be applied to teacher education?

References

Lynch, M., Norem–Hebeisen, A., & Gergen, K. (1981). *Self–concept: Advances in theory and research*. Cambridge, MA: Ballinger Publishing Company.

Myers I., & Myers, P. (1980). *Gifts differing*. Palo Alto, CA: Consulting Psychologists Press.

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Teacher Understanding of Student Understanding

USING TEACHER WRITTEN CASES TO EXPLORE STUDENTS' MATHEMATICAL THINKING: A VEHICLE FOR TEACHER INQUIRY GROUPS

Virginia Bastable, SummerMath for Teachers
Jill Lester, SummerMath for Teachers
Deborah Schifter, Education Development Center, Inc.

In recent years, the conviction has been growing that cases or stories may be more helpful than theoretical expositions to people who need to think in new ways about complex, context-dependent domains like teaching. The current mathematics education literature provides case studies of classroom teachers written by researchers, case studies conducted by university faculty who also teach mathematics to students in grades K -12 and make their own teaching the object of their research, and cases written by practicing teachers. We, the discussion-group proposers, extend this work by having teachers write cases in which they explore the mathematical thinking of their students to share with their colleagues in the context of our teacher-enhancement project.

The brief (2 -5 page) cases are used as the basis for discussions in which small groups of teachers work to understand the mathematical issues their students are grappling with. They provide a method for each teacher to inquire into his or her own practice and serve as a means to engage with peers in analyzing classroom process with particular focus on mathematical ideas and students' mathematical thinking.

This discussion session will address the questions: In what ways can such cases be used to foster a spirit of inquiry into students' mathematical thinking? In what ways does such work bring the findings of past decades in cognitive research to light for teachers?

Group organizers will set the context for discussion by providing a brief description of the project which generated the teacher written cases. Then participants will read a set of cases presenting classroom episodes which include student dialogue as they work on a particular mathematical idea. Participants will spend about five minutes in pairs sharing the issues that are evoked by the cases and then the whole group will list the various issues. We will then choose one or two to explore more deeply. The focus of this section of the discussion will be on the mathematics evoked by the cases.

Next there will be a period of reflection about the discussion itself and the role of the cases: In what ways did the cases stimulate our own thinking? In what ways could such discussions support teacher education/staff development efforts? What additional materials would be useful for teachers or teacher educators to help them use such cases to conduct similar discussions?

The session will close as group organizers provide examples which illustrate what teachers in their projects are learning from writing and discussing such cases.

AN ANALYSIS OF THE TEACHER'S PROACTIVE ROLE IN REDESCRIBING AND NOTATING STUDENTS' EXPLANATIONS AND SOLUTIONS

Kay McClain, Vanderbilt University
Joy Whitenack, Vanderbilt University

The analysis reported in this paper is part of a year-long, first-grade teaching experiment and focuses on the teacher's proactive role in supporting students' mathematical growth in an inquiry mathematics classroom. Within the project classroom, the teacher often redescribed and notated students' responses so that what students had done mathematically might become an explicit topic of conversation. As part of this process, she frequently introduced either informal or conventional notation to record students' explanations of their mathematical activity. The introduction of these notational schemes lead to students' development of ways of notating their own reasoning. In this way, the notation emerged from the students' activity while supporting shifts in their mathematical development.

The purpose of this paper is to document crucial aspects of one effective reform teacher's proactive role in initiating and guiding students' mathematical development. Particular attention will be given to how the teacher redescribed and notated student responses. This activity will be related both to shifts in discourse and to students' development of ways of notating their own reasoning. We will also attempt to clarify how the development of notational schemes became realized in the classroom by developing empirically grounded analyses of the teaching-learning process as it was interactively constituted in the classroom (Cobb, Wood, Yackel, & McNeal, 1992). The intent is, therefore, not to develop a prescriptive list of behaviors that purports to guarantee effective reform teaching. Instead, it is to develop a detailed account of one teacher's practice situated in a specific classroom. The reported analysis should be of more than local interest because it might serve as a paradigmatic case that can both help other teachers develop understandings of their own practice and contribute to the growing research literature on reform teaching.

In the following paragraphs, we first provide a description of the teacher and her classroom and then outline the data corpus. Against this background, we analyze the teacher's proactive role in supporting her students' mathematical growth by redescribing and notating their explanations.

Ms. Smith's Classroom

The majority of the eleven girls and seven boys in Ms. Smith's first-grade classroom were from middle or upper middle class backgrounds. There were no minority children in the classroom, although a small percentage attended the school.

The research reported in this paper was supported by the National Science Foundation under grant No. RED-9353587. The opinions expressed do not necessarily reflect the views of the Foundation.

The students in the class were representative of the school's general student population. Although not a parochial school, morals and values were considered to be part of the responsibility of schooling and children regularly participated in spiritual activities.

Ms. Smith's classroom is of particular interest because an analysis of videorecorded interviews conducted at the beginning and end of the school year indicates that the students' conceptual development in mathematics was substantial. Students who, at the beginning of the year, did not have a way to begin to solve the most elementary kinds of story problems posed with numbers of five or less had, by the end of the year, developed relatively sophisticated mental computation strategies for solving a wide range of problems posed with two-digit numbers.

The teacher, Ms. Smith, was a highly motivated and very dedicated teacher in her fourth year in the classroom. She had attempted to reform her practice prior to our collaboration and voiced frustration with traditional mathematics textbooks. She explained that her attempts to use a "center" approach left her without the benefits of productive whole class discussions. Although she valued students' ability to communicate, explain, and justify, she indicated that she had previously found it difficult to enact an instructional approach that both met her students' needs and enabled her to achieve her own pedagogical agenda. When we began working with Ms. Smith, it soon became apparent that she constantly assessed both the instructional activities she used and her own practice. In addition, she had a relatively deep understanding of both mathematics and her students' thinking. Ms. Smith was seeking guidance with her reform efforts; we were seeking a teacher with whom to collaborate as we developed sequences of instructional activities.

Data Corpus

Data were collected during the 1993-94 school year and consist of daily videotape recordings of 103 mathematics lessons from two cameras. During whole class discussions, one camera focused primarily on the teacher and on children who came to the whiteboard to explain their thinking. The second camera focused on the students as they engaged in discussions while sitting on the floor facing the whiteboard. Additional documentation consists of copies of all the children's written work; daily field notes that summarize classroom events; notes from daily debriefing sessions held with Ms. Smith; and videotaped clinical interviews conducted with each student in September, December, January, and May.

A method described by Cobb and Whitenack (in press) for conducting longitudinal analyses of videotape sessions guided the analyses. This method fits with Glaser and Strauss' (1967) constant comparative methods for conducting ethnographic studies. It involves constantly comparing data as they are analyzed with conjectures and speculations generated thus far in the data analysis. As issues arise while viewing classroom videorecordings, they are documented and clarified through a process of conjecture and refutation. Through this process, classroom accounts have been identified which can be used to clarify 1) how Ms. Smith's practices became realized in the classroom, and 2) how these practices contributed to her students' growth. This paper, which is part of a larger investigation will focus specifically on redescribing and notating students' explanations and solutions.

Classroom Analysis

Lesh, Post, and Behr (1987) observe that many students have "deficient understandings about the models and languages to represent and manipulate mathematical ideas" (p. 37). In our view, it is essential that students experience symbols in relation to their own mathematical activity if they are to develop grounded understanding of their meaning and use. Students might then view symbols and notations as ways of recording and communicating their thinking that they can use as the need arises.

In this particular classroom, Ms. Smith often attempted to initiate shifts in the level of classroom discourse so that what was done mathematically subsequently might become an explicit topic of conversation. As part of this process, she frequently drew pictures or used either informal or conventional notation as she redescribed students' explanations. Ways of notating therefore functioned as protocols of action (Dörfler, 1989) that grew out of the students' activity in a bottom-up manner (cf. Gravemeijer, in press). For example, students often solved an addition task such as 7 + 8 by partitioning the 8 into seven and one and reasoning, seven and seven is 14, and one more is 15. Ms. Smith devised a simple method of notating this activity by using an inverted "V" symbol that came to signify the partitioning of a number. Ms. Smith would typically follow the "V" notation with the number sentences that expressed the result of the partitioning (see figure 1).

```
7 + 8 =
    / \
    7  1

 7 + 7 = 14
14 + 1 = 15
```

Figure 1. Notating Decomposition of Numbers

On their own initiative, students often referred to this notation to explain their thinking to other children during whole class discussions. In addition, students began to use the records as a means of comparing solutions, thereby initiating shifts in the discourse such that features of their reasoning became an explicit topic of conversation. We speculate that the students' participation in such discourse supported their reflection on and mathematization of their prior activity. However, it is important to note that students were not obliged to use the notational schemes introduced by Ms. Smith. The children did in fact symbolize their thinking in a variety of different ways when they were asked to make records so that other children might understand how they had solved tasks. This lack of obligation allowed students to develop their own means of representing their reasoning by adapting what had been offered.

As an illustration, consider an incident which occurred on December 7. The task posed by Ms. Smith was a story problem involving a bus — *There are eight people on the bus and six more get on. How many people are on the bus now?* Students were asked to work individually on the task, making a record of their solution process so that others might understand their reasoning. The focus in this part of the lesson was effectively communicating their mathematical thinking — not imitating a given notational system. During the subsequent whole class discussion, Ms. Smith asked students to share their solution methods verbally as she redescribed and notated their activity. In this particular instance, the first offered solution involved using a doubles strategy:

> Kitty: I took one off the eight and I put it on to the six to make 7 plus 7 and I know 7 plus 7 makes 14.

Ms. Smith redescribed and notated (see figure 2).

```
  8 + 6 =
 / \
 7 1

 7 + 7 = 14
```

Figure 2. Notating Kitty's Solution

After questions and discussion, Ms. Smith asked for a different way. Jane explained that she partitioned the numbers differently.

> Jane: I stayed with the 6 but I broke it up into 3 and 3 and when it had the three it made 11 and three more ... it made ... uhm ... it made ... it made 13 and one more is 14.

Again, Ms. Smith redescribed and notated the solution (see figure 3), attempting to clarify to the students how Jane's explanation differed from Kitty's.

```
  8 + 6 =
     / \
     3 3

  8 + 3 = 11
 11 + 3 = 14
```

Figure 3. Notating Jane's Solution

In addition to Ms. Smith's verbal clarification of the difference in the solutions, symbolizing the two solutions offered opportunities for the other students to compare the two solutions as they attempted to clarify for themselves how these solutions compared to each other and to their own.

In examining the student work from the previously described task, it is important to note that there was diversity in the students' notational schemes. Although students often used elements of the teacher's notation scheme, they did so in original ways as they struggled to communicate their thinking. Even when students' verbal explanations were redescribed and notated by Ms. Smith in a manner consistent with her original scheme, the students worked to devise notational schemes to fit with their thinking that supported their interpretation of the solution process. As a consequence, although students might accept ways of talking about their activity that fit with the teacher's notation scheme, they continued to solve tasks using very different, personally meaningful notation schemes. This in turn made it possible for students to develop interpretations and strategies that reflected their current understanding while experiencing more sophisticated and efficient solutions in class discussions.

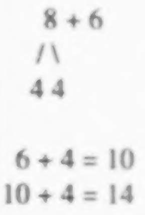


Figure 4.

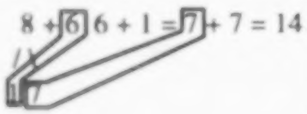


Figure 5.

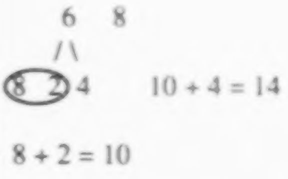


Figure 6.

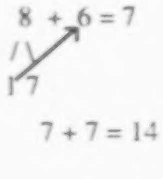


Figure 7.

In the sample records shown below (see figures 4 through 7), each child arrived at an answer of 14. However, only the first child's way of notating is consistent with that of the teacher. Although the other three used elements of the teacher's scheme, they adapted them in original ways.

It should be stressed that, from the students' perspective, Ms. Smith appeared to introduce notation almost in passing. In addition, the students were not obliged to follow—and never practiced—particular ways of notating. The notation was offered by Ms. Smith and became taken-as-shared only through a process of mutual negotiation between her and the students.

As a further note, the teacher's proactive role in guiding the development of ways of notating appears to have been critical in supporting her students' mathematical development. The children increasingly notated on their own initiative as they solved problems while working both individually and in groups. These records helped them distance themselves from their ongoing activity and thus reflect on what they were doing. Consequently, the use of notation contributed to the productiveness of whole class discussions by helping to make individual children's contributions explicit topics of conversation that could be compared and contrasted. It was as they participated in these discussions that the teacher guided her students' transition from informal, pragmatic problem solving to more sophisticated yet personally meaningful mathematical activity.

Conclusion

Throughout this paper, we have attempted to document crucial aspects of Ms. Smith's proactive role in redescribing and notating students' explanations and solutions. For Ms. Smith, the notational schemes emerged from the students' attempts to explain and justify their thinking; they were not predetermined schemes introduced into the classroom in a top-down manner. The analysis of Ms. Smith's role in introducing these schemes indicates that while they became taken-as-shared through a process of mutual negotiation, the teacher played a central role in initiating the development of notational schemes that served as protocols of students' mathematical activity.

The reform movement in mathematics education has emerged as a response to the consequences of traditional mathematics instruction. Often, reform teaching was characterized with reference to traditional teaching, and the emphasis was on what teachers should not do (e.g. funnel students to correct answers and script lesson plans). Although it was generally accepted that reform teachers should actively support their students' mathematics development, this was frequently characterized in vague terms such as facilitate or guide. It is only recently that explicit attention has been given to what specifically effective reform teachers do to support their students' development. This paper contributes to this growing literature by documenting the proactive actions of one teacher as she attempted to reform her practice.

References

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29*(3), 573-604.

Cobb, P., & Whitenack, J. (in press). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics*.

Dorfler, W. (1989). Protocols of actions as a cognitive tool for knowledge construction. In *Proceedings of the Thirteenth Conference of the International Group for the Psychology of Mathematics Education* (pp. 212-219). Paris, France: PME.

Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory*. Chicago: Aldine.

Gravemeijer, K. (in press). Mediating between concrete and abstract. In T. Nunes & P. Bryant (Eds.), *How do children learn mathematics?* Hillsdale, NJ: Lawrence Erlbaum Associates.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the learning and teaching of mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum Associates.

FACILITATING PROSPECTIVE TEACHERS' USE OF CHILDREN'S THINKING AND RESULTING CLASSROOM DISCOURSE

S. L. Sproule, The Florida State University
A. D. Thompson, The Florida State University

Our experience teaching prospective teachers and observing their field experiences suggests weaknesses in the learning experiences we offer them during methods courses. The primary areas of concern include the prospective teachers' limited understanding of children's thinking and the lack of depth of classroom discourse facilitated by the prospective teachers. We are currently addressing these concerns in the development of a methods course for prospective middle school mathematics teachers.

The primary vehicles we are using to deal with the limitations include tutoring and clinical interviews. In general, the purposes for the clinical interview experiences include facilitating the development of effective questioning techniques, encouraging the prospective teachers to listen to their students rather than dominating the discourse, give the prospective teachers an opportunity to deliberately model or explain children's mathematical thinking in a structured, organized and hopefully meaningful process.

The interview experience will be enhanced by the inclusion of a tutoring experience. We maintain that tutoring offers the prospective teacher a teaching environment where classroom management issues have been minimized. Thus the tutor (prospective teacher) has the opportunity to focus their attention on the student's conceptualization of mathematics and discourse inherent in the tutoring experience.

Diagnostic frameworks will be offered to the prospective teachers to complete a report on their understanding of the children's mathematical thinking experienced in the interviews and the tutoring sessions. In addition, the prospective teachers will be provided frameworks to facilitate their decision-making in the classroom. The frameworks are designed to encourage the prospective teacher to make their educational decisions regarding the use of calculators, alternative forms of assessment, problem solving, and grouping practices explicit.

Technology

THE DAY THE CALCULATOR CHANGED: VISUAL CALCULATORS IN PREALGEBRA AND ALGEBRA

John E. Gwens, East Carolina University

Multi-line-multi-operation calculators such as the TI-80 provide eighth-grade prealgebra and algebra students with significantly better computational tools for basic order-of-operation problems involving integers and signed rational numbers than do calculators offering only last-entry-or-result displays. Effects are more apparent for weaker students and in more complicated problems involving the distributive property.

The literature on calculator usage to date (with the exception of some recent studies of graphing capabilities) is based on Last-Entry-or-Result display (LER) calculators. Historically, studies have tended to focus on the effect of calculator usage on students' pencil & paper-based computational skills and attitudes toward mathematics rather than on the nature of students' interaction with the calculator (Hembree & Dessart, 1992). Calculators tend to be treated as "computational experts" useful for their ability to do quick and accurate arithmetic.

More recently calculators have begun to be investigated for a more meaningful role in the learning of mathematics. (Hirschhorn & Senk, 1992; Bitter & Hatfield, 1993). However, the question as to whether or not the visual feedback from the calculator might not be consistent with (or even in direct conflict with) students' written or mental representations of an expression seems, in these studies, not an object for investigation but an obstacle overcome by teaching students the calculator's mode of entry. This is quite understandable given the current nature of non-graphing calculators. But the fact that inexpensive dot-addressable displays are now becoming widely available suggests that this need no longer be the case—reasonably priced calculators can be designed that mimic hand and text-based operations.

Limited recognition of the potential of Multi-Line-Multi-Operation display (MLMO) calculators as teaching tools has begun to appear (Vonder Embse, 1992); but these calls are deficient in two ways: They lack a basis in theory and an empirical research base. This study is the first in a series of planned investigations into the use of MLMO calculators as tools for doing and learning mathematics and the nature of student interaction with various calculator displays.

As a first step, students in eighth-grade prealgebra and algebra were tested on selected skills involving order-of-operation problems with integers and rational numbers to determine what, if any, advantage MLMO displays have over LER displays.

Subjects

Participants in the study were four intact classes of eighth grade students at a local middle school. Two of the classes were first-year algebra (the only first-year algebra classes taught at the school); the others were two of the four eighth grade mathematics (primarily prealgebra) classes taken by all eighth-grade students not enrolled in algebra. All four classes under consideration were taught by the same teacher so as to minimize teacher effects. Data were collected during the final month of the 1994-95 school year.

All eighth-grade students in North Carolina take a state-constructed end-of-grade mathematics test consisting of eight components, seven of which are calculator-active. Texas Instruments' TI-12 Math Explorer is the recommended calculator and was used by all students during the year and for this test ("graphing" calculators have been excluded from use on this test on a state-wide basis). In addition, students in first-year algebra take a state-constructed end-of-course test requiring use of a graphing calculator.

Texas Instruments' TI-81 graphing calculators were used extensively during the year in the algebra class with minimal preparation on the use of TI-12s in preparation for the end-of-grade exam. Prealgebra classes used TI-12s extensively with some introduction to TI-81s.

Method

Students completed three forms of an instrument (see Instruments, below) designed to ascertain their proficiency with certain prealgebra and algebra skills. In each case, classes received approximately three days of review/instruction prior to the administration. The first instance of the instrument was completed manually (without calculator); half of the classes (one prealgebra, one algebra) completed the second instance using LER calculators and the third using MLMO calculators, while the other two classes reversed this sequence (in order to minimize possible sequencing bias). Administration of the instruments was untimed. Only students who completed all three sittings of the instruments (61 students—33 algebra and 27 prealgebra—approximately 75% of the original classes) were included in the analysis.

Instruments

The instrument consisted of 24 problems, four problems each (two using integers, two using rational numbers) in six groups: Simple addition/subtraction; simple multiplication/division; complex addition/subtraction; complex multiplication/division; simple distributive; and complex distributive (see Figure 1). Problems were selected and written to conform to the type and format of problems worked by the students during the year.

Three equivalent forms of the instrument were developed. In each case equivalent problems maintained signs and operations, changing only the numbers to reduce student reliance on memory of previous forms to generate answers. For example, the problem 2 - 3 on form 1 was changed to 1 - 2 on form 2 and 2 - 4 on form 3.

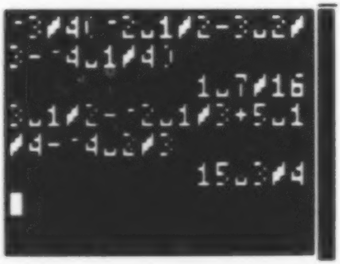
Equipment

Calculators used in the study were Texas Instruments' TI-12 Math Explorer and TI-80 graphing calculator. The TI-12 was chosen as the LER based on student

$\frac{1}{2} - -2\frac{1}{4}$	**Simple A/S**
$4(-2)$	Simple M/D
$2\frac{1}{3} - -3\frac{1}{2} + 4\frac{2}{3} - -3\frac{1}{4}$	Complex A/S
$-4 - -2 \div -1 + 3 \div -2$	Complex M/D
$3\frac{1}{3}\left(-2\frac{2}{3} - 3\frac{1}{4}\right)$	Simple Dist.
$2(-4 - -3) - 2(1 + -4)$	Complex Dist.

Figure 1. Sample Problems

familiarity with the calculator. The TI-80 is a new calculator, introduced in 1995, that combines the eight-line display capabilities of the TI-81 with the fraction operations of the TI-12 (see Figure 2). This calculator was chosen as the MLMO model based on student familiarity with TI-12 and TI-81 operations.



```
⁻3/4(⁻2␣1/2-3␣2/
3-⁻4␣1/4)
          1␣7/16
3␣1/2-⁻2␣1/3+5␣1
/4-⁻4␣2/3
         15␣3/4
█
```

Figure 2. Sample TI-80 Screen

Results

Results were analyzed using SPSS-PC v6.1. Comparing means by question indicated significant differences in all pairwise combinations (Manual vs LER, Manual vs MLMO, LER vs MLMO) using t-tests for paired samples (2-tail significance, $p<.0005$).

An analysis of variance for sex-related differences showed significance ($p<.0005$) only for manual calculations in prealgebra. It is interesting to note that in prealgebra classes girls outscored boys in each of the three instances (although only manual comparisons were significant). However, in the algebra classes boys outscored girls on manual computations, but were outscored by the girls on both LER and MLMO implementations (although no differences were significant).

Average time taken by students for LER and MLMO calculations were virtually identical, while times taken for manual operations were significantly longer for both algebra and prealgebra classes. Order of implementation (i.e., LER before MLMO or vice-versa) was not significant. (see Table 1).

Table 1

Mean and Standard Deviation

	Algebra		Prealgebra	
	Score	Time	Score	Time
Manual	.43/.13	37.65/12.59	.20/.11	26.74/6.44
LER	.82/.12	19.03/5.99	.62/.15	19.56/7.30
MLMO	.93/.10	20.00/6.36	.84/.12	20.33/5.60

More detailed analyses were carried out by question. Table 2 describes significant differences by question. (See also Figure 3 and Figure 4). Additional analyses by student were performed but are not reported in this paper.

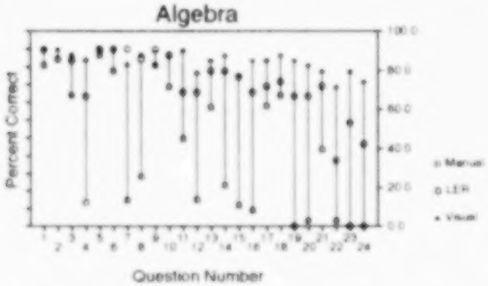


Figure 3. Algebra by Question

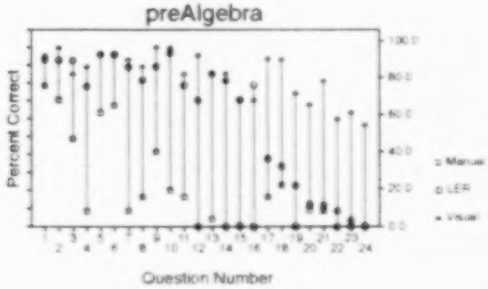


Figure 4. Prealgebra by Question

Table 2

Significant Differences by Question

Significant Differences by Question (p_.01)

	Simple Add/Sub				Complex Add/Sub				Simple Mult/Div				Complex Mult/Div				Simple Dist				Complex Dist			
	Integer		Rational		Integer		Rational		Integer		Rational		Integer		Rational		Integer		Rational		Integer		Rational	
Algebra	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
M vs L				*			*	*				*		*	*	*			*	*	*	*	*	*
M vs V			*	*			*	*			*	*	*	*	*	*	*	*	*	*	*	*	*	*
L vs V											*											*	*	*
preAlgebra	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
M vs L			*	*	*	*	*	*	*	*	*	*	*	*	*	*								
M vs V		*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
L vs V																	*	*	*	*	*	*	*	*

Discussion

Order-of-operation and signed number problems are common skills in prealgebra and algebra curriculums. This study suggests that computational gains on these problems using calculators are impressive—algebra scores rose from 43% to 82% to 93% and prealgebra from 20% to 62% to 84% (Manual to LER to MLMO)—and that MLMO calculators provide significant gains over LER calculators. These gains tend to be more pronounced with relatively weaker students (i.e., eighth graders in prealgebra versus those taking algebra) and in problems involving the distributive property.

Although a significant difference exists between LER and MLMO calculators for algebra students, an examination by question suggests that the bulk of the difference—for these students— lies in more complicated problems involving the distributive property and when several rational numbers are involved. For prealgebra students, the differences between MLMO and LER scores are more pronounced, extending over the full range of problems involving the distributive property.

Possible explanations for this phenomena are that eighth graders taking algebra are more able to deal with reduced visual feedback than their counterparts in prealgebra and/or that a better understanding of the concept of distribution facilitates calculator use (particularly with LER calculators). It is hypothesized that more complicated problems—order-of-operation problems with encapsulated brackets (e.g. 3(4-2(5+6))), problems requiring substitutions for variables, problems involving radicals or exponents, and so on—would further exacerbate the advantages of MLMO over LER calculators. The purpose of this study was to investigate points at which such calculators begin to make a difference in the prealgebra/algebra curriculum.

Student reaction to the calculators was pronounced. Prealgebra students who used the MLMO calculators first, followed by the LER model, were particularly vocal in their preference for the MLMO. One student from this group, while using the LER, complained, "You know she (the teacher) has those good (MLMO) calculators in the closet. Why can't we use them?" Another repeatedly got an answer on the LER she knew to be wrong (a simple problem she had worked manually) and complained, "I've put this problem in five times and can't get it to give me the right answer. I'm not going to do it again."

Interestingly, the time taken by students for LER and MLMO calculations was virtually identical. This seems to result from the observation that, unlike the young lady described above, students using LER's rarely re-entered a problem as a check against an answer. Students appeared to look at their calculation on the MLMO screen prior to execution, occasionally changing obvious mistakes.

Concerns and Implications

If calculators are to be used in middle grades to "(a) introduce new concepts, (b) provide a computational tool for use in discovery lessons, (c) simplify the computational aspects of real-life situational problems, and (d) assist students as they solve problems in group learning situations" (Bitter & Hatfield, 1993), then a calculator that provides better visual feedback—more in keeping with written symbolic notation—and allows students to obtain more accurate answers is of utmost importance. For the students in this study, MLMO calculators were superior tools for basic prealgebra and algebra computations and seem to better fit these purposes than do LER models.

A common concern with calculator use is that students can attain correct answers without understanding the underlying concepts, or without mastering basic pencil-and-paper skills. Several students in this study demonstrated lack of understanding (often writing "don't understand" next to a more complicated problem) or conceptual misconceptions (e.g., 4_ x 2_ = 4 x 2 + _ x _ = 8_) during the manual implementation, yet correctly answered equivalent problems using calculators. How calculators are used in helping students form concepts associated with order-of-operations and signed number operations must receive high priority in instructional planning if calculators are to become instructional tools in addition to computational experts.

References

Bitter, Gary G. & Mary M. Hatfield. (1993). Integration of the MATH Explorer into the mathematics curriculum: The Calculator Project Report. *Journal of Computers in Mathematics and Science Teaching, 12*(1), 59-81.

Hembree, Ray & Donald J. Dessart. (1992). Research on calculators in mathematics education. In James T. Fey (Ed.) *Calculators in mathematics education: 1992 yearbook of the National Council of Teachers of Mathematics* (pp. 22-31). Reston, VA: NCTM.

Hirschhorn, Daniel B. & Sharon Senk. (1992). Calculators in the UCSMP curriculum for grades 7&8. In James T. Fey (Ed.) *Calculators in mathematics education: 1992 yearbook of the National Council of Teachers of Mathematics* (pp. 78-80). Reston, VA: NCTM.

Vonder Embse, Charles. (1992). Concept development and prolem solving using graphing calculators in the middle school. In James T. Fey (Ed.) *Calculators in mathematics education: 1992 yearbook of the National Council of Teachers of Mathematics* (pp. 65-78). Reston, VA: NCTM.

THE EFFECT OF THE USE OF NUMBER LINES REPRESENTATIONS ON STUDENT UNDERSTANDING OF BASIC FUNCTION CONCEPTS

James R. Olsen, Western Illinois University

Researchers and educators are calling for increased use of technology and attention to function concepts in school mathematics. Students often have considerable difficulty gleaning pointwise and global information from Cartesian (R^2) representations of functions, whether they are hand- or machine-produced. Described here is an interactive computer-based learning environment (the Function Explorer) which provides dynamic, linked representations of functions. Table, parallel number lines, and perpendicular number lines representations dynamically display ordered pairs of the function. A randomized comparative experiment is described which was performed to test the effectiveness of the number lines representations for enhancing student understanding of basic function concepts.

The function concept is unifying and central to the understanding of mathematics and its applications. Research from the last two decades has detailed numerous and deep misconceptions and difficulties students have with the function concept (see Leinhardt, Zaslavsky, & Stein, 1990). This study was primarily concerned with the difficulties students have interpreting graphs. While students can often produce graphs from equations and tables, the reverse process of obtaining pointwise, and especially global, information from graphs is difficult for students (see Kieran, 1992). Educators and researchers see the use of computers as beneficial for the teaching of functional thinking (see Dubinsky & Tall, 1991).

Researchers raise two concerns which hinder the use of technology for teaching functional thinking. First, Kaput (1992) warns that the "retrofitting" of general application (expert) tools as tools for learning mathematics is not easy and may not be effective. Goldenberg (1991) and other researchers (see Moschkovich, Schoenfeld, & Arcavi, 1993) have found that students are not always gleaning the correct information, or insights, from the ("perfect") graphs presented by graphing software and graphing calculators. In fact, novices may pick up on the wrong aspects of what they see—what they see being effected by what they know. Secondly, the (static) Cartesian graph itself is particularly difficult for novices to interpret. Goldenberg, Lewis, and O'Keefe (1992) see "that the act of representing functions graphically [in R^2] has as much potential to produce confusion as enlightenment" (p. 240).

This researcher has developed an interactive computer program incorporating perpendicular number lines *and* parallel number lines to represent functions. The design of the representations, user-interface (the program is titled the *Function Explorer*), and learning activities used in this study are based on psychological, mathematical, and historical considerations. Piaget, Grize, Szeminska, and Bang (1977) found that the root of the function concept, *pairing*, is present in the minds of children at the preoperatory level. This elementary form of cognitive structuring allows the child to conceive of an action of starting with an object and determining a corresponding object (for example: child, mother; sheep, shepherd).

The primary purpose of the *Function Explorer* is, upon input by the user, to display a single ordered pair of the function. The *Function Explorer* has three representations where input of the value of the independent variable is possible: a table, parallel number lines, and perpendicular number lines. Input in the number lines representations is accomplished by moving the mouse pointer to the location on the input ("x") number line. Input via the table is accomplished by mouse-clicking the table and typing the value desired. For each new value of the independent variable, the corresponding function value is (instantaneously) updated. The input-output pair appears simultaneously in all three representations. In the case of the parallel number lines, the output ("y") is displayed on the output number line. The parallel number lines elaborate the ordered pair notion, providing the student a bridge between the table—which provides a *pair of numbers*—and the Cartesian graph (perpendicular number lines)—which provides a *single geometric point*. The parallel number lines are situated horizontally because this is the orientation students are familiar with in middle school and algebra textbooks.

The representations of the *Function Explorer* are dynamic. Interest in properties of moving bodies helped spurn development of the calculus in the 17th-century. Over time however, the *direction of motion* of a moving body at a point has evolved to be the *tangent to the curve* (Kline, 1972). Using the perpendicular number lines representation the student may witness and consider a point *moving* in the plane. (Computer speed of at least 33MHz is required for the three linked representations to update "simultaneously.")

Discrete points may be graphed and entered in the table permanently, by clicking the mouse button. However, the program does not produce a complete graph. The *Function Explorer* is designed to be a learning environment, by providing a scientific instrument for investigating functions—which is more like a microscope and less like a VCR. The program is intended to *aid understanding* and *not* designed to *produce human products* (i.e., complete _^{2} graphs).

Parallel number line representation of functions has been investigated by Friedlander, Rosen, and Bruckheimer (1982) and Arcavi and Nachmias (1993). In both these cases input and output values on the respective number lines are connected with lines, and multiple pairs are shown at the same time. The purpose of this representation was not to make connections to a table and Cartesian graph and the representation is not dynamic.

A parallel number line representation similar to that of the *Function Explorer* (input value on the top number line can be manipulated dynamically, and connection lines not used) is described in Goldenberg, Lewis, and O'Keefe's (1992) article and in O'Keefe's (1992) doctoral dissertation. O'Keefe found that the dynamic parallel number lines environment (*DynaGraph* program) accurately conveyed important features of functions, including dependence, relative change, and critical values. *DynaGraph* did not display a Cartesian graph.

Methodology and Data Sources[1]

To test the instructional effectiveness of the number lines representations, a randomized comparative experiment was performed. Four eighth-grade classes (n=74), all taught by one teacher, were used for this study (two Pre-Algebra and two Algebra I classes). Students in each class were randomly assigned to two treatment groups. Students in both treatment groups were in the same classroom, at the same time. To vary the treatment, each student was given a login name. When a student logged on to the computer, the correct version of the program was automatically loaded (depending on the assigned treatment group). The first treatment group (PNL group) used the *Function Explorer*, with all its representations displayed (table, parallel number lines, and perpendicular number lines). The second group (No PNL group) used a version of the program, which had the parallel number lines representation hidden (table and perpendicular number lines shown). All students used their version of the program to solve problems on worksheets—the content of which was taken from the curriculum in use. Data was gathered from a pre- and post-questionnaire on functions, an opinion survey, taped student interviews, and audit trails of learner interaction with the software (a design feature which keeps track of what representations the user is accessing, and when). A repeated-measures ANOVA ($p = .05$) was used to analyze the questionnaire results.

On day one of the experiment the pre-questionnaire was administered. On day two, a 5-10 minute introduction/demonstration of the *Function Explorer* was given. For the remainder of days two through six, students solved worksheet problems using the *Function Explorer*. On day seven, the post-questionnaire was administered. The questionnaire had five subtests. Content of the questionnaire was determined before the worksheet content and subjects were selected. All graphs on the questionnaire were Cartesian graphs.

Results

Questionnaire results showed that both groups showed significant improvement on the subtests involving pointwise interpretation and global interpretation of graphs. There was no significant change on subtests on the definition of function, use of letters in function notation to stand for varying quantities, and the relationship between the formula and the graph. There was no significant difference between the treatment groups, and no time-group interaction was found.

The audit trail data reports the number of uses and length of time a student used the three representations (table, parallel number lines, and perpendicular number lines). Since the parallel number lines representation had never been seen by the students previous to the study, it was unclear if the students would even use the representation. All the representations were used from the outset, and the five-day averages showed students in the PNL group used the parallel number lines representation 26% of the time, the perpendicular number lines 17% of the time, and the table 10% of the time. The five-day averages in the No PNL group showed the students using the perpendicular number lines 46% of the time and the table 10% of the time. Students reported in the surveys and interviews that they preferred the number line representations over the table because it was quicker (using the mouse) than typing in numbers. However, they *did use* the table for output. That is, they would mouse-point to an input value, and watch the table for the input-output pair.

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[1] This study is a dissertation study completed by the author at the University of Northern Colorado (Ph.D. in Educational Mathematics). Learning activities, function questionnaire, and additional numeric data from this experiment are available from the author.

Audit trail, interview, and survey results establish that the use of the representations varied significantly between students. Some students who had the parallel number lines, preferred rather to use the perpendicular number lines. On the other hand one student said she rarely used the perpendicular number lines and went so far as to say, "Take it out," because it was confusing. Some students felt the parallel number lines representation was clearer than the perpendicular numbr lines because the values of *x* and *y* were separate and more readable. The students did feel that the program helped them understand functions and graphs, and that the program was easy to use. The students liked the individual control the program offered. Those who compared the program to a graphics calculator, preferred the program because it was "easier to get exact answers," and the user control afforded by the mouse.

Discussion

This study has shown that interpretation of graphs can be taught successfully with number lines representations. One of the things that makes the global interpretation tasks difficult for students is that they must think about, and report sets of numbers and intervals of numbers, rather than one discrete number. The ability to use the mouse pointer to move back and forth within an interval to select (and input) values, helped students think about intervals of numbers.

No improvement was shown on the questionnaire subtests on the definition of function, use of letters in function notation to stand for varying quantities, and the relationship between the formula and the graph. First, these concepts were not taught during the study. Secondly, these notions are abstractions, and "just being in the presence" does not cause the student to construct these generalizations.

Future research is needed to better determine the appropriate use of multiple representations of functions. Many of the students in this study could answer the worksheet questions in the presence of the dynamic representations, but could not answer the analog questions reading a (static) Cartesian graph. Follow-up "off-line" learning activities may enhance the on-line activities. These representations need to be tested at other levels and with other concepts. These dynamic representations may be used to teach concepts from integer arithmetic to the calculus concepts of change.

The parallel number lines representation may give students who are not yet able to interpret Cartesian graphs access to function concepts. This may become increasingly important as school algebra is reorganized around the concepts of functions, families of functions, and mathematical modeling (see Heid, 1995). The *Function Explorer* is not designed to replace other graphing technologies, but is more as a forerunner, preparing students to more effectively interpret machine-produced graphs.

References

Arcavi, A., & Nachmias, R. (1993). What is your family name, Ms. function? Exploring families of functions with a non-conventional representation. *Journal of Computers in Mathematics and Science Teaching, 12*(3/4), 315-329.

Dubinsky, E., & Tall, D. (1991). Advanced mathematical thinking and the computer. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 231-248). Dordrecht, The Netherlands: Kluwer Academic.

Friedlander, A., Rosen, G., & Bruckheimer, M. (1982). Parallel Coordinate Axes. *Mathematics Teaching, 99,* 44-48.

Goldenberg, E. P. (1991). The difference between graphing software and educational graphing software. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 77-86). Washington, DC: MAA.

Goldenberg, P., Lewis, P., & O'Keefe, J. (1992). Dynamic representation and the development of a process understanding of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: aspects of epistemology and pedagogy* (pp. 235-260). Washington, DC: MAA.

Heid, M. K. (Ed.). (1995). *Algebra in a technological world.* Reston, VA: NCTM.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kline, M. (1972). *Mathematical thought from ancient to modern times.* New York: Oxford University Press.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions,* (pp. 69-100). Hillsdale, NJ: Lawerence Erlbaum.

O'Keefe, J. J. (1992). Using dynamic representation to enhance students' understanding of the concept of function. *Dissertation Abstracts International, 53,* 9301682.

Piaget, J., Grize, J., Szeminska, A., & Bang, V. (1977). *Epistemology and psychology of functions.* Dordrecht Holland: D. Reidel.

CONSTRUCTION OF CONCEPTS OF TRANSFORMATIONS USING TECHNOLOGY IN A LANGUAGE DIVERSE CLASSROOM

Juli K. Dixon, University of Nevada

The effects of instructional environment, English proficiency, and spatial visualization on eighth-grade students' ($N = 241$) construction of the rigid motion transformation concepts of reflection and rotation were investigated. Use of *The Geometer's Sketchpad* in a computer lab was compared to traditional classroom textbook-based instruction. Also investigated were the effects of instructional environment and English proficiency on students' two- and three-dimensional visualization.

The constructivist view of teaching and learning mathematics as well as Vygotsky's zone of proximal development were influential in the formation of a dynamic instructional environment based on use of *The Geometer's Sketchpad* in a computer lab. The constructivist view of teaching and learning mathematics combined with the zone of proximal development requires that the instructional environment of the learner be such that collaborative inquiry is emphasized and accommodated. Students working in pairs at computers to construct the properties of reflection and rotation using computer software that allows for student conjectures and provides visual feedback is consistent with these requirements. The choice of the computer program and design of associated student activities were based on Cummins' (1984) theory of context embedded versus context reduced instruction. *The Geometer's Sketchpad* provides in text clues through the dynamic, visual presentation of geometric properties. Together with activities, *The Geometer's Sketchpad* was used to aid in students' acquisition of the concepts and vocabulary involved with the rigid motion transformation concepts of reflection and rotation in a context embedded rather than a context reduced instructional environment.

A three factor, nonequivalent control-group design was used for the study. Validity and reliability were addressed during a pilot study. English proficiency was measured by performance on the Language Assessment Battery; visualization level (high, medium, or low) was assigned based on scores on the Paper Folding Test. After controlling for initial differences using the Card Rotation Test, it was concluded by analysis of covariance that students experiencing the dynamic environment significantly outperformed students experiencing a traditional environment on researcher made and expert content-validated measures of the concepts of reflection and rotation as well as on a measure of two-dimensional visualization (the Card Rotation Test). The students' environment did not significantly affect their performance on a measure of three-dimensional visualization (the Paper Folding Test). English proficiency was not a significant predictor for any of the dependent variables.

Reference

Cummins, J. (1984). *Bilingualism and special education: Issues in assessment and pedagogy.* San Diego, CA: College-Hill Press.

USING CLASSROOM DISCOURSE TO FACILITATE MATHEMATICS UNDERSTANDING

Ramakrishnan Menon, Nanyang Technological University

This study was conducted in a Grade 3 (n = 9, 3 girls & 6 boys) remedial class to see whether classroom discourse through discussion, student-constructed questions (e.g. Menon, 1994; Silverman, Winograd, & Strohauer, 1992) and calculator-based pattern-searching could help students' motivation for, and understanding of, mathematics. The teacher, with 6 years teaching experience, taught the class after she had discussed some approaches with me.

By the end of the study, all students generated, and solved, meaningful one-step word problems and had improved multiplication and word problem skills. They found interesting patterns for certain multiples, realized the connection between multiplication and repeated addition, mastered their multiplication tables and, after some time, even refused to use the calculator as it was much faster to compute mentally. They also stated they liked and understood mathematics better.

One interesting aspect of the study was that students' improved understanding in class seldom transferred to test situations. They reverted to old habits (e.g., computing blindly, without checking on the reasonableness of the answer) when confronted with a test paper, but when they were reminded to look for similarities between the test questions and the exercises they had done in class, they did much better. Possibly, the test questions were in a context different from that of the class—very much like students who in classroom situations could not do computations which were structurally the same as those they could successfully do in a real-life situation, such as shopping (e.g. Stigler & Baranes, 1988).

While the teacher in this study had a number of advantages compared to a teacher in a normal class (e.g., a small, almost homogeneous group of underachievers, and a two-hour block in which to try out activities and exercises), some of the approaches might be successfully implemented in a "regular" classroom.

References

Menon, R. (1994). The role of context in student-constructed questions. *Focus on Learning Problems in Mathematics, 17*(1), 26-34.

Silverman, F.L., Winograd, K., & Strohauer, D. (1992). Student-generated story problems. *Arithmetic Teacher, 39*(8), 6-12.

Stigler, J.W., & Baranes, R. (1988). Culture and mathematics learning. In E.Z. Rothkopf (Ed.), *Review of Research in Education*, 15, (pp. 253-306). Washington, DC: American Educational Research Association.

THE GOLDEN RATIO—MATHEMATICS OF BEAUTY: A LEARNING SOFTWARE IN HYPERCARD

Kyoko Suzuki and Eric G. Tobiason
University of Illinois at Urbana-Champaign

This presentation intends to demonstrate a software written in HyperCard 2.1 for learning "the Golden Ratio" for secondary school children aged 12 - 18. The objectives of this study are:

1) creating learning materials of mathematical concepts with fun and easy access for secondary school children,

2) providing interactive activities using computers, and

3) providing supplemental learning materials of general mathematical topics bridging mathematics and daily life.

The Golden Ratio is a well known mathematical topic, which relates to various fields in the real-world phenomena. There are some books for adults describing the topic, but few books for secondary school children offering fun and easy access. Easily accessible environment is an important feature of learning materials for those who express mathematics anxiety. This is a big reason to choose cartoon characters as "buttons" in HyperCard when developing the software. Cartoon characters also can motivate children to work on the learning material.

What are big differences between computers and books? First, this software can provide non-linear sequences of information so that users can select any order of information retrieval. There are two panels which allow users to choose a topic to study: general panel and math panel (see Figure 1). Second, computer software can provide animation which entertains users. This software is freeware.

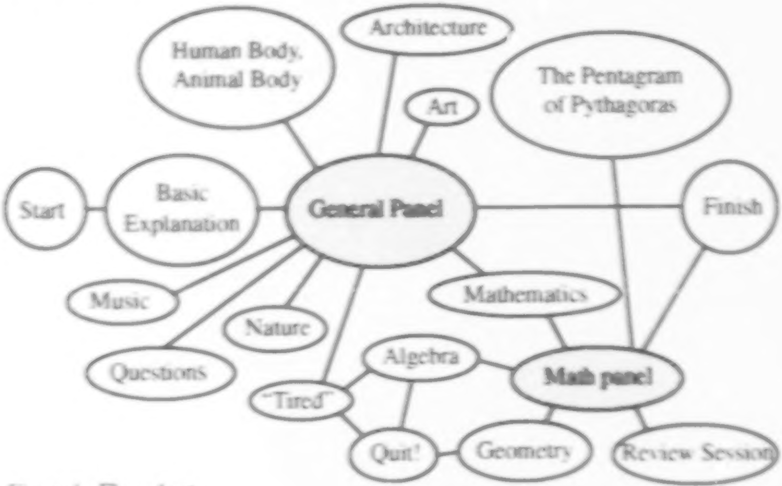
Architecture
Human Body, Animal Body
The Pentagram of Pythagoras
Art
Start
Basic Explanation
General Panel
Finish
Music
Mathematics
Nature
Questions
Algebra
Math panel
"Tired"
Quit!
Geometry
Review Session

Figure 1. Flowchart

STUDENTS' CONCEPTIONS OF LIMITS AND LIMIT EXPLORER.

William J. Hardin, Syracuse University

This poster session will describe a research project to investigate the effects of differing pedagogical approaches to teaching the mathematical concept of limit on students' conception of limits and on procedural ability. The session will also include a demonstration of *Limit Explorer,* a software package that is designed to provide interactive animations of the standard pictures related to limits. Conceptual understanding is viewed in light of Tall and Vinner's (1981) notion of concept image. Procedural ability is the ability to apply procedures and does not always require conceptual understanding.

Studies have been done on students' conceptions of limits (Davis & Vinner, 1986; Williams, 1991), but these studies did not compare differing pedagogical approaches. These studies found that students could have incorrect conceptions of limit and yet still solve traditional limit problems. Other studies (Heid 1988, Palmiter 1992), compared the effects of using computers to teach calculus with a traditional pencil-and-paper approach, but did not specifically address limits. Thus, there is a lack of comparative studies on students' conceptions of limits.

This study will add to the existing body of knowledge of students' conceptions of limits and the use of calculators and computers to teach calculus by addressing the lack of comparative studies on students' conceptions of limits. This study may also yield valuable information for those educators who wish to improve students' conceptual understanding of limits. It is hoped that I can refine Williams' limit classification model to include graphical depictions of limit as well as Movshovitz-Hadar, Inbar and Zaslavsky's (1987) error classification model. Finally, the *Limit Explorer* software, developed specifically for this study, will be a new tool for instruction of limits as well as other calculus concepts.

References

Davis, R., B. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior,* 5, 281-303.

Heid, M. K. (1988). Re sequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education,* 19, 3-25.

Movshovitz-Hadar, N., Inbar, S., & Zaslavsky, O. (1987). An empirical classification model for errors in high school mathematics. *Journal for Research in Mathematics Education,* 18, 3-14.

Palmiter, Jeannette (1992). Effects of computer algebra systems on concept acquisition in calculus. *Journal for Research in Mathematics Education,* 22, 151-156.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics,* 12, 151-169.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education,* 22, 219-236.

TEACHING MATHEMATICS IN A COMPUTER-INTENSIVE ENVIRONMENT:[1] ONE TEACHER'S MATHEMATICAL UNDERSTANDINGS AND USE OF TOOLS

Glen Blume, The Pennsylvania State University

This case study[2] examined a teacher's knowledge of mathematics and use of computing tools when solving mathematics problems in a computer-intensive environment and their relation to his implementation of a reformulated, computer-intensive first-year algebra curriculum (Fey et al., 1991).

Neal, an experienced mathematics teacher, was one of 30 teachers who participated in a four-week summer institute focusing on computer-intensive mathematics, understanding students' understandings, and teaching and learning issues related to implementing computer-intensive mathematics curricula; implemented a year-long computer-intensive algebra course (Fey, et al., 1991); and participated in a one-week institute the following summer.

The author and members of the research team[3] administered task-based interviews to Neal prior to and after the four-week institute and during the institute the following summer; administered three pedagogy scenario interviews, and conducted eight observations of Neal's classes (whole group and computer-lab interactions) that included pre- and post-observation conferences with Neal. Verbatim transcripts of the interviews, classroom observations, and conferences were prepared and analyzed.

Neal demonstrated an ability to explore mathematics in a computer-intensive environment and to explore an unfamiliar tool. His exploration of the tool overshadowed his exploration of mathematics, however, and while Neal stated that he encouraged students to experiment, that experimentation was largely confined to learning capabilities of the tool rather than exploring the mathematics inherent in the problem situations. Neal's students used a computer algebra system but he is hesitant to implement computer symbolic manipulation prior to by-hand manipulation. Neal's emphasis on curve fitting reflects a focus on the curve fitter and the process of curve fitting rather than on the fitted function and its characteristics and questions related to issues of mathematical modeling.

Reference

Fey, J. T., Heid, M. K., Good, R., Sheets, C., Blume, G., & Zbiek, R. M. (1991). *Computer-Intensive Algebra*. College Park, MD: The University of Maryland and The Pennsylvania State University.

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[1] The computer-intensive environment in this study included student access, primarily in pairs, to Macintosh computers with the computer algebra system Calculus *T/L II*, distributed by Brooks/Cole Publishing Co., Pacific Grove, CA.

[2] This material is based on work supported by the National Science Foundation under Grant No. TPE 91-55313. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

[3] M. Kathleen Heid, Penn State University; Rose Mary Zbiek, The University of Iowa; Sarah Eaton, Fall Mountain High School; Pete Johnson, Colby-Sawyer College.

Visualization

VISUALIZATION AND STUDENTS' PERFORMANCE IN TECHNOLOGY-BASED CALCULUS

Enrique Galindo, Indiana University

The relationship between college students' preferred mode of processing mathematical information—visual or nonvisual—and their performance in calculus classes with and without technology was investigated. Students elected one of three different versions of an introductory differential calculus course: using graphing calculators, using the computer algebra system Mathematica®, or using no technology. A total of 139 students participated in the research. Presmeg's Mathematical Processing Instrument (MPI) was used to determine students' visual processing preference. The interactions of students of different visual processing preferences with the software Mathematica were also investigated using task-based interviews. Results from the sections using graphing calculators suggest that appropriate uses of technology may equally benefit students of different cognitive styles.

> Lie was an intuitionalist; this might have been doubted in reading his books, [however] no one could doubt it after talking with him; you saw at once that he thought in pictures. Madame Kovalevski was a logician. Among our students we notice the same differences; some prefer to treat their problems 'by analysis,' others 'by geometry.' *(Poincaré, 1900/1907, p. 17)*

In the midst of calculus reform in the U.S.A. many of the new approaches to the calculus make use of computers or calculators with graphing capabilities. However, individuals vary in their preferences for using visual methods of solution when solving mathematics problems. Among mathematicians, both visualizers and nonvisualizers have made important contributions to the progress of mathematics, as illustrated by the quote from Poincaré (1900/1907). Furthermore, there seems to be agreement on the importance of mental imagery in thinking and in the act of creation (Hadamard, 1945; Koestler, 1967; Shepard, 1978). Among students, according to Krutetskii (1968/1976), the ability to visualize abstract mathematical relationships and the ability for spatial geometric concepts do not determine the extent of mathematical giftedness but only its type. However, several research studies suggest that there is a negative association between students' degree of visuality and their performance in school mathematics (Lean & Clements, 1981; Presmeg, 1986). This previous research has been conducted in classes that use no technology, and the differences in mathematical achievement favoring students who are nonvisualizers have been observed both at the senior high school level, and at the freshman college level. Nevertheless, the author's research on calculus courses suggests that technology, and software with multiple-representation capabilities, can be used to promote conceptual understanding and equally favor both visualizers and students who are nonvisualizers (Galindo, in press, 1994). Some results from this research will be discussed in this paper. Students' interactions with the software Mathematica will also be examined.

Importance and Status of Visualization in Mathematics Education

With an increased emphasis on the study of patterns (National Research Council, 1989; Steen, 1988), visualization is acquiring an important role in mathematical endeavor. Computer-generated graphs are enabling the mathematician to visualize the content of abstract theorems (Pool, 1992), and new conjectures are suggested by the eye (Mandelbrot, 1983). The current status of visualization in mathematics is best summed up by Steen (1990) as follows:

> Thanks to computer graphics, much of the mathematician's search for patterns is now guided by what one can really see with the eye, whereas nineteenth-century mathematical giants like Gauss and Poincaré had to depend more on seeing with their mind's eye.... For centuries the mind has dominated the eye in the hierarchy of mathematical practice; today the balance is being restored as mathematicians find new ways to see patterns, both with the eye and with the mind. *(p.2)*

It seems thus natural to think that if we want students to do mathematics the same way as mathematicians do it, computers and visualization should also have an important role in mathematics education. However, it seems that visualization has had for a long time a low status in school mathematics.

Although it is generally accepted that visual representations offer a powerful introduction to the complex abstractions of mathematics (Bishop, 1989), and that for some subjects such as geometry it is believed that visualization is a necessary tool in concept formation (Hershkowitz, 1989), there are a number of students' difficulties with visualization that have been reported in the mathematics education literature (Clement, 1985, 1989; Goldenberg, 1988, 1991; Yerushalmi & Chazan, 1990). Not only do students have difficulty visualizing concepts and interpreting graphs, but instances of students' reluctance to use visual methods have been reported (Balomenos, Ferrini-Mundy & Dick, 1988; Eisenberg & Dreyfus, 1991; Vinner, 1989). Dreyfus (1991) points out that teachers and educators contribute to the low status of visualization in school mathematics:

> The message is that visualization may be a useful and efficient learning aid for many topics in high school and college mathematics, but nevertheless an aid, a crutch, a step, sometimes a necessary and important step, but only a step on the way to the real mathematics. *(p. 34)*

One of the possible consequences of the low status of visual methods in school mathematics is the differential performance in mathematics courses of visualizers and nonvisualizers. Lean and Clements (1981) found that first year engineering students who preferred to process mathematical information by verbal-logical means tended to outperform more visual students on mathematical tests. Presmeg (1986) found that visualizers are seriously under-represented among high mathematical achievers at the senior high school level, and she provides some explanations for this phenomena. The question then arises of whether these differences in mathematical performance in favor of the nonvisualizer student can also be observed in classes that use computers and graphing calculators, that is, technology with multiple-representation capabilities.

Visualization and Mathematical Performance in Technology-Based Calculus Classes

Participants in this study were enrolled in the first course of a three–quarter calculus and analytic geometry sequence for science and engineering majors. The purpose of the course was to provide students with a solid foundation in one–variable differential calculus. Students elected one of three different versions of the course. One approach used graphing calculators and the textbook *Calculus a Graphing Approach* (Finney, Thomas, Demana, & Waits, 1993); another used the computer algebra system Mathematica® and the textbook *Calculus and Mathematica* (Brown, Davis, Porta, & Uhl, 1992); and the last used no technology explicitly and the textbook *Calculus* (Finney & Thomas, 1991). Eighteen out of twenty–six sections of the calculus course participated in the study, with approximately 25 students enrolled in each section. The eight sections using graphing calculators and the eight sections using no technology used the lecture–recitation format. Performance in these classes was evaluated using three midterms, several quizzes, and a final exam. The two sections using Mathematica® had five 48–minute sessions every week in the computer laboratory. The students in these sections were evaluated considering individual and group homework, literacy sheets, participation, 6 in–class quizzes, one midterm and one final exam. The 18-item Mathematical Processing Instrument (Presmeg, 1985, 1986) was used to determine students' visual processing preference.

From the three calculus approaches, a total of 139 students participated in the research. Out of 36 possible points, MPI scores varied from 6 to 29, with a median of 17. It was found that the MPI scores were normally distributed and the cognitive styles of visualizers, nonvisualizers, and students of the harmonic type, were found among students in every calculus approach. One research question investigated the relationship between college students' preferred mode of processing mathematical information and their performance in calculus classes with and without technology. It was found that students who are nonvisualizers obtained significantly better scores than visualizers in the calculus sections using no technology, and in the calculus sections using the software Mathematica. On the other hand, there were no significant differences in the calculus scores obtained by visualizers and nonvisualizers in the sections using graphing calculators. These results and their implications for the use of technology in mathematics education are discussed elsewhere (Galindo, in press).

Students' Interactions With The Software

Another research question investigated the interactions of students of different visual processing preferences with the software Mathematica. Task–based interviews of students of each cognitive style from the sections using the computers were conducted. Students to be interviewed were selected using purposeful sampling, in particular two strategies: theory based, or operational construct, sampling and maximum variation sampling. The theoretical construct used for the selection process was *mathematical visuality*. Two students of each cognitive style were selected for the interviews, thus a total of 6 students were interviewed.

There were two goals for the task–based interviews. The first goal was to gather more information about the student's preferred mode of solving mathematics problems—visual or nonvisual. The MPI was used early in the course to this end, but it was desired to investigate if students' work in the calculus would give evidence about their visual orientation that was in agreement with the MPI results. A second goal of the task–based interviews was to look at the ways in which students of different degrees of visuality use the software when solving mathematics problems. Mathematica software has different types of tools—tools to graph functions, tools to solve equations symbolically, and tools to do numerical calculations. The course puts great emphasis on graphical methods for solving problems and encourages visual thinking. It was desired to investigate if students would use the software in ways that reflect their visual orientation, or if they would mostly rely on the graphical methods emphasized in class.

The problems solved during the interviews were analyzed and scored using the same point system used in the MPI. A problem was given 2 points if it was solved using visual methods. A problem solved by numerical or symbolic methods was assigned 0 points, and problems solved by a combination of methods or problems where it was hard to tell the method used, were given 1 point. Students' work during the task–based interviews provided further evidence about their visual orientation. After scoring students' solutions to the interview problems using the MPI rubric described above, it was found that the methods of solution used by the students during the task–based interviews reflect the visual orientation indicated by the MPI. Students who obtained a high MPI score tended to use graphic methods of solution and preferred to use the plotting capabilities of the software, whereas students with low MPI scores used numeric and symbolic methods of solution and the corresponding software commands.

As for the interaction of the students with the software, it was found that the tools used by the students did correspond to their visual preferences, with visually oriented students preferring to use graphical methods to solve problems, and nonvisualizers preferring to use commands such as Solve, or N[Solve]. If students tend to use software tools that correspond to their visual preference, why is that nonvisualizers seem to outperform visualizers in the Mathematica sections? Some possible explanations are examined in the next section.

Conclusions

The data obtained from both the sections using Mathematica and the sections using no technology provided evidence of a negative relationship between MPI score and total calculus score for the students taking these approaches. These results show that differences in mathematics final scores between visualizers and nonvisualizers prevail at the college level and that they are not easily removed. The fact that no significant differences were found between calculus performance and degree of visuality in the sections using graphing calculators, suggests that appropriate uses of technology may equally benefit students of different cognitive styles. Mathematics education research should seek to investigate the appropriate conditions for this to take place.

It was also found that the students' visual orientation observed during task-based interviews and the software tools they use correspond to the degree of visuality indicated by the MPI. Furthermore, the negative association between course scores and degree of visuality found in the Mathematica sections seems to be the result of the long symbolic sentences that students must enter for Mathematica to plot a graph. The visualizers in these sections needed to go through an analytic expression in order to take advantage of the graphs. Thus, educators and software designers should be aware of the restrictions that Computer Algebra Systems may impose on students of different cognitive styles, as well as of their effects on students' performance in mathematics.

Another important variable that must be considered is the role of the teacher. The present study was repeated for the Mathematica sections during the following quarter, when an experienced Ph.D. in mathematics taught the sections using this calculus approach; no significant differences were found this time in the calculus performance of students of different cognitive style. Furthermore, the interactions between the teacher's cognitive style and the student's visual preference, as well as their effect on student's performance, need to be investigated in computer-based environments that encourage visual thinking.

Presmeg (1985), identified 17 classroom aspects which are reported in the literature to be facilitative of formation and use of visual imagery in school mathematics. Among such aspects we find: conscious teacher attempts to generate imagery in pupils by the use of instruction to form images, and the creation of dynamic situations to think in moving pictures, (b) teacher formation and use of their own imagery, and (c) a pictorial presentation of the topics. The classroom aspects conducive to the students' formation of mental imagery in courses that make use of technology need to be investigated.

References

Balomenos, R., Ferrini-Mundy, J., & Dick, T. (1988). Geometry for calculus readiness. In M. Lindquist & A. Schulte (Eds.), *Learning and teaching geometry, K-12* (pp. 185-209). Reston, VA: National Council of Teachers of Mathematics.

Bishop, A. J. (1989). Review of research on visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1), 7-16.

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Brown, D. P., Davis, W., Porta, H., & Uhl, J. J. (1992). *Calculus & Mathematica®* (preliminary ed.). Reading, MA: Addison–Wesley.

Clement, J. (1985). Misconceptions in graphing. In L. Streefland (Ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education,1*, 369–375.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics, 11*(2), 77–87.

Dreyfus, T. (1991). On the status of visual reasoning in mathematics and mathematics education. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education, 1*, 33–48.

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (MAA Notes 19, pp. 25–37). Washington, DC: The Mathematical Association of America.

Finney, R. L., & Thomas, G. B., Jr. (1991). *Calculus* (rev. printing). Reading, MA: Addison–Wesley.

Finney, R. L., Thomas, G. B., Jr., Demana, F. D., & Waits, B.K. (1993). *Calculus A Graphing Approach* (preliminary ed.). Reading, MA: Addison–Wesley.

Galindo, E. (in press). *Visualization and students' performance in mathematics: Implications for the use of technology in mathematics education.* Proceedings of the Seventh Annual International Conference on Technology in Collegiate Mathematics. Reading, MA: Addison-Wesley.

Galindo, E. (1994). *Visualization in the calculus class: Relationship between cognitive style, gender, and use of technology.* Doctoral dissertation. The Ohio State University.

Goldenberg, E. P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behavior, 7*, 135—173.

Goldenberg, P. (1991). The difference between graphing software and educational graphing software. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (MAA Notes 19, pp. 77-86). Washington, DC: The Mathematical Association of America.

Hadamard, J. (1945). *An essay on the psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Hershkowitz, R. (1989). Visualization in geometry: two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61–76.

Koestler, A. (1967). *The act of creation.* New York: Macmillan.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children.* (J. Teller, Trans.). Chicago, IL: University of Chicago Press. (Original work published 1968)

Lean, G., & Clements, M. A. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics, 12*, 267–299.

Mandelbrot, B. (1983). *The fractal geometry of nature.* San Francisco, CA: Freeman.

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press.

Poincaré, H. (1907). *The value of science* (G. B. Halsted, Trans.). New York: The Science Press. (Original work published 1900)

Pool, R. (1992). Mathematicians join the computer revolution. *Science, 256*, 52–53.

Presmeg, N. C. (1985). *The role of visually mediated processes in high school mathematics: A classroom investigation.* Unpublished doctoral dissertation, University of Cambridge, England.

Presmeg, N. C. (1986). Visualisation in high school mathematics. *For the Learning of Mathematics, 6*(3), 42–46.

Shepard, R. N. (1978). Externalization of mental images and the act of creation. In B. S. Randhawa & W. E. Coffman (Eds.), *Visual learning, thinking, and communication* (pp. 133–189). New York: Academic Press.

Steen, L. A. (1988). The science of patterns. *Science, 240*, 611–616.

Steen, L. A. (1990). Pattern. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 1–10). Washington, DC: National Academy Press.

Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on Learning Problems in Mathematics, 11*(2), 149–156.

Yerushalmi, M., & Chazan, D. (1990). Overcoming visual obstacles with the aid of the Supposer. *Educational Studies in Mathematics, 21*(3), 199–219.

EXPLORING STUDENTS' RESPONSES TO VISUAL PROBLEMS AT THE HIGH SCHOOL CALCULUS LEVEL

Preety Nigam, Syracuse University

In recent years, there has been a spate of research articles aimed at "reforming" calculus at the undergraduate level (Confrey, 1993; Heid, 1988). Much of this research has focused primarily on two areas: visualization and the use of technology. It is perhaps a happy coincidence that the two areas have become significant simultaneously, as much of the technology available for teaching today, of which the graphing calculator is an example, offers features which allow students to draw and analyze graphs easily. This study focuses its attention on high school students in order to see whether visualization plays a role in their understanding of concepts in calculus and whether they can use visual models to solve problems. Further, it intends to explore their attitude towards visual-based mathematical problems and the use they make of the graphing calculator in solving such problems.

The National Council for Teachers of Mathematics' (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (1989) emphasizes the fact that understanding in mathematics is making connections between ideas, facts and procedures, where the definition of "ideas" includes external representations of a concept. Further, it advocates the use of multiple representations while teaching. Visualization is one such form of representation which could be particularly useful since it can provide a concrete aspect of what may be some very abstract ideas.

For this study, volunteers were sought from two sections of an introductory calculus course in a suburban high school. The students were randomly assigned to small groups of one, two or three students. The researcher then conducted task-based interviews with the students, where the tasks consisted of a series of mathematical problems with strong visual content. The students were encouraged to discuss the problems and explain in detail the approach they were using to investigate the problem and arrive at a solution.

While the data from this study is in the process of analysis, it is expected that the results from this study will add to the body of research aimed at revising the curriculum in calculus in a significant way. By documenting students' responses towards visual problems, it will help to illuminate the visual models that students construct in order to solve mathematical problems and the possible difficulties they may encounter while constructing them. This information will prove useful to educators who wish to enhance their students' conceptual understanding. In addition, the study will also aid in studying the usefulness of one of the features of the graphing calculator namely, graphing.

Whole Numbers

OPERATIONAL SENSE IN FIRST GRADE ADDITION

David Slavit, Washington State University

This paper outlines a theoretical perspective for studying student understandings of the concept of addition. The notion of operational sense is defined as a way to describe the notion of addition as a mathematical object, paving the way for an application of the theory of reification at this level. Previous frameworks relative to problem solving are also incorporated. The report of a year-long investigation in a first-grade classroom is then provided. It was found that understandings of specific aspects of operational sense were beneficial to successful problem solving strategies on part-unknown action tasks. These understandings were also beneficial to the ability to transfer knowledge of addition to a finite group setting (clock arithmetic). Hence, a connection was found between specific kinds of knowledge of arithmetic and the students ability to model the actions of a problem. Limitations of the framework and study are also discussed.

In this study, I examine if aspects of the theory of reification (Sfard and Linchevski, 1994) can be used to operationalize student understandings of early arithmetic. Reification involves the transitioning of understandings in line with actions and processes to more permanent understandings in line with mathematical objects. Gray and Tall (1994) provide several nice examples of where reification may occur in a student's mathematical career, including the reification of the counting process into object-oriented conceptions of number.

Operational Sense

Because I was interested in student understanding, and not just problem solving behaviors, I attempted to lay out a theoretical basis that would be 1) consistent with the theory of reification, 2) useful in exploring student understandings of addition, and 3) useful in relating these understandings to existing theories of problem solving (Riley, Greeno, and Heller, 1983; Briars and Larkin, 1984; Carpenter, 1985). I defined *operational sense* in an effort to satisfy these requirements. A base definition of operational sense could involve the ability to use the operation on at least one set of mathematical objects (such as the ability to add positive integers). But this is clearly a minimal conceptualization. I maintain that operational sense which promotes deep understandings of the operation involves various kinds of flexible conceptions which can be interrelated by the learner. From this perspective, operational sense could involve (additive components in parentheses):

1. *A conceptualization of the base components of the process.* (This involves an understanding of the decomposition of addition tasks into uniform counting or, perhaps later, a devised strategy such as: 7+8=15 since 8=3+5, so 7+8=7+3+5=10+5=15).

2. *Familiarity with properties which the operation is able to possess* (commutativity, invertibilty, associativity, existence of an identity).

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3. *Relationships with other operations.* (In addition to the relationships the operation of addition has with its inverse (subtraction), the distributive property in any field provides a means of connecting two operations, such as addition and multiplication. Further, multiplication is often initially understood as repeated addition).

4. *An awareness of the various symbol systems associated with the operation* (digits, +, >, etc.).

5. *Familiarity with operational contexts.* (The use of action, comparison, and part-whole situations are familiar contexts for the operation of addition).

6a. *Ability to use the operation on abstract objects.* (This involves the use of addition without a reliance on concrete quantities. Here, the process of addition is being used to act on clearly understood quantities, with no need to rely on the base components of the process, such as counting, or on concrete representations, such as unifix blocks).

6b. *A knowledge of operational facts.*

7. *Ability to relate the use of the operation across different mathematical objects.* (Addition on integers, modular systems of different bases, fractions, decimals, finite groups, variable expressions (symbolic functions), graphs (graphic functions), vectors, and sequences all share a fundamental relationship in regard to the process, even though the mathematical objects are very different. The ability to see connections across these systems can be quite powerful in establishing an advanced operational sense of addition).

8. *Ability to move back and forth between the above conceptions.*

Some of the above dimensions of operational sense have been previously investigated. For example, Cobb and Wheatley (1988) discuss, in considerable detail, the manner in which second grade children use operational sense to solve two-digit addition tasks, as well as how these children were able (or unable) to reify the notion of ten. Much work has been conducted on the effects of situations and contexts on problem solving strategies (e.g., Carpenter, 1985). However, the author knows of no such study which takes the perspective outlined above in collectively investigating first graders' development and use of operational sense. Further, because the evidence at this level suggests that mathematical behaviors will not always fit in to predispositioned frameworks, I expected to modify the above perspective as the data accumulated.

Since the study was conducted at the first grade level, it was obvious that many students would not develop some of the above components of operational sense to any relevant degree. Components 1-4, 6, and 8 were specifically investigated in this study.

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Project Design

Data Collection Schedule

This project investigated the degree to which an operational sense of addition was achieved by students in a first grade classroom in a small, northwestern city. Videotaped interviews were conducted with 17 students a total of 5 times from October through May. Transcripts of some of the interviews were made. Classroom observations and instructor interviews were conducted approximately once per week to provide descriptions of the instruction.

Interview Data

Each of the student interviews contained a series of tasks designed to elicit the students' operational sense of addition. Number size was in line with the current level of instruction. Student responses to the tasks assisted in designing future interviews. The tasks were analyzed for patterns in student problem solving tendencies and the level of operational sense displayed. Each interview contained an addition story task of each of the forms a+b= _ , a+ _ =c, and _ +b=c. These tasks were designed to provide information regarding computational ability and problem solving strategies. The stories were predominantly action type tasks that included names and objects familiar to each specific child. Probing or clarifying questions were consistently given during and following the working of the tasks in order to elicit more detailed information about the solution strategy and the understandings held by the student. Frequency counts were generated in regard to correctness and classification of solution strategy. Additional analysis was conducted on these tasks in regard to the level of operational sense present in the solution strategies. For example, a counting backward strategy implicates the child in regard to an ability to invert the operation, and counting strategies and invented algorithms provide evidence as to the manner in which addition was understood in regard to its base processes. A variety of other tasks, some of which are described in the discussion below, provided more detailed information in regard to the development of various aspects of operational sense.

Results

Instruction

The instructor was an experienced teacher with a Master's Degree in Elementary Education. The students grew quite close to her throughout the year. In my view, the instructor provided an educational context quite conducive to learning mathematics. This included opportunities for the students to make conjectures, discuss solution strategies, experience mathematics in a variety of situations and representations, and to reflect on the mathematics being discussed. The instructor rarely forced an algorithm or universal way of doing mathematics into the discussion, leaving this to her students. It is worth noting that, despite these qualities, I was most impressed with her ability to positively stimulate the development of her students as young boys and girls.

The instructor incorporated classroom activities that I would classify as promoting an operational sense of addition. These included numerous counting, estimation, and pattern activities, general addition tasks that explore different ways of adding two numbers to achieve the same result, skip counting, story problems in a variety of contexts, discussions of zero, and tasks which discuss commutativity. In addition, aspects of operational sense were found in student-initiated comments during classroom discussion.

Development of an Operational Sense of Addition

A discussion of the data addressing the students' development of specific aspects of operational sense will now be given. This will be followed by a more inferential reporting of the students' overall development of an operational sense of addition.

Counting. All students demonstrated at least some degree of knowledge regarding the role of counting as a base process for addition. Counting strategies, facts, or heuristics were used by all of the students on result-unknown tasks (a+b=_) during each interview. This was also the case on part-unknown tasks (a+_=c and _+b=c) with the exception of three students who consistently guessed or repeated the total, making no use of their counting abilities. However, this was clearly a result of their inability to understand the problem context rather than a limitation of their ability to relate the counting process to the additive situation.

Properties. Commutativity, invertibility, and the zero identity were additive properties which the tasks were designed to specifically address. Tasks addressing commutativity involved a result-unknown task stated in its two commutative forms. The manner in which the student solved the second task was observed. An immediate answer to the second question followed by discussion that appropriately addressed the order-irrelevant nature of the numbers involved implicated the student on the use of commutativity in the solution process. The use of a counting strategy on the second task suggested that commutativity was not used. Four students consistently recognized commutative situations and applied these understandings in problem solving situations throughout the year. Ten students had difficulty during the initial interview, but showed uses of commutativity the remaining time. Three students held very unstable notions of commutativity, using this property sporadically throughout the year on these tasks. In addition, four students used count-on strategies on initial-unknown tasks, suggesting that commutativity was used to support the count-on procedure.

The identity property was investigated through the use of tasks of the form (a+_=a): "What's your favorite number (FN)? I'm going to figure out my FN by adding a number to your FN. Is there anyway that I could have the same FN as you? (if needed, a specific number was introduced into the wording of the task)." No student was successful on all four of the Identity tasks from Interviews 1,2,4, and 5. Ten of the students expressed knowledge of the zero identity in a very sporadic manner across the interviews, seemingly forgetting and remembering the zero property from one interview to the next. Two students did not use the property at any time.

A subtraction task immediately following a related addition task (e.g., 3+_=9 and 9-3=_) given during Interviews 3 and 4 were used to analyze the students' ability to invert the operation of addition and relate it to subtraction. No student used invertibility to answer the subtraction task during Interview 3, with most successful strategies involving counting backwards with blocks. Three students made explicit references to the inverse relation among the two related tasks during Interview 4, and three others made comments suggesting some connection was made. Analysis of subtraction strategies on all part-unknown additive tasks were also made, and 6 of the 17 students used subtraction techniques to answer the additive tasks at some point throughout the year.

Relationship to other operations. The ability to relate addition to the operation of subtraction has just been discussed. A multiplication task was given during Interviews 3 (repeated addition) and 5 (array). Five of the 17 students were successful on the repeated addition task, and 9 (including the previous 5) students were successful on the array task. Count-all with blocks and heuristics were the most frequent strategies used.

Knowledge of symbol system. Before the first interview, 10 of the 18 students could count to 100, 2 other students missed one decade, and the remaining five students could not count higher than 30. Only two students were able to correctly write all ten digits, with 5,7, and 9 the most common digits to be written backwards. All but four students could produce the symbol "+". These limitations quickly dissipated and were not a noticeable barrier in student development.

Use on abstract objects. Five students relied on concrete objects when working each task throughout the year. The remaining students showed varying degrees of an ability to perform the process of addition at a more abstract level, including the use of facts and heuristics on additive tasks. However, only one student made consistent use of heuristics throughout the year, with most of the other student uses emerging during Interview 5.

Other tasks provide additional data on this aspect of operational sense. One task during Interview 3 asked to determine which was bigger: 10+3+5 or 13+5. Seven students stated 10+3+5 was larger because "it has more numbers." Of the four students who correctly stated their equivalence, only two explicitly mentioned the equivalence of 10+3 and 13, while the other two found each sum. During this interview the students were also asked if they knew 4+4, and then were immediately asked if they knew 4+5. Of the 12 students who immediately knew 4+4, 2 used a fact to answer 4+5, 5 used a heuristic, and 5 students had to use a counting strategy. The students were also asked to state how many ways they knew to add two numbers together so that they equal nine. During Interview 1, 8 students could think of no number pairs, 6 students stated one number pair (5 of these said 4+5), and two, four, and five number pairs were stated by one student each. But during Interview 4, 6 students gave 8 or more number pairs, and all but one student gave at least two number pairs, including 6 students who also involved subtraction. Only four students used blocks or fingers on this task during Interview 4.

Overall Analysis. It appears that flexible understandings of addition in regard to base processes are central in the development of other aspects of operational sense. However, some students developed notions such as commutativity before such development. It is hypothesized that these students developed very fragile notions of number as an object which they could apply to a commutative setting, irrespective of counting strategies. All of the investigated additive properties were mentioned during instruction, but commutativity was acquired much more readily than identity or invertibility.

Six of the seven students who exhibited correct strategies on initial-unknown tasks at least 80% of the time also displayed solid understandings of commutativity or invertibility (or both). Because these students either performed a counting back strategy or commuted the number sentence and used a counting on strategy, an understanding of these properties was vital in their solutions. These students also made the greatest use of heuristics. The five students who did not show one correct strategy on the part-unknown tasks throughout the year also showed no understanding of invertibility and little or no use of heuristics. In addition, a transfer task given during Interviews 4 and 5 asked the students to determine what time it would be 5 hours after 9:00 A.M. A picture of a clock was given to those students who initially showed difficulty, and the class had just completed a few lessons on telling time. Though not universal, a pattern in the data was found between success on this task, correct solution strategies, and knowledge of additive properties.

Implications

This study attempted to combine students' ability to reify addition with an analysis of problem solving strategies in studying the mathematical behaviors of first graders encountering addition. Problems did arise from the use of this framework. These included measures of specific understandings of operational sense as well as the narrow scope of a first grade curriculum in relation to the broad definition of operational sense. However, the numerous kinds of data and the connections between specific components of understanding and problem solving allowed for some conclusions to be made.

Riley et al. (1983) suggest that there exists specific knowledge about additive structures that affects problem solving behavior. Others (Briars and Larkin, 1984; Carpenter, 1985) suggest that problem solving behaviors more closely relate to the actions and contexts of the problem. It appears that some connection exists between types of knowledge associated with addition and the types of strategies students use. Knowledge of specific properties allowed students to better model the actions present in the tasks. This study provides both a means of combining the above two perspectives and data to support the compatibility of the theories.

References

Briars, D.J. & Larkin, J.H. (1984). An integrated model of skill in solving elementary word problems. *Cognition and Instruction, 1,* 245-296.

Carpenter, T.P. (1985). Learning to add and subtract: An exercise in problem solving. In E.A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17-40). Hillsdale, NJ: Erlbaum.

Cobb, P. & Wheatley, G. (1988). Children's initial understanding of ten. *Focus on Learning Problems in Mathematics, 10(3),* 1-28.

Gray, E.M., & Tall, D.O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25(2),* 116-140.

Riley, M.S., Greeno, J.G., & Heller, J.I. (1983). Development of children's problem solving ability in arithmetic. In H.P. Ginsburg (Ed.), *The development of mathematical thinking* (153-196). New York: Academic Press.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification — The case of algebra. *Educational Studies in Mathematics, 26,* 191-228.

PLACE-VALUE: PROBLEM-SOLVING AND WRITTEN ASSESSMENT USING DIGIT-CORRESPONDENCE TASKS

Sharon Ross, California State University, Chico
Elisa Sunflower, California State University, Chico

The purpose of this study was to examine the effects of researcher-developed lessons on students' understanding of two- and three-digit numeration. Digit-correspondence tasks, often used for individual interview assessment of place value understanding, were adapted to be used as problem-solving tasks. The tasks were presented to three classes, grades 3-5. Students were given ample opportunities, in cooperative groups and as a whole class, to discuss and exchange points of view. In the selected classrooms the social norms established by the teacher encouraged such exchanges. A scoring rubric was developed for a whole-class, digit-correspondence task requiring individual written responses. Only 18% were successful on the preassessment. Of the 58 students initially unsuccessful, 71% were successful after the instructional intervention as measured by a delayed postassessment.

In digit-correspondence tasks, students are asked to construct meaning for the individual digits in a multidigit numeral by matching the digits to quantities in a collection of objects. As measured by such tasks, even in the fourth and fifth grades no more than half the students demonstrate an understanding that the "5" in "25" represents five of the objects and the "2" the remaining 20 objects (Kamii, 1982; Ross, 1986; Ross, 1990).

Constance Kamii has argued that young students' developing understanding of place value is eroded by traditional algorithmic instruction in addition and subtraction, where individual digits are all treated as "ones" (Kamii & Lewis, 1993). Significant gains in conceptual understanding of place value have been demonstrated among first and second grade children participating in full-year studies where students are encouraged to invent their own methods for multidigit addition and subtraction (cf. Fuson & Smith, 1994; Hiebert & Wearne, 1992; Kamii, 1989).

In this study we examined the learning among older students who in prior grades had experienced traditional algorithmic instruction for multidigit addition and subtraction. In earlier work we had individually interviewed numerous students using digit-correspondence tasks, and had often wondered how children would react if they heard the ideas of other students and had an opportunity to react. We designed this study with two questions in mind. What would students learn from their peers if they share their thinking about the meanings of the digits in digit-correspondence tasks? What could be learned about student thinking by examining their individual responses to a written whole-class assessment instead of by using individual interviews.

Method

We worked in a fifth grade classroom of 22 students, a fourth grade classroom of 20 students, and a combination class that included 19 third and 10 fourth graders. The three heterogeneous classrooms were selected because of the teachers' experience and expertise with problem-based instruction. The teachers had successfully established social norms to encourage students to exchange points of view with respect to their mathematical thinking. All three teachers had worked collaboratively with the university-based research team over a period of three years in a grant-supported teacher leadership program. In the program they studied constructivist theories of learning mathematics and collaboratively designed curriculum and practiced instructional strategies to be consistent with those theories.

The instructional intervention was conducted in February and March, when students were accustomed to the classroom routines and to problem-based instruction. We were in each classroom over a period of five or six consecutive days. Written assessment tasks and four 90-minute lessons were presented.

Written Digit Correspondence Assessment Tasks

We designed tasks that could be administered to the whole class, rather than in individual interviews. Each student received a picture of 35 objects. Aided by an overhead projector transparency of the picture, the researcher elicited a consensus that the number of objects in the picture was 35. The researcher then wrote "35" on the transparency and said "Thirty-five stands for the 35 beans (or squares). She then circled the "5" in one color and asked the students to do the same. She then asked, "What does this part of 35 have to do with how many beans are in the picture? Write down what you think and color the picture to show what you mean. After allowing for response time, she circled the "3" with another color, asked students to do the same and asked "How about this part? What does THIS part have to do with how many beans are in the picture?"

For the preassessment, the picture of 35 objects arranged in a rectangular five-by- seven array. A second version, picturing 35 objects in an ungrouped collection, was administered at the close of the instructional period, and again in June which was three months after the instruction.

Lessons

Each lesson began with a problem-solving task to set the stage for the digit-correspondence (experimental) task, which was to decide what the parts of the number had to do with how many objects are in a collection. The stage-setting tasks were designed to reflect typical intermediate-grades curriculum (topics included area, multiplication, and division), and to provide entry for all students. Manipulative materials and/or drawings were part of all the tasks. A set of detailed lesson descriptions including samples of student work is available from the authors.

144 Squares. Students were asked to decide, in groups, whether or not three gridded paper shapes were the same amount of paper (area). The rectangular shapes were 12cm x 12cm (144), 13cm x 11cm (143), and a shape 6cm x 24 cm with one square centimeter cut off each corner (140). Students reached consensus that the yellow square was the largest, with an area of 144 square centimeters. In the digit-correspondence task, students were asked "what does this part of 144 (circling each individual digit beginning with the 4 in the units place, then the tens digit and finally the "1") have to do with how many square centimeters are in the yellow shape?" We provided each group a transparency picture of the 12 x 12 square for preparing their presentation to the whole class.

124 Cubes. In this lesson we used a "factory" context of filling orders for cubes. Base ten blocks were available as models for cubes "prepackaged" in sets of ten and one hundred. We asked, "How many ways can you fill an order for 124 cubes?" Making a list was modeled as a problem-solving strategy. For the digit-correspondence task, each group was assigned one of the non-standard ways to fill the order (e.g., seven long packages and 54 individual cubes) and asked to "decide which blocks would fill the order for each of the three parts of the number" (digits).

26 Wheels. Students were asked to determine how many toy wheels were contained in a clear plastic bag, based on the following two clues: "There are enough for six cars. There are two left over." After students reached a consensus that there would be 26 wheels we asked what each part of 26 (the "6" and then the "2") had to do with "how many you have." A diagram of the six cars (each with its four wheels) and the remaining two wheels was provided each student as they worked individually and a transparency version was provided to each group.

62 Wheels. "If each car has four wheels, how many cars can be fitted with 62 wheels?" After arriving at a consensus of 15 cars, we asked, What does each part (the "2" and the "6") of 62 had to do with how many wheels you have? Students made their own drawings.

Typically, a member of the research team presented the task, researchers and the classroom teacher circulated among groups during the cooperative-group work time, and the classroom teacher led the whole-class discussion while groups presented their results on overhead transparencies. Teachers used questions and comments to focus attention on differences and similarities among the ideas presented, and often asked students to elaborate by showing with the picture. Special care was taken to provide neither any direct instruction about the "tens and ones" meanings of the digits nor any judgments about the correctness of the ideas presented.

Transcripts of the lessons were based on note-taking by a trained observer and audio recordings. All individual written work and group work, which was usually in the form of overhead projector transparencies, were collected for analysis.

Results and Discussion

We sorted the individual written assessment papers into categories of similar responses and developed a descriptive rubric of nine distinct categories. Reading the 71 preassessment papers was discouraging. Twelve students failed to respond to the questions, and 32 invented meanings that gave no hint of the "3" representing 30. One student gave the response that the "5" meant five squares and the "3" stood for three squares. All the other students attempted to account for the whole collection of 35 squares; "rows of five" and "counting by threes" (even accounting for the remainder) were common responses. Fourteen students used the language of tens and ones in their written responses, but we could not be sure they were talking about the collection of squares in the picture or simply describing the names they had learned for the "coloms" (sic). Eight students wrote responses that strongly suggested that they might understand the meanings of the digits, but included no pictorial evidence of a 30 and 5 partitioning. Only five students gave truly convincing written and pictorial evidence of understanding.

With only one or two students in each classroom demonstrating understanding at the beginning of the instruction, we were concerned that there would be insufficient numbers of knowledgeable peers for the social-interaction design to produce changes in student conceptions. However, students found the lessons engaging and were soon immersed in making sense out of the digits. They examined many ideas, and lively debate often occurred as students exchanged points of view about the meaning of the individual digits. The task that elicited the most heated debate was "26 Wheels." One viewpoint was that the "2" stood for twenty wheels (usually in five cars), and that the "6" stood for the remaining six wheels. Other students were equally adamant that the "2" stood for the two wheels left over and the "6" stood for the six cars or the wheels on the six cars.

The responses on the postassessments were generally both more correct and more expansive than those on the preassessment. In the delayed postassessment, 23 students described the "3" in 35 as representing not simply 30, but also as three sets of ten; 10 of the 23 partitioned the accompanying picture into sets of ten while the remaining 13 partitioned it into 30 and 5. An additional 29 students wrote that the digits represented five squares and 30 squares; 15 of these included pictures. We concluded that there might be three reasons for the improvement. One is that students constructed meanings for the individual digits in a multidigit numeral that were more consistent with our place-value numeration system than those they held before the instructional intervention. Another is that they became better at expressing their mathematical thinking after the experience of talking and writing about their ideas, and hearing and seeing other students' ideas. Finally, because relatively few students used the pictures to show what they meant in the preassessment, we chose to prompt the use of coloring the pictures more assertively in administering the postassessments.

Some small groups seemed to get stymied with a single incorrect interpretation because it was the viewpoint of a student in their group who was a respected leader in the classroom. Students in these groups might have benefited more had we changed the groups so that they could have experienced a more fluid exchange of ideas. Although we were constrained to present the lessons on consecutive days, the lessons might have been more effective if spaced across the school year, because teachers change the composition of the collaborative of groups every few weeks.

To evaluate changes in student thinking about the digits, we compared the preassessments with the postassessments in terms of success. All responses that related the "5" in 35 to a set of five objects and the "3" in 35 to the remaining set of 30 objects or three sets of ten objects were defined as "successful." On the preassessment, the work of only 13 of the 71 students (18%) demonstrated that they knew that the "3" represented thirty of the objects; these 13 were also successful on both the immediate and delayed postassessments. On the immediate postassessment, 45 additional students (63%) were successful, falling to 41 (58%) on the delayed postassessment. Among those who were initially unsuccessful, 65% of the third graders, 76% of the fourth graders, and 63% of the fifth graders were successful on the delayed postassessment. Two were absent and 15 (21%) remained unsuccessful on the postassessments.

Conclusions

The digit-correspondence instructional tasks used in this study are "worthwhile" as defined in the NCTM Professional Teaching Standards (National Council of Teachers of Mathematics, 1991). By presenting a few digit-correspondence tasks in a problem solving mode and allowing students to exchange points of view, teachers may be able to help more students in grades three through five construct an understanding of the meanings of the digits in a multidigit numeral.

In the NCTM's *Mathematics for the Young Child*, Thompson recommends that teachers use digit-correspondence tasks to interview individual students as a way to diagnose place-value understanding (1990, 106-107). Teachers of older students may find the whole-class, written format described in this study to be a useful alternative.

Understanding place value is important to achieving good number sense, estimating and mental math skills, and to an understanding of multidigit operations. The results of this study contribute to a growing body of evidence that students can construct important mathematical concepts and structures through social interaction and communication with their peers about worthwhile mathematical tasks.

References

Fuson, K. C., & Smith, S. (April, 1994,). *Supporting Latino First Graders' Ten-Structured Thinking in Urban Classrooms.* Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans.

Hiebert, J., & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in mathematics Education, 23*(2), 98-122.

Kamii, C. (1989). *Young Children Continue to Reinvent Arithmetic, Second Grade: Implications of Piaget's theory.* New York: Teachers College Press.

Kamii, C., & Lewis, B. (1993). The Harmful Effects of Algorithms in Primary Arithmetic. *Teaching Pre K-8, 23*(4), 36-38.

Kamii, M. (1982). *Children's graphic representation of numerical concepts: a developmental study.* Unpublished doctoral dissertation, Harvard University.

National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Ross, S. H. (April, 1986). *The Development of Children's Place-Value Numeration Concepts in Grades Two through Five.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco CA.

Ross, S. H. (1990). Children's acquisition of place-value numeration concepts: The roles of cognitive development and instruction. *Focus on Learning Problems in Mathematics, 12*(1), 1-17.

Thompson, C. S. (1990). Place value and larger numbers. In J. N. Payne (Ed.), *Mathematics for the Young Child*, (pp. 89-108). Reston VA: National Council of Teachers of Mathematics.

RURAL STUDENTS' INFORMAL KNOWLEDGE OF DIVISION

Sarah B. Berenson, North Carolina State University
Draga Vidakovic, North Carolina State University

This preliminary study examined the meanings, models, and strategies of rural students in grades 3-8 for solving simple, whole number division problems. Findings suggested that students have multiple meanings for division words such as *share* and *fair share*. Cultural factors and diversity in the classroom may be associated with these multiple word meanings. Students in this study worked from three division models: partitive, quotative, and splitting. The majority of students selected the partitive model as their model of choice. Younger students selected the quotative model more than older students. Among four strategies, older students used division facts more often than younger students. Younger students depended on addition/subtraction strategies more than older students. A number of students at every grade level used multiple strategies.

The division ideas of rural students in grades 3-8 were examined in this preliminary study. Previous studies investigated children's and teachers' division concepts and processes in relation to a number of external variables including problem type, context, number type, representations, and rule violations or misconceptions (Harel, Behr, Post, & Lesh, 1994; Greer, 1992; Tirosh & Graeber, 1990). Children in grades 1-3 were interviewed by Kouba (1989) to identify the division strategies used to solve simple division problems. Fischbein, Deri, Nello, and Marino (1985) concluded that the model of choice among students in grades 5 and 7 was the partitive division model. The quotative model influenced only grade 9 students' choices. Others have proposed that there are more division models, and of particular interest to us was Confrey's (1994) *splitting* model.

Division is one strand of the multiplicative conceptual field described by Vergnaud (1994). He asserted that the multiplicative conceptual field should first consider the intuitive, implicit, or informal mathematical knowledge of students. The theoretical framework of Vergnaud (1994) reflects the constructivism of Vygotsky who included a described teaching as the mediated meanings of situations, words, and symbols between teacher and student (Vygotsky, 1986). Teaching provides the social interactions, but from a constructivist perspective the learning is a personal process, unique to every student. Vygotsky (1986) observed that children before adolescence have very different meanings than those of adults for situations, words, and symbols (Voight, 1994). To better understand the multiple meanings, models, and strategies that students bring to the classroom, this study investigated the informal knowledge used by rural children in grades 3-8 to solve simple, whole number division problems.

Method

The students in this preliminary study were from a large, rural county in Southeastern United States. Within the county there was a wide range of cultures. Many students' families have lived in their farming communities for generations, other families are transient or migrant agricultural workers. In this county school system there are 16,067 K-12 students of which 72% are white and 23% are African American. Hispanics (3%) make up a small percent of the school population followed by Native Americans and Asian students who together account for less than 1% of the student population. In March 1995, 35% of the K-12 school population received free or reduced lunch indicating low socio-economic status.

Data for this study were collected in two ways. Teachers of grades 3-8 conducted structured video interviews of some of their students (n=55) and gave a paper/pencil assessment to all of their students (n=451). In the structured interviews, students were encouraged to talk about *sharing* in their lives. Then teachers asked students to determine if and explain why a particular distribution of candy was a *fair share*. The interview also included a simple, open-ended division problem that suggested *sharing division* and involved dividing a bag of candy. The first item on the paper/pencil assessment asked students to explain in words and pictures how they would solve a simple, partitive division problem. Next students were asked to write a story problem from a picture that suggested partitive division.

Analyses and Results

Data from the interviews and the paper/pencil instrument were examined to determine the meanings that 3-8 students gave for *share* and *fair share*, their models for division, and the strategies that they used to solve simple, partitive division problems. The analyses of the data involved multiple sorts to code the responses. Agreement between the categories established by previous studies and our categories provided information for final adjustments to the categories. Then the data were tabulated to reflect the frequency of each category. Student profiles were used to report the frequency of strategy responses because students had more than one strategy for a single item.

Meanings

Sharing is recommended in methods textbooks as a meaningful way to introduce young children to partitive division. However, Vygotsky (1986) stated that children, up until adolescence, have very different meanings of words than adults, although they have learned how to use the words to communicate with adults. In the interviews, students were asked about their definitions of *sharing*, and to explain if the teacher's arbitrary distribution of candy was an example of *fair share*. Multiple sorts of these responses were coded to establish the following categories of meanings for both *share* and *fair share*: 1) equal amounts in meaning, 2) mathematical meaning, and 3) cultural meaning. The first category was tabulated if students mentioned equal shares. The second category was tabulated if students gave a description of division. If the student's definitions of *share* and *fair share* indicated that cultural factors were involved, then tabulation was made in the third category.

Among 55 interviewed students, there were 15 students whose definitions of *sharing* included ideas of equality, and 31 students whose definitions of *fair sharing* included ideas of equality. Older students tended to include ideas of equality more than younger students, for the meanings of both words. The analysis of whether students associated division with *sharing* and *fair share* gave the following results. Only 4 students included division in their definitions of *sharing* (1 for 3-4 gr.; 2 for 5-6 gr.; 1 for 7-8 gr.). Two students included division as part of their definition of *fair share*. Many students, who gave non-mathematical definitions, showed great diversity in their meanings of *share*, and indicated a wide variety of cultural values. They included:

- To loan personal items such as toys, clothes, etc.
- Taking turns with another person.
- Doing things with another person.
- To keep a secret with a friend or sibling.
- If they have nothing, you give them something of yours.
- Giving to someone if they deserve it.
- To give others the same amount, including one's self so nobody gets upset.
- If you have something that the other person wants, and they have something you want, then you give each other those things.
- To give the other person more because I can get whatever I want anytime.
- Everyone is treated fairly when you have something.
- Older people, like parents or older siblings, can get more than younger children.
- Not sharing all the candy so that you can save some for another day. Only take a few.

Among students in grades 3-4, 71% included cultural factors in their definitions of *sharing*. For students in grades 5-6 and grades 7-8, 74% and 59% respectively included cultural factors in their definitions. When asked if a particular distribution was a *fair share*, the evidence of cultural values decreased greatly among all students' meanings (5% overall) and as previously seen, a corresponding increase in notions of equality among groups was noted.

Models

Primitive models used in solving division problems were reported by Fischbein et al. (1985) and included the partitive and the quotative models of division. Fischbein described the partitive model as *sharing division*, where something or a collection of things is divided into a number of equal parts or groups. He defined the quotative model as *measurement division*, where the student is required to find how many times a given quantity is contained in a larger quantity. In addition, Confrey (1994) has proposed a third primitive model of *splitting division*. Much like a binary tree, the students make successive halvings or pairings in the splitting process to produce multiple versions of the original. These new versions are created by either magnifying or shrinking the original version. Interview transcripts in our study were examined to tabulate how many students used each of the three models. Also, note was made of the students who determined the dividend before solving the open-ended problem.

Table 1 presents the frequency distributions of the different models in percent for grades 3-4, 5-6, and 7-8. We found that more than half of the students (51%) did not count the candy to determine the dividend. Sixty-four percent of the students in grades 3-4, 56% of the students in grade 5-6, and 39% of the students in grades 7-8 did not know the dividend before starting their solutions.

Table 1

Frequency of Primitive Models Used by Students in Grades 3-4, 5-6, and 7-8 to Solve an Open-Ended Division Problem

	% Categorical Responses (n)							
Model	3rd & 4th grade (n=14)		5th & 6th grade (n=18)		7th & 8th grade (n=23)		totals (n=55)	
Partitive	50	(7)	67	(12)	78	(18)	67	(37)
Quotative	36	(5)	28	(5)	22	(5)	27	(15)
Splitting	14	(2)	6	(1)	–		5	(3)

The partitive model for division was used more frequently by students at all grade levels. This may be because the problem involved dividing candy, suggesting a sharing model of division. However, 27% of the students selected a quotative model for the problem. Few students (5%) used the splitting model. A larger percent of the older students used the partitive model while the younger students used it less. Among those selecting a quotative model, there was a larger percent of younger students and a smaller percent of older students who selected the model. No students in grades 7-8 selected the splitting model.

Strategies

The paper/pencil instrument was used to collect data about the division strategies of students in grades 3-8. Multiple sorts established the categories of the solution strategies of simple, partitive division problems. These categories were compared to those reported by Kouba (1989) and modified as follows: 1) addition/subtraction, 2) dealing out, 3) multiplication, and 4) division. Two dealing out strategies were noted among the students – one in which the student distributed one or two to each group (divisor) consecutively, and the other in which the student dealt out all the objects (quotient) at one time to a number of groups. The first dealing out process required multiple circuits of the groups up to the limit of the dividend. The second dealing out process required only one circuit of dealing out where the number of groups formed were limited by the dividend. Since students may have used more than one strategy within a response, student profiles were used to represent the different combinations.

The profiles of students' division strategies are reported below in Table 2. In the student profile, a "1" indicated that a particular strategy was present, while a "0" indicated the absence of the strategy. Reading from left to right the cell combinations are addition/subtraction, dealing out, multiplication, and division. For our study, a "1001" profile indicated the student employed an addition/subtraction strategy and a division strategy to solve simple, partitive division problems. Also, the profile indicates that the student did not use any dealing out or multiplication strategies. Of the 451 students who took the paper/pencil instrument, there were 134 who gave incomplete or no response. The percents shown in Table 2 are calculated on the 317 students whose answers indicated a particular strategy.

Table 2
Student Profiles for Grades 3-4, 5-6, and 7-8 and the Frequency of Strategies Used to Solve Partitive Division Word Problems

	% Categorical Responses (n)							
Profile	3rd & 4th grade (n=64)		5th & 6th grade (n=102)		7th & 8th grade (n=151)		totals (n=317)	
0001	22	(14)	47	(47)	56	(84)	46	(145)
0010	23	15)	25	(25)	18	(27)	21	(67)
0100	8	(5)	5	(5)	3	(4)	4	(14)
1000	31	(20)	6	(6)	5	(8)	11	(34)
0011	5	(3)	8	(8)	9	(14)	8	(25)
0101	–	1	(1)	3	(4)	1	(5)	
0110	–	1	(1)	–	< 1	(1)		
1001	6	(4)	4	(4)	4	(6)	4	(14)
1010	5	(3)	1	(1)	1	(2)	2	(6)
1100	–	2	(2)	–	1	(2)		
1011	–	2	(2)	–	1	(2)		
0111	–	–	1	(2)	1	(2)		

Generally, older students selected division as a strategy more than younger students. Younger student preferred addition/subtraction strategies more than older students. There were a total of 55 students who had more than one strategy to explain their solutions to the partitive division word problems.

Discussion and Summary

This was a preliminary study to examine the informal meanings, models, and strategies of rural students in grades 3 - 8 when solving primitive division problems. Primitive is a descriptor used in the sense that both the quotient and the divisor were whole numbers greater than one but smaller than the whole number dividend (Harel, et al., 1994; Fischbein et al., 1985). It appears, that for some rural students there were strong cultural values associated with and affecting their meanings of *sharing*. Giving someone more because one has enough may be indicative of the value that farmers place on helping one's neighbors. Saving some candy for another day seemed to us to be a survival strategy for a child living in poverty. The multiple meanings of *share* and *fair share* suggested that teachers cannot assume that all of their students have the same meanings of words used in the mathematics classrooms, especially if the students are culturally diverse. We view the different meanings of words of teachers and students as an interesting research area to pursue. Our findings concerning models are somewhat different than Fischbein et al. (1985) with respect to older students' success with quotative division. This may be because Fischbein's quotative problems involved decimal numbers while ours were whole numbers. We found that a higher percent of the younger students selected the quotative model when compared to the percent of older students. A future research question for investigation, as suggested by Kouba (1989), is to study individual students' use of multiple models of division, and in particular, the splitting model. We found that students used multiple strategies to solve these division problems, and perhaps they use multiple models as well. The extent to which students can apply the strategies identified in this study to division problems involving other models and division of rational numbers are other areas for future research.

References

Confrey, J. (1994). Splitting, similarity, and rate of change: a new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 293-332). Albany, NY: State University of New York Press.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16* (1), 3-17.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.). *Handbook on research on mathematics teaching and learning* (pp. 276-295). New York: MacMillan.

Harel, G., Behr,M., Post, T., & Lesh, R. (1994). The impact of the number type on the solution of multiplication and division problems: Further investigations. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 365-388). Albany, NY: State University of New York Press.

Kouba, V. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20* (2), 147-158.

Tirosh, D., and Graeber, A. O. (1989). Preservice elementary teachers; explicit beliefs about multiplication and division. *Educational Studies in Mathematics*, 20, 79-96.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.) *The development of multiplicative reasoning in the learning of mathematics* (pp. 41-60). Albany, NY: State University of New York Press.

Voight, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies in Mathematics, 26*, 275-298.

Vygotsky, L. (1986). *Thought and language*. Cambridge, MA: MIT Press.

ADDITION OF WHOLE NUMBER SCHEMES FOR STANDARD TWO AND STANDARD THREE PUPILS

Noraini Binti Idris, The Ohio State University

The main purpose of this study is to recognize the addition of whole number schemes mastered by primary school pupils, on the basis of their behaviour and explanation when solving problems in the addition of whole numbers. Four categories of whole number addition problems were given to the pupils: mental illustration, open-sentence problems, place value concept and "box" problem.

The subjects of this study consisted of three standard two pupils and three standard three pupils from one of the national primary schools in Kuala Lumpur, Malaysia. These pupils were chosen by their class teacher based on their performance on the 1989 First Term Examination.

The analysis and behaviour descriptions of the pupils were carried out based on videorecorded interviews and comments made by the researcher while conducting the interviews. Four whole number addition schemes have been recognized: counting all, counting on, counting on from the biggest addend and algorithm technique.

Several pupils were acknowledged to be using the basic commutative concept, place value concepts and the number concepts that are related to solving the problems of whole number addition .

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Address List

Sergei Abramovich
105 Aderhold Hall
University of Georgia
Athens, GA 30602-7124
sergey@moe.coe.uga.edu
1-111

Verna M. Adams
Dept. of Teaching & Learning
Washington State University
Pullman, WA 99164-2132
verna_adams@wsu.edu
1-216, 1-268, 1-319

Sunday A. Ajose
East Carolina University
Greenville, NC 27858
maajose@ecuvm.cis.ecu.edu
2-42

Maureen Albright
105 Aderhold Hall
University of Georgia
Athens, GA 30602
malbrigh@moe.coe.uga.edu
2-193

Sitsofe E. Anku
Nanyang Technological University
School of Science
469 Bukit Timah Road
Singapore 1025
ankus@nievax.nie.ac.sg
1-181

Peter Appelbaum
Curriculum and Instruction
William Patterson College of New Jersey
Wayne, NJ 07470
appelbaum@zodiac.rutgers.edu
2-244

Bridget Arvold
105 Aderhold Hall
University of Georgia
Athens, GA 30602
arvold@moe.coe.uga.edu
2-193

David Ashley
4750 Collegiate Drive
Florida State University
Panama City, FL 32405
tclbay12@mail.firn.edu
2-133

Diane S. Azim
722 Second St.
Cheney, WA 99004
2-226

Virginia Bastable
Mount Holyoke College
South Hadley, MA 01075
vbastabl@mhc.mtholyoke.edu
2-289

Michael T. Battista
404 White Hall
Kent State University
Kent, OH 44242
mbattist@kentvm.kent.edu
1-192, 1-413

Joanne Rossi Becker
Dept. of Mathematics and Computer Science
San Jose State University
San Jose, CA 95192
2-255

Nadine Bednarz
Universite du Quebec a Montreal
Montreal, Quebec, Canada H3C 317
2-203

Ronald Benbow
500 W. Reade Ave.
Taylor University
Upland, IN 46989
rnbenbow@tayloru.edu
2-136

Sarah B. Berenson
CRMSE
Box 7801
315 Poe Hall
North Carolina State University
Raleigh, NC 27695-7801
berenson@unity.ncsu.edu
1-297, 2-344

Maria L. Blanton
Mathematics Education Dept.
Hyde Hall
North Carolina State University
Raleigh, NC 27695
mlblanto@eos.ncsu.edu
1-297

Glen Blume
169 Chambers
The Pennsylvania State University
University Park, PA 16802
bti@psuvm.psu.edu
2-317

Steven Blumsack
Florida State University
Tallahassee, FL 32306-3032
blumsack@math.fsu.edu
1-247

Caroline Van Auken Borrow
32750 Seneca Drive
Solon, OH 44139
1-413

Daniel J. Brahier
EDCI - 529 Education Building
Bowling Green State University
Bowling Green, OH 43403
brahier@bgnet.bgsu.edu
2-55

Theresa Brar
School of Education
University of California - Santa
Barbara
Santa Barbara, CA 93106
1-229, 1-373

Mary E. Brenner
School of Education
University of California - Santa
Barbara
Santa Barbara, CA 93106
betsy@edstar.gse.ucsb.edu
1-164, 1-229, 1-373

George W. Bright
School of Education
UNC at Greensboro
Greensboro, NC 27412-5001
brightg@steffi.uncg.edu
1-299

Patricia A. Brosnan
College of Education
253 Arps Hall
1945 N. High Street
The Ohio State University
Columbus, OH 43210-1172
brosnan.1@osu.edu
1-235

Dawn L. Brown
14219 Otter Run Rd.
Tallahassee, FL 32312
DawnB51686@aol.com
2-13

Judith Curran Buck
Hyde Hall
Mathematics Dept.
Plymouth State College
Plymouth, NH 03264-1600
jcurran@oz.plymouth.edu
1-298

Gail Burrill
National Center for Research in
Mathematical Sciences Education
1025 W. Johnson St.
University of Wisconsin - Madison
Madison, WI 53706
gburrill@macc.wisc.edu
1-95

Jinfa Cai
Dept. of Mathematics, Statistics, &
Computer Science
Marquette University
Box 1881
Milwaukee, WI 53201
jinfac@marque.mscs.mu.edu
1-359, 2-24

Jennifer Chauvot
Aderhold Hall Room 105
The University of Georgia
Athens, GA 30602
jchauvot@moe.coe.uga.edu
2-115, 2-155

Douglas H. Clements
593 Baldy Hall
State University of New York at
Buffalo
Buffalo, NY 14260
Clements@acsu.buffalo.edu
1-192

Paul Cobb
Peabody College Box 330
Vanderbilt University
Nashville, TN 37203
cobbp@ctrvax.vanderbilt.edu
1-3

Jere Confrey
Cornell University
Dept. of Education
422 Kennedy Hall
Ithaca, NY 14853-4203
jc56@cornell.edu
1-395, 1-421

Michael L. Connell
CUIN 5872
University of Houston
Houston, TX 77204-5872
connell@uh.edu
1-187

Jose (Alberto) Contreras
College of Education
1945 N. High St.
The Ohio State University
contreras.8@osu.edu
2-245, 2-240

Joe Cook
Paige Academy
Boston, MA
joec@media.mit.edu
1-291

Thomas J. Cooney
105 Aderhold Hall
University of Georgia
Athens, GA 30602
tcooney@moe.coe.uga.edu
2-91, 2-109

Rebecca Corwin
29 Everett St.
Lesley College
Cambridge, MA 02138-2790
rebecca_corwin@terc.edu
2-97

Ruth Cossey
3130 Maxwell Ave.
Oakland, CA 94619
1-54

Wendy N. Coulombe
201 Penchant Ct.
Cary, NC 27513
wendy_coulombe@ncsu.edu
1-297

Kathleen Cramer
University of Wisconsin
107 Ames Hall
River Falls, WI 54022
1-377

Suzanne Damarin
101A Ramseyer Hall
29 W. Woodruff Ave.
The Ohio State University
Columbus, OH 43210
1-30

Emily Dann
CMSCE
10 Seminary Pl.
Rutgers University
New Brunswick, NJ 08903
2-276

Linda Ruiz Davenport
Education Development Center
55 Chapel St.
Newton, MA 02158
ldavenport@edc.org
2-187

Genevieve A. Davis
404 White Hall
Kent State University
Kent, OH 44242
GDavis@Kentvm.kent.edu
2-199

Yolanda De La Cruz
Education & Social Policy
2115 N. Campus Drive
Northwestern University
Evanston, IL 60208-2610
delacruz@merle.acns.nwu.edu
1-331

Juli K. Dixon
Dept. of Instruction and Curricular Services
4505 S. Maryland Pkwy.
University of Nevada - Las Vegas
Las Vegas, NV 89154-3005
dixonj.1@nevada.edu
2-313

Helen M. Doerr
215 Carnegie Hall
Syracuse University
Syracuse, NY 13244-1150
hmdoerr@mailbox.syr.edu
1-117

Angeles Dominguez
215 Carnegie Hall
Syracuse University
Syracuse, NY 13210
adomingu@mailbox.syr.edu
2-66

Susan J. Doubler
Lesley College
Cambridge, MA 02138
sue_doubler@terc.edu
2-97

Richard Duran
School of Education
University of California - Santa
Barbara
Santa Barbara, CA 93106
duran@edstar.gse.ucsb.edu
1-229, 1-373

Christine L. Ebert
105G Pearson Hall
University of Delaware
Newark, DE 19716
cebert@math.udel.edu
1-349, 2-209

Thomas Edwards
Teacher Education Division
295 Education Bldg.
Wayne State University
Detroit, MI 48202
tedwards@cms.cc.wayne.edu
2-149

Rita L. Eisele
Math/ematicsComputer Science Dept.
Indiana State University
Terre Haute, IN 47809
maeisele@root.indstate.edu
2-50

Patrick J. Eggleton
Mathematics Dept.
5014 Berry College
Mt. Berry, GA 30149
eggleton@berry.edu
2-79

Charles E. Emenaker
9555 Plainfield Rd.
University of Cincinnati
Cincinnati, OH 45235
Charles.Emenaker@uc.edu
2-135

George Emese
Mathematics Dept.
Rowan College
Glassboro, NJ 08028
gemese@mars.rowan.edu
1-143

Diana Erchick
257 Arps Hall
1945 North High Street
The Ohio State University
Columbus, OH 43210
Erchick.1@osu.edu
2-48

David R. Erickson
School of Education
University of Montana
Missoula, MT 59812
erickson@selway.umt.edu
2-202

Warren W. Esty
Dept. of Mathematical Sciences
Montana State University
Bozeman, MT 59715
esty@math.montana.edu
1-147

David Feikes
1401 S. U.S. 421
Purdue University - North Central
Westville, IN 46391
2-175

M. Jayne Fleener
College of Education
University of Oklahoma
Norman, OK 73019
DAS2496@UOKMVSA.BACKBONE.UOKNOR.EDU
2-104

Susan N. Friel
School of Education
Peabody Hall
UNC - Chapel Hill
Chapel Hill, NC 27599-3500
sfriel@email.unc.edu
1-299

Pamel G. Fry
University of Oklahoma
Norman, OK 73019
2-104

Iddo Gal
University of Pennsylvania
Philadelphia, PA 19104
gal@literacy.upenn.edu
1-401

Enrique Galindo
201 Rose Avenue
Indiana University
Bloomington, IN 47405
egalindo@indiana.edu
2-321

Aurora Gallardo
Research and Advanced Studies
Centre
Nicolas San Juan 1421
Col. Del Valle
Mexico City
D.F., 03100 Mexico
1-158

Herendira Garcia-de-Galindo
2379 S. Worthington Dr.
Indiana University - Bloomington
Bloomington, IN 47405
hgalindo@indiana.edu
2-286

William E. Geeslin
Mathematics Dept.
Kingsbury Hall
University of New Hampshire
Durham, NH 03824
weg@christa.unh.edu
2-67

Elizabeth Ann George
10 Oakhurst Circle
Pittsburgh, PA 15215
bgeorge@vms.cis.pitt.edu
1-204

Lynda Ginsburg
Graduate School of Education
University of Pennsylvania
3910 Chestnut Street
Philadelphia, PA 19104-3111
ginsburg@literacy.upenn.edu
1-401

Nina R. Girard
2855 R. Bedford St. Apt. F
Johnstown, PA 15904
nina@vms.cis.pitt.edu
1-232

Lynn T. Goldsmith
Education Development Center, Inc.
55 Chapel St.
Newton, MA 02158
2-249

Marshall Gordon
The Park School of Baltimore
Old Court Rd.
Brooklandville, MD 21022
mgordon@umd5.umd.edu
1-230

Shirley B. Gray
Dept of Mathematics and Computer Science
California State University - Los Angeles
Los Angeles, CA 90032
sgray@calstatela.edu
2-49

Diana Underwood Gregg
Dept. of Mathematics, Computer Science, & Statistics
Purdue University - Calumet
diana@sage.cc.purdue.edu
2-36

Barbara MacKay Gupta
SW 845 Mies
Pullman, WA 99163
54222663@wsuvm1.csc.wsu.edu
1-166

Oakley D. Hadfield
Box 30001; Dept. 3CUR
New Mexico State University
Las Cruces, NM 88003-0001
hadfield@nmsu.edu
2-131

Susan Hagen
Dept. of Mathematics
Virginia Tech
Blacksburg, VA 24061-0123
hagen@math.vt.edu
1-300

Jim Hammerman
Education Development Center
55 Chapel St.
Newton, MA 02158
JHammerman@edc.org
2-268

Howard Hansen
Dept. of Mathematics
1 University Circle
Western Illinois University
Macomb, IL 61455
mfhlh@uxa.ecn.bgu.edu
1-184

William J. Hardin
114 Linwill Terrace Apt. 4
Syracuse University
Suracuse, NY 13206
wjhardin@mailbox.syr.edu
2-316

Michael D. Hardy
Dept. of Curriculum and Instruction
Rm 209 Carothers Hall
Florida State University
Tallahassee, FL 32306-3032
mdhardy@mailer.acns.fsu.edu
1-332

Delwyn L. Harnisch
University of Illinois at Urbana-Champaign
Champaign, IL 61820
1-183

Kathleen Heid
171 Chambers Bldg.
Penn State University
University Park, PA 16803
ik8@psuvm.psu.edu
2-221

Terese A. Herrera
College of Education
257 Arps; 1945 N. High St.
Columbus, OH 43210
therrera@pacific.mps.ohio-state.edu
2-206

Catherine Hill
Godby High School
2-31

Ted Hodgson
Dept. of Mathematical Sciences
Montana State University
Bozeman, MT 59717
Hodgson@mathfs.math.montana.edu
1-371

Mary Ann Huntley
2349 Computer & Space Sciences Bldg.
University of Maryland at College Park
College Park, MD 20742
mhuntley@wam.umd.edu
2-137

Noraini Binti Idris
Dept. of Educational Theory & Practice
1945 N. High St.
The Ohio State University
Columbus, OH 43210
2-351

Kathy M.C. Ivey
Dept. of Mathematics
Western Carolina University
Cullowhee, NC 28723-9049
kivey@wpoff.wcu.edu
1-301

Patricia A. Jaberg
Illinois State University
Normal, IL 61790
jaberg@math.ilstu.edu
2-143

Elizabeth Jakubowski
236 Stone Bldg.
Florida State University
Tallahassee, FL 32306-3010
ejakubow@garnet.acns.fsu.edu
1-247, 2-262, 2-200

Todd M. Johnson
317 Stevenson Hall
Campus Box 4520
Illinois State University
Normal, IL 61790-4520
Johnson@math.ilstu.edu
2-3, 2-134

Graham A. Jones
Mathematics Dept.
Illinois State University
Normal, IL 61790-4520
jones@math.ilstu.edu
1-169

Constance Kamii
School of Education
University of Alabama at Birmingham
Birmingham, AL 35294-1250
edu5080@uabdpo.dpo.uab.edu
1-222

Rochelle G. Kaplan
William Paterson College
Wayne, NJ 07470
kaplan_r@spc.wilpaterson.edu
2-244

James J. Kaput
Dept. of Mathematics
University of Massachusetts - Dartmouth
North Dartmouth, MA 02747
jkaput@umassd.edu
1-71

Patricia Ann Kenney
Learning Research and Development
Center
University of Pittsburgh
Pittsburgh, PA 15260
kenney@vms.cis.pitt.edu
1-175

Thomas E. Kieren
Dept. of Secondary Education
University of Alberta
Edmonton, Alberta, Canada T6G
2G5
tom_kieren@ualberta.ca
1-123

Georgianna Klein
Math/Stat Dept.
Grand Valley State University
Allendale, MI 49401
kleing@gvsu.edu
1-303

Eric J. Knuth
School of Education
Campus Box 249
University of Colorado at Boulder
Boulder, CO 80309-0249
1-304

Trish Koontz
404 White Hall
Kent State University
Kent, OH 44242
TKoontz@Phoenix.Kent.edu
2-199

Victoria Boller LaBerge
Dept. of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115
laberge@math.niu.edu
2-132

Andrea Lachance
Cornell University
Dept. of Education
420 Kennedy Hall
Ithaca, NY 14853-4203
aml13@cornell.edu
1-395

Lionel N. LaCroix
John Fraser Secondary School
2665 Erin Centre Blvd.
Mississauga, Ontario, Canada L5H
5H6
lionlacr@village.ca
2-64

Diana V. Lambdin
School of Education, 3266
Indiana University
Bloomington, IN 47405
lambdin@indiana.edu
2-168

Cynthia W. Langrall
Mathematics Dept.
Illinois State University
Normal, IL 61790-4520
langrall@math.ilstu.edu
1-169

Doug Lapp
Dept. of Mathematics
Box 986
Morehead State University
Morehead, KY 40351
lapp.1@osu.edu
2-70

Brenda Lee
127 An Lok St.
Chia Yi
Wu Feng Institute of Technology and
Commerce
Taiwan, Republic of China
1-141

Annette Ricks Leitze
Dept. of Mathematical Sciences
Robert Bell Bldg., Rm. 465
Ball State University
Muncie, IN 47306-0490
00aeleitze@bsu.edu
2-138

Lawrence Lesser
Dept. of Mathematical Sciences
University of Northern Colorado
Greeley, CO 80639
lmlesse@bentley.univnorthco.edu
1-302

Jill Bodner Lester
Mount Holyoke College
South Hadley, MA 01075-1441
jlester@mhc.mtholyoke.edu
2-289

Gavrielle Levine
Dept. of Curriculum and Instruction
C. W. Post Campus
Long Island University
Brookville, NY 11548
GALEVINE@HORNET.LIUNET.EDU
2-285

Bradford F. Lewis
Curriculum and Instruction
209 Carothers Hall
Florida State University
Tallahassee, FL 32306-3032
2-69

Bey-bey Li
Dept. of Teaching and Learning
Washington State University
Pullman, WA 99164-2132
09970610@wsuvm1.csc.wsu.edu
1-216

Yeping Li
624 LRDC
University of Pittsburgh
Pittsburgh, PA 15260
yeping+@pitt.edu
1-231

Gwen Lloyd
610 East University Ave., 1228 SEB
University of Michigan
Ann Arbor, MI 48109
gwend@umich.edu
2-162, 2-233

Jane-Jane Lo
Dept. of Mathematics
Cornell University
18 Smugglers Path
Ithaca, NY 14850
JANEJANEL@AOL.COM
1-383

Chris Lovell
Old Dominion University
Virginia Beach, VA 23456
2-73

Cheryl A. Lubinski
Dept. of Mathematics
Campus Box 4520
Illinois State University
Normal, IL 61761
cal@math.ilstu.edu
2-143

Armando M. Martinez-Cruz
National University of Mexico
Ofnas, Admas. 2, ler Piso,
Ave. Universidad 3000
Mexico City, Mexico, D.F. 04510
mcruz@servidor.unam.mx
1-277

Joanna O. Masingila
Dept. of Mathematics
215 Carnegie Hall
Syracuse University
Syracuse, NY 13244-1150
jomasing@sued.syr.edu
1-325

Marguerite M. Mason
405 Emmet St.
Ruffner Hall
University of Virginia
Charlottesville, VA 22902
mmm4v@virginia.edu
1-307

Susann M. Mathews
Dept. of Mathematics and Statistics
Wright State University
Dayton, OH 45435
smathews@desire.wright.edu
1-165

Sue Tinsley Mau
IUPUI
Indianapolis, IN 46227
smau@indyvax.iupui.edu
2-62

Richard Mayer
School of Education
University of California - Santa Barbara
Santa Barbara, CA 93106
mayer@psych.ucsb.edu
1-229, 1-373

Robert Mayes
Dept. of Mathematical Sciences
University of Northern Colorado
Greeley, CO 80639
rmayes@hopper.univnorthco.edu
1-302

Kay McClain
Box 330 GPC
Vanderbilt University
Nashville, TN 37203
MCCLAIKL@ctrvax.vanderbilt.edu
2-290

J. Randy McGinnis
University of Maryland at College Park
College Park, MD 20742
jm250@umail.umd.edu
2-137

Douglas B. McLeod
Math Sciences/CRMSE
San Diego State University
San Diego, CA 92182-0315
dmcleod@sciences.sdsu.edu
1-240

Betsy McNeal
Graduate School of Education
3700 Walnut St.
University of Pennsylvania
Philadelphia, PA 19104-6216
betsym@nwfs.gse.upenn.edu
1-261

David E. Meel
999 Killarney Drive Apt. #1
Carnegie Mellon University
Pittsburgh, PA 15234-2652
meel@vms.cis.pitt.edu
1-142

Melissa Mellissinos
San Diego State University
San Diego, CA 92182
mmelliss@sunstroke.sdsu.edu
1-240

Ramakrishnan Menon
School of Science
National Institute of Education
469 Bukit Timah Road
Nanyang Technological University
Singapore 1025
menonr@nievax.nie.ac.sg
2-314

Harold Mick
Dept. of Mathematics
Virginia Tech
Blacksburg, VA 24061-0123
mick@math.vt.edu
1-300

Michael Mikusa
401 White Hall
Kent State University
Kent, OH 44240-2549
mmikusa@Phoenix.kent.edu
1-129

Mathew Mitchell
School of Education
2130 Fulton St.
University of San Francisco
San Francisco, CA 94117
mitchellm@usfca.edu
1-337, 1-339, 2-51, 2-63

Rosemary Mitchell
Straley Elementary
Princeton, NJ 24740
1-427

Anne Morris
133C Willard Hall
Educational Development
University of Delaware
Newark, DE 19716-0101
abmorris@strauss.udel.edu
2-17

Bryan Moseley
Dept. of Educational Psychology
University of California - Santa
Barbara
Santa Barbara, CA 93106-9490
bryan@edstar.gse.ucsb.edu
1-164, 1-229, 1-373

Judit Moschkovich
Institute for Research on Learning
2550 Hanover St.
Palo Alto, CA 94304
judit_moschkovich@irl.org
1-333

John C. Moyer
Dept. of Mathematics, Statistics, &
Computer Science
P.O. Box 1881
Marquette University
Milwaukee, WI 53201-1881
johnm@mscs.mu.edu
1-359

Patrick Mwerinde
Mathematical Sciences Dept.
105E Pearson Hall
University of Delaware
Newark, DE 19716
mwerinde@math.udel.edu
1-349

Ricardo Nemirovsky
TERC
2067 Massachusetts Ave.
Cambridge, MA 02144
Ricardo_Nemirovsky@terc.edu
1-291

Sacra Nevaire Nicholas
Room 114 Collings Hall
Oklahoma University
Norman, OK 73069
2-85

Marsha P. Nicol
Eisenhower National Clearinghose
for Mathematics and Science Education
1929 Kenny Road
The Ohio State University
Columbus, OH 43210
mnicol@enc.org
2-275, 2-201

Preety Nigam
Dept. of Mathematics
215 Carnegie Hall
Syracuse University
Syracuse, NY 13244
pnigam@mailbox.syr.edu
2-328

Tracy Noble
TERC
2067 Massachusetts Ave.
Cambridge, MA 02144
Tracy_Noble@terc.edu
1-291

Evangelina Diaz Obando
219 MCH Mathematics Education
Florida State University
Tallahassee, FL 32306-3032
ediaz@irazu.una.ac.cr
2-262, 2-200

James R. Olsen
Mathematics Dept.
1 University Circle
Western Illinois University
Macomb, IL 61455
jr-olsen@bgu.edu
2-308

Judith Olson
Dept. of Mathematics
1 University Circle
Western Illinois University
Macomb, IL 61455
olsonj@ccmail.wiu.bgu.edu
1-184

Melfried Olson
Dept. of Mathematics
1 University Circle
Western Illinois University
Macomb, IL 61455
olsonm@ccmail.wiu.bgu.edu
1-184

Albert D. Otto
Campus Box 4520
Illinois State University
Normal, IL 61790-4520
otto@math.ilstu.edu
2-143

John E. Owens
129 Austin
East Carolina University
Greenville, NC 27858
maowens@ecuvm.cis.ecu.edu
2-301

Ralph Pantozzi
154 Hana Rd.
Edison, NJ 08817-2045
pantozzi@gandalf.rutgers.edu
2-276

Barbara J. Pence
Dept. of Mathematics and Computer Science
San Jose State University
San Jose, CA 95192
1-210, 2-255

Randolph A. Philipp
6475 Alvarodo Road
Suite 206
San Diego, CA 92120
RPhilipp@mail.sdsu.edu
2-122

Elizabeth Phillips
Dept. of Mathematics
101 Wills House
Michigan State University
East Lansing, MI 48824-1050
ephillips@math.msu.edu
1-101

Susan Pirie
Faculty of Education
2125 Main Mall
University of British Columbia
Vancouver, BC, Canada V6T 1Z5
spirie@unixg.ubc.ca
1-123

Louise Poirier
Departemente de didactique
Universite de Montreal
C.P. 6128
Succursale Centre-Ville
Montreal, Quebec, Canada H3C 3J7
2-203

Dianne Pors
East Side Union High School District
2-255

Mary K. Porter
St. Mary's College
South Bend, IN
1-325

Donald T. Porzio
Dept. of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115-2888
dporzio@math.niu.edu
1-228

Thomas R. Post
175C Peik Hall
University of Minnesota
Minneapolis, MN 55455
postx001@maroon.tc.umn.edu
1-377

Roland G. Pourdavood
University of Oklahoma
Norman, OK 73019
2-104

Norma C. Presmeg
Curriculum & Instruction
Florida State University
Tallahassee, FL 32306
npresmeg@garnet.acns.fsu.edu
1-314, 2-31

Ronald V. Preston
Austin Bldg. 129
East Carolina University
Greenville, NC 27858
rvpresto@indiana.edu
2-168

Richard M. Preyer, Jr.
Dept. of Education
Kennedy Hall
Cornell University
Ithaca, NY 14853-4203
rmp5@cornell.edu
1-430

Terry M. Price
Dept. of Teaching and Learning
Washington State University
Pullman, WA 99164-2132
tprice@mail.wsu.edu
1-268, 1-319

Susan Prion
School of Education
1880 Grove St.
University of San Francisco
San Francisco, CA 94117
prions@usfca.edu
1-337, 1-339, 2-51, 2-63

Sidney L. Rachlin
Dept. of Mathematics
129 Austin
Greenville, NC 27858
marachli@ecuvm.cis.ecu.edu
1-343

Thomas G. Ralley
Dept. of Mathematics
The Ohio State University
Columbus, OH 43210
1-235

Anne M. Raymond
Dept. of Elementary Education
Indiana State University
Terre Haute, IN 47809
eeraymon@befac.indstate.edu
1-182, 2-14

Barbara R. Smith Reed
School of Education
University of California - Santa Barbara
Santa Barbara, CA 93106
reedr@bcvms.bc.edu
1-229, 1-372, 1-373

Michelle K. Reed
ERIC Clearinghouse for Science, Mathematics, and Environmental Education
1929 Kenny Rd.
The Ohio State University
Columbus, OH 43210
reed.253@osu.edu
2-4

David A. Reid
Dept. of Secondary Education
341a Education South
University of Alberta
Edmonton, Alberta, Canada T6G 2G5
reidd@gpu.srv.ualberta.ca
1-123

Beverly S. Rich
Campus Box 4520
Illinois State University
Normal, IL 61790-4520
bsrich@math.ilstu.edu
2-143

Billie F. Risacher
Mathematics and Computer Science Dept.
One Washington Square
San Jose State University
San Jose, CA 95192-0103
risacher@jupiter.sjsu.edu
2-16

Cheryl Stitt Roddick
705 Tall Oaks Blvd, #20
Auburn Hills, MI 48326
cstitt@magnus.acs.ohio-state.edu
1-134

Sharon Ross
1652 Park Vista Dr.
Dept. of Math & Statistics
California State University - Chico
Chico, CA 95928
sross@oavax.csuchico.edu
2-338

Adalira Saenz-Ludlow
The University of North Carolina at Charlotte
Charlotte, NC 28223-0001
sae@unccvm.uncc.edu
1-152

Umaru A. Saleh
109 Poucher Hall
SUNY at Oswego
Oswego, NY 13126
Saleh@oswego.oswego.edu
1-140

Manuel Santos T.
Cinvestav-IPN-Mexico
Dakota 379; Col. Napoles
D.F. 03810, Mexico
Lsantos@mvax1.red.cinvestav.mx
1-365

Annette M. Sassi
Education Development Ctr.
55 Chapel St.
Newton, MA 02158
asassi@edc.org
2-249, 2-187

Grace Hotchkiss Scarano
Dept. of Education
Kennedy Hall
Cornell University
Ithaca, NY 14853-4203
gmh5@cornell.edu
1-421

Bonnie Schappelle
San Diego State University
San Diego, CA 92182
BSchappe@sunstroke.sdsu.edu
1-240

Deborah Schifter
Education Development Center
55 Chapel St.
Newton, MA 02158-1060
dschifter@edc.org
2-289

Mary Ellen Schmidt
307 Ovalwood Hall
1680 University Drive
The Ohio State University
Mansfield, OH 44906
schmidt.22@osu.edu
2-68

Elizabeth Senger
Dept. of Curriculum & Instruction
Louisiana State University
Baton Rouge, LA 70808
senger@asterix.ednet.lsu.edu
2-205

Kenneth L. Shaw
4750 Collegiate Drive
Florida State University
Panama City, FL 32405
kshaw@garnet.acns.fsu.edu
2-133

Barry E. Shealy
593 Baldy Hall
SUNY at Buffalo
Buffalo, NY 14260-1000
bshealy@ubvms.cc.buffalo.edu
2-109

Edward A. Silver
Learning Research and Development Center
3939 O'Hara Street
Pittsburgh, PA 15260
eas@vms.cis.pitt.edu
1-61

Linda Simonsen
Dept. of Mathematical Sciences
Montana State University
Bozeman, MT 59717-0240
simonsen@math.montana.edu
2-242

Rose Sinicrope
Dept. of Mathematics
East Carolina University
Greenville, NC 27858
masinicr@ecuvm.cis.ecu.edu
1-427

Arthur R. Skelly, Jr.
Rickards High School
2-31

David Slavit
322 Neill Hall
Washington State University
Pullman, WA 99164-3113
dslavit@wsu.edu
1-284, 2-331

Vladimir Sloutsky
College of Education &
Center for Cognitive Science
The Ohio State University
Columbus, OH 43210
vsloutsk@magnus.acs.ohio-state.edu
2-17

Margaret Schwan Smith
706 LRDC
3939 O'Hara St.
Pittsburgh, PA 15260
pegs@vms.cis.pitt.edu
2-181

Vincent Snipes
159 Herlong Dr., Apt. 1
Tallahassee, FL 32310
vsnipes@garnet.acns.fsu.edu
2-69

Alejandro Solano
Mathematics Education
219 Carothers Hall
Florida State University
Tallahassee, FL 32306-3032
asolano@garnet.acns.fsu.edu
1-314

Jesse Solomon
Brighton High School
Boston, MA
jsolomon@k12.oit.umass.edu
1-291

S. L. Sproule
Curriculum and Instruction
Florida State University
Tallahassee, FL 32306-3032
ssproule@mailer.fsu.edu
2-297

Robert E. Stake
University of Illinois
stake@ux1.cso.uiuc.edu
1-240

Elena Steencken
10 Seminary Place
Rutgers University
New Brunswick, NJ 08903
steencke@gandalf.rutgers.edu
2-276

Mary M. Sullivan
Dept. of Mathematics
1071 Blue Hill Ave.
Curry College
Dedham, MA 02026
msulliva@curry.edu
1-338

Elisa Sunflower
California State University - Chico
Chico, CA 95929
2-338

Kyoko Suzuki
210 Education Bldg.
1310 S. Sixth St.
University of Illinois at Urbana-Champain
Champaign, IL 61820
k-suzuki@students.uiuc.edu
1-183, 2-315

Sylvia R. Taube
University of Texas - Pan American
West University Dr.
Edinburg, TX 78539
SRTAUBE@PANAM.EDU
1-428

Anne R.Teppo
Dept. of Mathematical Sciences
Montana State University
Bozeman, MT 59715
teppo@math.montana.edu
1-147

Marcia L. Tharp
3409 Sandpiper Rd.
Old Dominion University
Virginia Beach, VA 23456
MLT100f@eagle.cc.odu.edu
2-73

A. D. Thompson
Curriculum and Instruction
Florida State University
Tallahassee, FL 32306-3032
2-297

Carol A. Thornton
Mathematics Dept.
Illinois State University
Normal, IL 61790-4520
thornton@math.ilstu.edu
1-169

Sandra Davis Trowell
Teacher Education
Box 870213
The University of Alabama
Tuscaloosa, AL 35487-0231
strowell@bamaed.ua.edu
2-65

Pamela Turner
105 Aderhold Hall
University of Georgia
Athens, GA 30602
pturner@moe.coe.uga.edu
2-115, 2-155

Marta Elena Valdemoros
Centro de Investigaciony Estudios
Avanzados del IPN
Nicolas San Juan 1421
Colonia del Valle
Mexico, Distrito Federal, 03100
1-407

Laura Van Zoest
Dept. of Mathematics
Western Michigan University
Kalamazoo, MI 49008
laura.vanzoest@wmich.edu
2-130

Draga Vidakovic
CRMSE
Box 7801; 315 Poe Hall
North Carolina State University
Raleigh, NC 27695-7801
draga@poe.coe.ncsu.edu
2-344

Avril von Minden
608 Allen Hall
Dept. of Ed. Foundations
Horrabin Hall-80
Macomb, IL 61455
avon@wvnvm.wvnet.edu
2-243

Sharon B. Walen
SIMMS Project
Montana State University
Bozeman, MT 59717-2281
walen@math.montana.edu
2-6

Tad Watanabe
Towson State University
Dept. of Mathematics
Towson, MD 21204
tad@midget.towson.edu
1-383, 1-390, 2-137

David Webb
School of Education
University of California - Santa Barbara
Santa Barbara, CA 93106
1-229, 1-373

Grayson H. Wheatley
Mathematics Education
219 Carothers Hall
Florida State University
Tallahassee, FL 32306-3032
gwheatle@garnet.acns.fsu.edu
1-247, 1-314, 2-13, 2-262, 2-200

Joy Whitenack
Vanderbilt University
Nashville, TN 37203
WHITENJW@ctrvax.vanderbilt.edu
2-290

Steven R. Williams
Dept. of Mathematics
Brigham Young University
Provo, UT
williams@math.byu.edu
2-6

Linda Dager Wilson
Dept. of Educational Development
University of Delaware
Newark, DE 19711
ldwilson@strauss.udel.edu
1-253

Melvin (Skip) Wilson
610 E. University, 1228 SEB
University of Michigan
Ann Arbor, MI 48109
skipwils@umich.edu
2-162, 2-233

Patricia S. Wilson
105 Aderhold Hall
University of Georgia
Athens, GA 30602
pwilson@moe.coe.uga.edu
2-91

Erna Yackel
Purdue University - Calumet
Dept. of Mathematics, Computer Science, & Statistics
Hammond, IN 46323
yackeleb@nwi.calumet.purdue.edu
1-3

Vicki Zack
5822 Einstein Avenue
Montreal, Quebec
Canada H4W 2X6
vicki_z@ceo.sts-systems.ca
1-354

Richard A. Zang
Mathematics Dept.
University of New Hampshire at Manchester
Manchester, NH 03102
raz@christa.unh.edu
1-199

Rose Mary Zbiek
289 Lindquist Center North
University of Iowa
Iowa City, IA 52242
rzbiek@education-po.education.uiowa.edu
2-214